On composition operators on the Wiener algebra of Dirichlet series

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Abstract. We show that the symbol of a bounded composition operator on the Wiener algebra of Dirichlet series does not need to belong to this algebra. Our example even gives an absolutely summing (hence compact) composition operator.

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1 Introduction

In [3] (see also [5]), composition operators on the Wiener algebra $\mathcal{A}^+$ of all absolutely convergent Dirichlet series were studied.

Recall that $\mathcal{A}^+$ is the space of all analytic maps $f\colon \mathbb{C}_0 \to \mathbb{C}$ which can be written

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \text{with} \qquad \|f\|_{\mathcal{A}^+} := \sum_{n=1}^{\infty} |a_n| < +\infty \,,$$

where, for $\theta \in \mathbb{R}$, we note $\mathbb{C}_\theta = \{z \in \mathbb{C}\,;\ \Re\mathrm{e}\, z > \theta\}$. If $\phi\colon \mathbb{C}_0 \to \mathbb{C}_0$ is an analytic function, the composition operator $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ of symbol ϕ on this space is defined as $C_\phi(f) = f \circ \phi$. Gordon and Hedenmalm, for the Hilbert space $\mathscr{H}^2$, showed in [6] that such a symbol has necessarily the form

$$\phi(s) = c_0\, s + \varphi(s)\,, \tag{1.1}$$

where $c_0 \geq 0$ is an integer and φ is a convergent Dirichlet series with values in $\mathbb{C}_0$, that is $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_0$ is an analytic function which can be written $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ for $\Re\mathrm{e}\, s$ large enough. Moreover, this Dirichlet series is uniformly convergent in $\mathbb{C}_\varepsilon$ for all $\varepsilon > 0$ ([13, pages 1625–1626 and Theorem 3.1]; see also [12, Theorem 8.4.1, page 245]).

It is shown in [3, Theorem 2.3] that C_ϕ is bounded on $\mathcal{A}^+$ if and only if $\sup_{N\geq 1} \|N^{-\phi}\|_{\mathcal{A}^+} < +\infty$, and that it is compact if and only if $\|N^{-\phi}\|_{\mathcal{A}^+} \underset{N\to\infty}{\longrightarrow} 0$. Note that it is actually proved in [6, Theorem 4] that if $N^{-\phi}$ is a Dirichlet series for all $N \geq 1$, then ϕ as necessarily the form (1.1). Then $\|N^{-\phi}\|_{\mathcal{A}^+} = \|N^{-\varphi}\|_{\mathcal{A}^+}$, so c_0 plays no role, so we assume in the sequel that $c_0 = 0$.

When X is a Banach space of analytic functions that contains the identity map $u\colon z \mapsto z$, and $C_\phi\colon X \to X$ is a composition operator, then $\phi = C_\phi(u)$ belongs to X. For $X = \mathcal{A}^+$, it is not the case, so it is natural to ask if $\varphi \in \mathcal{A}^+$ when $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is a bounded composition operator. The object of this short note is to give a negative answer (Theorem 2.1).

Let us point out that it is proved in [3, Proposition 2.9] that $\varphi \in \mathcal{A}^+$ does not suffice to have a bounded composition operator on $\mathcal{A}^+$; the symbol is even a Dirichlet polynomial

$$\varphi(s) = c_1 + c_r r^{-s} + c_{r^2} r^{-2s}$$

where $r \geq 2$ is an integer and $c_r, c_{r^2} > 0$. For such a Dirichlet polynomial, it is proved that C_φ is not bounded if $\Re\mathrm{e}\, c_1 < \frac{(c_r)^2}{8\, c_{r^2}}$ and $c_r \leq 4\, c_{r^2}$ (for example, $c_r = 4$, $c_{r^2} = 1$ and $\Re\mathrm{e}\, c_1 < 2$).

2 Main result

Recall that $\varphi\colon \mathbb{C}_0 \to \mathbb{C}$ is a convergent Dirichlet series, if φ is analytic on $\mathbb{C}_0$ and we can write $\varphi(s) = \sum_{n=1}^\infty a_n n^{-s}$ for $\Re\mathrm{e}\, s$ large enough.

Theorem 2.1. *There exists a convergent Dirichlet series φ inducing a bounded composition operator $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$, but such that $\varphi \notin \mathcal{A}^+$. Moreover, $\varphi \in \mathscr{H}^p$ for all $p < \infty$ and C_φ is compact and absolutely summing.*

Let us recall the definition of the Hardy space $\mathscr{H}^p$ of Dirichlet series, following [2]. That uses the Bohr representation of Dirichlet series. Let $(p_j)_{j\geq 1}$ be the increasing sequence of all the prime numbers (so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on). If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the decomposition of the integer n in prime factors, to the Dirichlet series $\varphi(s) = \sum_{n=1}^\infty a_n n^{-s}$ is associated the Taylor series $(\Delta\varphi)(z) = \sum_{\boldsymbol{\alpha}} a_n z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_r^{\alpha_r}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r, 0, 0, \ldots)$. Due to Kronecker's theorem, φ is bounded if and only if $\Delta\varphi$ is bounded, and $\|\varphi\|_\infty = \|\Delta\varphi\|_\infty$. The Hardy space $\mathscr{H}^p$ is the space of all convergent Dirichlet series φ for which $\Delta\varphi$ belongs to the Hardy space $H^p(\mathbb{T}^\infty)$, with the norm $\|\varphi\|_{\mathscr{H}^p} = \|\Delta\varphi\|_{H^p}$.

Note that $\mathcal{A}^+$ is isometrically isomorphic, by this map Δ, to the Wiener algebra $A^+(\mathbb{T}^\infty)$.

Let us also recall that a bounded linear map $u\colon X \to Y$ between two Banach spaces X and Y is r-summing ($1 \leq r < \infty$) if there is a positive constant K such that

$$\bigg(\sum_{k=1}^n \|u(x_k)\|^r\bigg)^{1/r} \leq K \sup_{\xi \in B_{X^*}} \bigg(\sum_{k=1}^n |\xi(x_k)|^r\bigg)^{1/r}$$

for all $x_1, \ldots, x_n \in X$, $n \geq 1$, and where B_{X^*} is the unit ball of X^*. For $r = 1$, these operators are also said absolutely summing.

Proof of the theorem. We are going to take a symbol φ of the form $\varphi(s) = \sum_{k=1}^{\infty} c_k\, 2^{-ks} = f(2^{-s})$, where $f\colon \mathbb{D} \to \mathbb{C}_0$ is an analytic function such that $\sup_{N\geq 1} \|N^{-f}\|_{A^+(\mathbb{D})} < +\infty$, but $f \notin H^\infty$.

Recall that $A^+ = A^+(\mathbb{D})$ is the space of all analytic functions $u\colon \mathbb{D} \to \mathbb{C}$ such that $u(z) = \sum_{n=0}^{\infty} a_n z^n$, with $\|u\|_{A^+(\mathbb{D})} := \sum_{n=0}^{\infty} |a_n| < +\infty$.

We choose for f a conformal map sending the unit disk $\mathbb{D}$ onto the half-strip

$$R = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 1 \text{ and } |\mathfrak{Im}\, z| < \pi\}\,.$$

Explicitly, we take $f = \tau_1 \circ L \circ h \circ c \circ T$, where

$$T(z) = \frac{1+z}{1-z}\,;\quad c(z) = \mathrm{e}^{i\pi/4}\sqrt{z}\,;$$
$$h(z) = \frac{iz+1}{z+i}\,;\quad L(z) = -2\log z\,;$$

and $\tau_1(z) = z+1$. T maps the unit disk $\mathbb{D}$ onto the right-half plane; then c sends the right-half plane onto the first quadrant; h the first quadrant onto the right-half of $\mathbb{D}$; L this right-half of $\mathbb{D}$ onto the half-strip $\{|\mathfrak{Im}\, z| < \pi, \mathfrak{Re}\, z > 0\}$, and finally the translation τ_1 sends this half-strip onto the half-strip R.

This map is clearly not in H^∞, but, for every $\beta \in (0, \pi/2)$, there is a positive constant C_β such that $R + C_\beta$ is contained in the angular sector of vertex 0 and of opening β; it follows (see [4, Theorem 3.2]) that $f + C_\beta \in H^p$ for all $p < \pi/\beta$; so $f \in H^p$ for all $p < \infty$. We can also see that

$$f(\mathrm{e}^{it}) = \alpha\, \log|\, i - \mathrm{e}^{it}| + g(t)\,,$$

with $g \in L^\infty$ and α a constant, so that $f \in H^{\Psi_1}$, the Hardy-Orlicz space attached to the Orlicz function $\Psi_1(x) = \mathrm{e}^x - 1$, and

$$\|f\|_p = O\,(p)$$

as p goes to infinity.

Since $f \notin H^\infty$, we a fortiori have $f \notin A^+$, so $\varphi \notin \mathcal{A}^+$. However $\varphi \in \mathscr{H}^p$ for all $p < \infty$ since $f \in H^p$ for these values of p.

We now have to show that $N^{-\varphi} \in \mathcal{A}^+$, i.e. $N^{-f} \in A^+$, for all $N \geq 1$. This is clear for $N = 1$. For $N \geq 2$, we have $N^{-f} = \exp\,(-f \log N)$, and the range of $f \log N$ is the half-strip

$$R_N = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > \log N \text{ and } |\mathfrak{Im}\, z| < \pi \log N\}\,.$$

Under the exponential map e^{-z}, ∂R_N is transformed as follows:

1) the vertical segment $[\log N - i\pi \log N, \log N + i\pi \log N]$ is sent onto the circle of center 0 and radius $1/N$, browsed $\log N$ times;

2) the half-line $\{t\, \log N + i\pi \log N\,;\ t \geq 1\}$ is one-to-one mapped onto the radius $(0, \mathrm{e}^{-i\pi \log N}/N]$;

3) the half-line $\{t\, \log N - i\pi \log N\,;\ t \geq 1\}$ is one-to-one mapped onto the radius $(0, \mathrm{e}^{i\pi \log N}/N]$.

Hence, if $F_N = N^{-f}$, we have

$$\int_0^{2\pi} |F_N'(\mathrm{e}^{it})| \, dt = \frac{2}{N} + 2\pi \, \frac{\log N}{N} < +\infty \, ,$$

so $(N^{-f})' \in H^1$. By Hardy's inequality (see [4, Corollary page 48]), it follows that $N^{-f} \in A^+$, and there exists a positive constant C such that $\|N^{-f}\|_{A^+} \leq C \log N/N$. In particular, $\|N^{-\varphi}\|_{\mathcal{A}^+} = \|N^{-f}\|_{A^+} \underset{N\to\infty}{\longrightarrow} 0$, so $C_\varphi \colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact, by [3, Theorem 2.3].

To end the proof, remark that, writing $u_N(s) = N^{-s}$, we have

$$\sum_{N=2}^{\infty} \|C_\varphi(u_N)\|_{\mathcal{A}^+}^2 = \sum_{N=2}^{\infty} \|N^{-\varphi}\|_{\mathcal{A}^+}^2 \leq C^2 \sum_{N=2}^{\infty} \frac{(\log N)^2}{N^2} < +\infty \, .$$

Hence, using the Cauchy-Schwarz inequality, we can define a bounded linear operator $S \colon \ell_2 \to \mathcal{A}^+$ by sending the N-th vector e_N of the canonical basis of ℓ_2 to $N^{-\varphi}$, and we have the factorization $C_\varphi = ST$, where $T \colon \mathcal{A}^+ \to \ell_2$ is defined by setting $T(u_N) = e_N$. But $\mathcal{A}^+$ is isometrically isomorphic to ℓ_1, and the canonical injection from ℓ_1 to ℓ_2 is 1-summing (this was first remarked by Pietsch [10, § 1, Satz 10], and it is a particular case of the Grothendieck theorem). Let us recall why that holds. To each $(\alpha_k)_{k\geq 1} \in \ell_1$, we associate the L^∞ function $\sum_{k=1}^\infty a_k r_k$, where $(r_k)_{k\geq 1}$ is the sequence of the Rademacher functions on $[0,1]$; the canonical injection from $L^\infty(0,1)$ into $L^1(0,1)$ is absolutely summing (see [11, top of page 11], or [1, Remark 8.2.9]) and, by Khintchin's inequalities (see [8, Chapitre 0, Théorème IV.1], or [9, Chapter 1, Theorem IV.1]), the L^1-norm of $\sum_{k=1}^\infty \alpha_k r_k$ is equivalent to $\big(\sum_{k=1}^\infty |\alpha_k|^2\big)^{1/2}$.

It follows that C_φ is 1-summing. □

Note that, since $\mathcal{A}^+ \cong \ell_1$ has the Schur property, and since every q-summing operator is weakly compact, every q-summing operator into $\mathcal{A}^+$ is compact.

A slight modification of the proof gives a variant of Theorem 2.1.

Theorem 2.2. *For every $p \in (1, \infty)$, there exists a convergent Dirichlet series φ such that $\varphi \in \mathscr{H}^q$ for all $q < p$, but $\varphi \notin \mathscr{H}^p$, and such that φ induces a bounded composition operator $C_\varphi \colon \mathcal{A}^+ \to \mathcal{A}^+$. Moreover, C_φ is compact and is absolutely summing.*

Proof. We replace the conformal map f of Theorem 2.1 by a conformal map f from $\mathbb{D}$ onto the intersection of the angular sector $\{z \in \mathbb{C}_0 \, ; \ |\arg z| < \pi/2p\}$ with the half-plane $\mathbb{C}_1$. We have $f \notin H^p$ though $f \in H^q$ for all $q < p$ (see [7, top of page 237]). We set $\varphi(s) = f(2^{-s})$ for $\Re\mathrm{e}\, s > 0$. We have $\varphi \in \mathscr{H}^q$ for all $q < p$, but $\varphi \notin \mathscr{H}^p$.

For all $N \geq 1$, we have $N^{-f} \in A^+$. This is clear for $N = 1$. For $N \geq 2$, let $\beta = \pi/2p$ and $\gamma_\pm(t) = \exp(-\,\mathrm{e}^{\pm i\beta} t)$, with $t \geq \log N/\cos\beta$, then the boundary of the range of $F_N = N^{-f}$ is the union of γ_+ and γ_-, and of the circle of radius $1/N$ browsed $(1/\pi)\,(\tan\beta)\,\log N$ times. Since

$$\int_{\log N/\cos\beta}^{+\infty} |\gamma'_\pm(t)|\,dt = \int_{\log N/\cos\beta}^{+\infty} \mathrm{e}^{-(\cos\beta)t}\,dt = \frac{1}{\cos\beta}\,\frac{1}{N}\,,$$

we get that

$$\int_0^{2\pi} |F'_N(\mathrm{e}^{it})|\,dt = \frac{2}{\cos\beta}\,\frac{1}{N} + \frac{\tan\beta}{\pi}\,\frac{\log N}{N} < +\infty\,,$$

so $F'_N \in H^1$ and $F_N = N^{-f} \in A^+$.

Moreover, $\|N^{-\varphi}\|_{\mathcal{A}^+} = \|N^{-f}\|_{A^+} \lesssim \log N/N \underset{N\to\infty}{\longrightarrow} 0$, so C_φ is compact on $\mathcal{A}^+$.

Since $\sum_{N=1}^\infty \|N^{-\varphi}\|^2_{\mathcal{A}^+} < +\infty$, we get, as in the proof of Theorem 2.1, that C_φ is 1-summing. □

3 Another result

Let us remark that the example of [3, Proposition 2.9] quoted in the Introduction is a Dirichlet polynomial φ such that $N^{-\varphi} \in \mathcal{A}^+$ for all $N \geq 1$, though the associated composition operator C_φ is not bounded from $\mathcal{A}^+$ into itself.

Theorem 3.1. *For any non-negative number $A \in \mathbb{R}_+$, there exists a convergent Dirichlet series φ such that $\varphi(\mathbb{C}_0) \subseteq \mathbb{C}_A$, but such that, for any $N \geq 2$, we have $N^{-\varphi} \notin \mathcal{A}^+$.*

In particular, the composition operator C_φ is not bounded from $\mathcal{A}^+$ into itself.

That will follow from the following result.

Lemma 3.2. *Let $N \geq 2$ and let $\varphi\colon \mathbb{C}_0 \to \mathbb{C}$ be an analytic function such that $N^{-\varphi} \in \mathcal{A}^+$. Then, for every $a \in \mathbb{R}$, either $\varphi(s)$ has a limit as s tends to ia, or $\mathfrak{Re}\,\varphi(s)$ tends to $+\infty$ as s tends to ia.*

Proof. Since $N^{-\varphi}$ belongs to $\mathcal{A}^+$, it is continuous on $\overline{\mathbb{C}_0}$; hence it has limits at every point $ia \in i\mathbb{R}$. If this limit is 0, that means that $\mathfrak{Re}\,\varphi(s) \underset{s\to ia}{\longrightarrow} +\infty$. If not, we have $\lim_{s\to ia} N^{-\varphi(s)} = c \neq 0$. Therefore, if $r < |c|$, there is some open disk V centered at ia such that $N^{-\varphi(s)} \in D(c,r)$ when $s \in V$. Let F be a determination of the logarithm in $D(c,r)$. Then

$$\psi(s) := F[N^{-\varphi(s)}] \underset{s\to ia}{\longrightarrow} F(c)\,.$$

Since

$$\exp[-\varphi(s)\,\log N] = N^{-\varphi(s)} = \exp[\psi(s)]\,,$$

there exists $k = k(s) \in \mathbb{Z}$ such that $\psi(s) = -\varphi(s)\,\log N + 2k(s)\pi i$. But φ and ψ are continuous on $V \cap \mathbb{C}_0$; it follows that $k(s)$ is constant. Therefore

$$\varphi(s) = -\psi(s) + 2k\pi i \underset{s\to ia}{\longrightarrow} -F(c) + 2k\pi i\,.$$ □

Proof of Theorem 3.1. Let

$$\varphi(s) = A + 1 + \exp\left(- \frac{1 + 2^{-s}}{1 - 2^{-s}} \right) \cdot$$

Then φ is a convergent Dirichlet series and maps $\mathbb{C}_0$ into $\mathbb{C}_A$. However, $N^{-\varphi} \notin \mathcal{A}^+$ because φ does not have a limit as s goes to 0. $\square$

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