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ABSTRACT. Let F be a field of characteristic 2 and K a purely inseparable modular extension of F. Our aim in this paper is to give a complete classification of anisotropic semisingular F-quadratic forms φ that have over K a maximal Witt index and a defect index at least equal to the half of the dimension of the quasilinear part. The case of totally singular quadratic forms will be also treated. Our method also allows us to classify the forms φ under the unique hypothesis of maximality of the Witt index over K. This extends a recent result of Sobiech studying the hyperbolicity of nonsingular F-quadratic forms over K [17]. Based on our classifications, we are able to give necessary and sufficient conditions under which an anisotropic semisingular F-quadratic form has a given Witt index over K. We also study the quasi-hyperbolicity of semisingular F-quadratic forms over function fields of certain irreducible polynomials and extend to such forms many results established by the first author in [11].

1. INTRODUCTION

Throughout this paper F denotes a field of characteristic 2. Let φ be an anisotropic quadratic form over F, and K a field extension of F. An important problem in the algebraic theory of quadratic forms is to study the behavior of φ after extending scalars to K. A question in this sense consists in giving the conditions under which the form φ becomes isotropic over K. Similarly, when φ is nonsingular (*resp.* singular), we also ask for the hyperbolicity (*resp.* quasi-hyperbolicity) over K.

Singular quadratic forms split into two classes: Totally singular forms and semisingular forms (see Section 2 for details). The notion of quasi-hyperbolicity is a generalization of hyperbolicity to singular quadratic forms. Recall that a totally singular quadratic form φ is called *quasi-hyperbolic* if $i_d(\varphi) \geq \frac{\dim \varphi}{2}$, where $i_d(\varphi)$ is the defect index of φ. Similarly, a semisingular quadratic form φ is called *quasi-hyperbolic* if $i_t(\varphi) \geq \frac{\dim \varphi}{2}$, where $i_t(\varphi)$ is the total index of φ.

In the particular case when φ is an anisotropic semisingular quadratic form and $K = F(p)$ is the function field of an irreducible polynomial $p \in F[x_1, \cdots, x_n]$, the quasi-hyperbolicity of φ over K implies that p is inseparable, meaning that $p \in F[x_1^2, \cdots, x_n^2]$, and the condition $i_t(\varphi_K) \geq \frac{\dim \varphi}{2}$ is equivalent to saying that the Witt index $i_W(\varphi_K)$ is maximal (i.e., $i_W(\varphi_K) = \frac{\dim \varphi - \dim \mathrm{ql}(\varphi)}{2}$) and the quasilinear part $\mathrm{ql}(\varphi)$ of φ is quasi-hyperbolic over K (Proposition 2.3). This equivalence is no longer true for a field extension given by the compositum of function fields of irreducible polynomials as the simple example 2.2 shows. For this reason we introduce the following definition: *A semisingular quadratic form φ is called strictly quasi-hyperbolic if the Witt index $i_W(\varphi)$ is maximal and $\mathrm{ql}(\varphi)$ is quasi-hyperbolic.*

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Our aim in this paper is to study the strict quasi-hyperbolicity of anisotropic semisingular quadratic forms over purely inseparable modular extensions. This is motived by a recent work of Sobiech [17], which gave a complete answer to the hyperbolicity of nonsingular F-quadratic forms over purely inseparable extensions of F (not necessarily modular). Here we restrict ourselves to purely inseparable modular extensions due to the fact that we can't control the quasi-hyperbolicity of totally singular forms over purely inseparable extensions which are not modular. Corollary 3.2 gives a complete answer to the quasi-hyperbolicity of anisotropic totally singular forms over purely inseparable modular extensions completing a previous result of Hoffmann on the isotropy for the same forms and extensions [4, Theorem 5.9]. The main result of this paper is Theorem 7.1 that gives a complete answer to the strict quasi-hyperbolicity of semisingular quadratic forms over purely inseparable modular extensions of exponent > 1. The case of exponent 1, i.e., multiquadratic purely inseparable extensions is answered in Theorem 6.1.

Theorem 7.1 is a generalization to semisingular quadratic forms of a result of Sobiech on the hyperbolicity of nonsingular quadratic forms over purely inseparable extensions. More precisely, the generator forms mentioned in equation (3.2) will play a crucial role in our classifications, and already appear in [17]. For his method, Sobiech first computed the kernel of purely inseparable extensions in the setting of Kato-Milne cohomology and then translated it to quadratic forms with the help of Kato's isomorphism giving the connection between nonsingular quadratic forms and differential forms [8]. For our proof of Theorem 7.1, we take a completely different method specific to singular quadratic forms since there is no connection established between such forms and differential forms. In a way, situation is a bit subtle and rigorous in case of singular quadratic forms since many well known results like Witt cancellation are not applicable for singular forms.

To prove Theorem 7.1, we first treat the case of a simple purely inseparable extension and working on the same lines extend the proof for any purely inseparable modular extension with slightly more technical computations. For the convenience of the reader, we mention the proof of both cases. We give here the outline of proof for a simple purely inseparable extension $F(\sqrt[2^n]{d})$, $n \geq 2$. We divide this proof into two major steps: We first study the case where the semisingular form φ is of the type $(1, s)$ and becomes quasi-hyperbolic over $F(\sqrt[2^n]{d})$ (Proposition 5.2). In this first step, we use an induction on n, and we also take help of a recent result of ours [15, Theorem 1.1], which gives us that φ represents the polynomial $x^{2^n} + d$ up to a scalar represented by φ. Further, we use the Cassel-Pfister theorem (Proposition 4.2). The second step deals with an induction on the dimension of the nonsingular part of the semisingular form using a tricky argument based on the so-called completion lemma (Lemma 2.6).

This paper is organized as follows. In section 2, we recall some definitions and results on quadratic forms in characteristic 2. In Section 3, we give some preliminaries concerning the quasi-hyperbolicity of totally singular forms over purely inseparable modular extensions, and introduce some background on the generator forms in equation (3.2). Section 4 is devoted to computations on these generator forms when they are defined over an inseparable quadratic extension of F, and to some representation results. In section 5, we prove Theorem 5.1 that treats the quasi-hyperbolicity over purely inseparable simple extensions. Section 6 contains the proof of Theorem 6.1 concerning multiquadratic purely inseparable extensions, and then in Section 7 we give the proof of our main result (Theorem 7.1). As an application we completely answer the isotropy of semisingular quadratic forms over purely inseparable modular extensions (Theorem 8.4). In the last section, we introduce the notion of a strong Pfister set, in the spirit of what was done in [11], and give some domination results when a semisingular quadratic form represents certain inseparable polynomials. We also give a complete classification of quadratic forms that become quasi-hyperbolic over the function fields of the following irreducible polynomials $x^4 + ax^2 + b$, $x^4 + ay^2 + b$ and $x^4 + ay^4 + bx^2$.

2. Background on quadratic forms

Let φ be a quadratic form over F with underlying vector space V. The radical of φ is the F-vector space

$$\mathrm{Rad}(\varphi) := \{v \in V \mid B_\varphi(v, w) = 0 \text{ for all } w \in V\},$$

where B_φ is the polar form of φ. The restriction of φ to $\mathrm{Rad}(\varphi)$ is given by a diagonal quadratic form $\sum_{i=1}^{s} c_i x_i^2$ that we denote $\langle c_1, \dots, c_s\rangle$. Obviously, this form is unique up to isometry, we call it the quasilinear part of φ and denote it $\mathrm{ql}(\varphi)$.

Over the space V, the form φ can be written as follows:

$$\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \dots \perp [a_r, b_r] \perp \mathrm{ql}(\varphi), \tag{2.1}$$

where $\simeq$ and $\perp$ denotes the isometry and orthogonal sum of quadratic forms, and $[a, b]$ denotes the quadratic form $ax^2 + xy + by^2$. In this case, we say that φ is of type (r, s). As in equation (2.1), the form φ is called:

- nonsingular if $s = 0$,
- totally singular if $r = 0$,
- semisingular if $r > 0$ and $s > 0$.
- singular if $s > 0$.

We denote by $\dim \varphi$ the dimension of φ. We say that φ represents α if there exists $v \in V$ such that $\varphi(v) = \alpha$. We denote by $D_F(\varphi)$ the set of values in F^* represented by φ. We let $D_F^0(\varphi) = D_F(\varphi) \cup \{0\}$.

For $a, b \in F$ and $\alpha \in F^*$, let $[a; \alpha; b]$ denote the binary quadratic form $ax^2 + \alpha xy + by^2$.

For $p \in F[x_1, x_2, \dots, x_n]$ an irreducible polynomial, let $F(p)$ be the field of fractions of the quotient ring $F[x_1, \dots, x_n]/(p)$. We call it the function field of p.

A scalar $\alpha \in F^*$ is called a norm of φ if $\varphi \simeq \alpha\varphi$.

For a field extension K/F and φ an F-quadratic form, let φ_K denote the form φ considered as a form over K by scalar extension.

2.1. **Witt decomposition.** We say that φ is isotropic if there exists $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$, otherwise φ is called anisotropic.

For any integer $n \geq 0$, the quadratic form $\underbrace{\varphi \perp \dots \perp \varphi}_{n \text{ times}}$ is denoted by $n \times \varphi$.

The quadratic form φ uniquely decomposes as follows:

$$\varphi \simeq \varphi_{an} \perp i \times [0, 0] \perp j \times \langle 0\rangle,$$

where φ_{an} is an anisotropic quadratic form. We call φ_{an} the anisotropic part of φ, and the integer i (*resp.* j) is called the Witt index (*resp.* the defect index) of φ. The integer $i+j$ is called the total index of φ. We denote i, j and $i+j$ by $i_W(\varphi)$, $i_d(\varphi)$ and $i_t(\varphi)$, respectively. The form φ is called nondefective when $i_d(\varphi) = 0$.

Two quadratic forms φ_1 and φ_2 are Witt-equivalent, denoted by $\varphi_1 \sim \varphi_2$, if $\varphi_1 \perp m \times [0,0] \simeq \varphi_2 \perp n \times [0,0]$ for some $m, n \in \mathbb{N}$.

Definition 2.1. *Let φ be a singular quadratic form.*
(1) If φ is totally singular, then φ is called quasi-hyperbolic if $i_d(\varphi) \geq \frac{\dim \varphi}{2}$.
(2) If φ is semisingular, then φ is called quasi-hyperbolic (resp. strictly quasi-hyperbolic) if $i_t(\varphi) \geq \frac{\dim \varphi}{2}$ (resp. $i_W(\varphi) = \frac{\dim \varphi - \dim \mathrm{ql}(\varphi)}{2}$ and $\mathrm{ql}(\varphi)$ is quasi-hyperbolic).

Obviously, a semisingular quadratic form which is strictly quasi-hyperbolic is in particular quasi-hyperbolic. But the converse is not always true as the following easy example shows:

Example 2.2. *Consider the quadratic form $\varphi = [t_1, t_2] \perp \langle 1, t_3, t_4, t_5 \rangle$ defined over the rational function field $K = F(t_1, \cdots, t_5)$ in the indeterminates $t_1, \cdots, t_5$. Extending the scalars to $L = K(\sqrt{t_3}, \sqrt{t_4}, \sqrt{t_5})$ yields $\varphi_L \simeq [t_1, t_2] \perp \langle 1 \rangle \perp 3 \times \langle 0 \rangle$. Thus, $i_t(\varphi_L) \geq \frac{\dim \varphi}{2}$ but $i_W(\varphi_L) = 0$.*

However, the following proposition shows that the two versions of quasi-hyperbolicity coincide over the function field $F(p)$ for any irreducible $p \in F[x_1, \cdots, x_n]$.

Proposition 2.3. *(*[15, Prop. 2.2]*) Let $p \in F[x_1, \cdots, x_n]$ be an irreducible polynomial. Let φ be an anisotropic semisingular quadratic form. If φ is quasi-hyperbolic over $F(p)$, then p is inseparable and $\varphi_{F(p)}$ is strictly quasi-hyperbolic.*

We recall the following cancellation result:

Proposition 2.4. *(*[9, Proposition 1.2]* for (1),* [6, Lemma 2.6] *for (2)) Let φ_1, φ_2 be two quadratic forms (possibly singular). Suppose that one of the following conditions holds:*
(1) $\varphi_1 \perp \psi \simeq \varphi_2 \perp \psi$ for some nonsingular form ψ,
(2) $\varphi_1 \perp s \times \langle 0 \rangle \simeq \varphi_2 \perp s \times \langle 0 \rangle$ for some integer $s \geq 0$ and φ_1, φ_2 nondefective.
Then, $\varphi_1 \simeq \varphi_2$.

The following isometries are easy to prove and will be used in our proofs:

$$[a,b] \perp [c,d] \simeq [a+c, b] \perp [c, b+d],$$
$$[a,b] \perp \langle c \rangle \simeq [a+c, b] \perp \langle c \rangle\,,$$
$$[1,a] \simeq [1,a^2],$$
$$\alpha[a,b] \simeq [\alpha a, \alpha^{-1} b],$$
$$[a;\alpha;b] \simeq [a, b\alpha^{-2}],$$

for all $a, b, c \in F$ and $\alpha \in F^*$. In particular, the first isometry gives the equivalence $[a,b] \perp [a,d] \sim [a, b+d]$.

2.2. **Pfister forms.** For $a_1, \ldots, a_n \in F^*$, let $\langle a_1, \ldots, a_n \rangle_b$ be the diagonal bilinear form defined by:

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sum_{i=1}^{n} a_i x_i y_i.$$

Let $W(F)$ (*resp.* $W_q(F)$) be the Witt ring of regular symmetric F-bilnear forms (*resp.* the Witt group of nonsingular F-quadratic forms). The group $W_q(F)$ is endowed with a $W(F)$-module structure as follows: To any regular symmetric F-bilinear form B on a vector space U and nonsingular F-quadratic form φ on a vector space V, we associate a nonsingular quadratic form $B \otimes \varphi$ defined on $U \otimes_F V$ by:

$$B \otimes \varphi(u \otimes v) = B(u, u)\varphi(v) \text{ for any } (u, v) \in U \times V$$

and whose polar form is $B \otimes B_\varphi$ [2].

An n-fold bilinear Pfister form is a form of type $B = \langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b$ for some $a_i \in F^*$; we write $B = \langle\langle a_1, \cdots, a_n \rangle\rangle_b$ for short. If $Q : x \mapsto B(x, x)$ is the totally singular quadratic form associated to B, we write $Q = \langle\langle a_1, \cdots, a_n \rangle\rangle$ and call it an n-fold quasi-Pfister form. An $(n+1)$-fold quadratic Pfister form is a nonsingular quadratic form of type $\langle\langle a_1, \cdots, a_n \rangle\rangle_b \otimes [1, b]$ for some $a_i \in F^*$, $b \in F$, we denote it $\langle\langle a_1, \cdots, a_n; b]]$. Both Pfister and quasi-Pfister forms are round, i.e., $x \in F^*$ is represented by a Pfister (or quasi-Pfister) form π if and only if $\pi \simeq x\pi$ [2, Th. 2.4, page 95], [6, Prop. 8.5]. The set of forms isometric (*resp.* similar) to n-fold quadratic Pfister forms will be denoted by $P_n(F)$ (*resp.* $GP_n(F)$).

Let Q be a quadratic Pfister form and B a bilinear form. Using the roundness of Q, we prove the following: If $B \otimes Q$ is isotropic, then there exists an isotropic bilinear form B' such that $B \otimes Q \simeq B' \otimes Q$. In particular, $(B \otimes Q)_{an} \simeq C \otimes Q$ for some bilinear form C and $i_W(B \otimes Q)$ is divisible by $\dim Q$. Similarly, if we take Q a quadratic form and B a bilinear Pfister form, then $(B \otimes Q)_{an} \simeq B \otimes Q'$ for some quadratic form Q' and $i_W(B \otimes Q)$ is divisible by $\dim B$. For totally singular forms, we also have the following: If Q' is a quasi-Pfister form and ρ a totally singular form, then $(Q' \otimes \rho)_{an} \simeq Q' \otimes \delta$ for some totally singular form δ, and $i_d(\rho \otimes Q')$ is divisible by $\dim Q'$.

We also have the following fact: If Q be a quadratic Pfister form, B a bilinear form and $\alpha \in D_F(B \otimes Q)$, then there exists a bilinear form B' such that $B \otimes Q \simeq (\langle \alpha \rangle_b \perp B') \otimes Q$. Similarly, If Q' is a quasi-Pfister form, ρ a totally singular form and $\alpha \in D_F(Q' \otimes \rho)$, then there exists a totally singular form ρ' such that $Q' \otimes \rho \simeq Q' \otimes (\langle \alpha \rangle \perp \rho')$. These two facts are due to the roundness property of Pfister forms.

2.3. **Dominated forms.** Let φ and ψ be quadratic forms with underlying vector spaces V and W, respectively. We say that φ is dominated by ψ, denoted $\varphi \prec \psi$, if there exists an injective F-linear map $\sigma : V \longrightarrow W$ such that $\varphi(v) = \psi(\sigma(v))$ for all $v \in V$. We say that φ is weakly dominated by ψ if there exists $\alpha \in F^*$ such that $\alpha\varphi \prec \psi$. An explicit translation of the domination relation is given by the following theorem.

Theorem 2.5. ([6, Th. 3.4]) *Let φ and ψ be quadratic forms over F. Then, φ is dominated by ψ is there exist nonsingular quadratic forms φ_r and τ, nonnegative integers $s' \leq s \leq s''$,*

$c_i \in F (1 \leq i \leq s'')$ *and* $d_j \in F (1 \leq j \leq s')$ *such that* $\varphi \simeq \varphi_r \perp \langle c_1, \ldots, c_s \rangle$ *and*

$$\psi \simeq \varphi_r \perp \tau \perp [c_1, d_1] \perp \ldots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \ldots, c_{s''} \rangle .$$

We also mention the completion lemma due to Hoffmann and Laghribi which will play a crucial role in many proofs.

Lemma 2.6. *(*[6, Lemma 3.9]*) Let* R *and* R' *be nonsingular quadratic forms over* F*, and* $c_i, c_i', d_i \in F$*,* $1 \leq i \leq n$*. Suppose that* $R \perp \langle c_1, \ldots, c_n \rangle \simeq R' \perp \langle c_1', \ldots, c_n' \rangle$*. Then, there exist* $d_1', \ldots, d_n' \in F$ *such that* $R \perp [c_1, d_1] \perp \ldots \perp [c_n, d_n] \simeq R' \perp [c_1', d_1'] \perp \ldots \perp [c_n', d_n']$.

We derive from Lemma 2.6 some specific corollaries that we will need.

Corollary 2.7. *Let* $\varphi = R \perp \mathrm{ql}(\varphi)$ *be an anisotropic quadratic form over* F*, and* π *a quasi-Pfister form such that* $\mathrm{ql}(\varphi) \simeq \pi \otimes \gamma$ *for some totally singular form* γ*. Let* $c_1, \cdots, c_s \in D_F(\pi)$ *be such that* $\langle c_1, \cdots, c_s \rangle \prec \varphi$*. Suppose that* $1 \in D_F(R)$*. Then, there exist* $d_1, \cdots, d_s \in F$ *and* R' *a nonsingular form such that* $\varphi \simeq [c_1, d_1] \perp \cdots \perp [c_s, d_s] \perp R' \perp \mathrm{ql}(\varphi)$*. In particular,* $\dim R \geq 2s$*.*

Proof. First we claim that $D_F^0(\langle c_1, \cdots, c_s \rangle) \cap D_F^0(\mathrm{ql}(\varphi)) = \{0\}$. In fact, suppose we have a nonzero scalar $c \in D_F(\langle c_1, \cdots, c_s \rangle) \cap D_F(\mathrm{ql}(\varphi))$. Using the roundness of π, we get $\mathrm{ql}(\varphi) \simeq \pi \otimes (\langle c \rangle \perp \cdots)$. Since, $\langle c_1, \cdots, c_s \rangle$ is anisotropic as it is dominated by φ, we get $\langle c_1, \cdots, c_s \rangle \subset \pi$. Hence, $c \in D_F(\pi)$. Consequently, the condition $\mathrm{ql}(\varphi) \simeq \pi \otimes (\langle c \rangle \perp \cdots)$ implies that $1 \in D_F(\mathrm{ql}(\varphi))$, a contradiction because $1 \in D_F(R)$ and φ is anisotropic. Since $D_F^0(\langle c_1, \cdots, c_s \rangle) \cap D_F^0(\mathrm{ql}(\varphi)) = \{0\}$, the domination condition $\langle c_1, \cdots, c_s \rangle \prec \varphi$ ensures the existence of a nonsingular form R' and scalars $c_1', \cdots, c_s', d_1', \cdots, d_s' \in F$ such that $\langle c_1, \cdots, c_s \rangle \simeq \langle c_1', \cdots, c_s' \rangle$ and $\varphi \simeq [c_1', d_1'] \perp \cdots \perp [c_s', d_s'] \perp R' \perp \mathrm{ql}(\varphi)$ (Theorem 2.5). Moreover, by the completion lemma, there exist $d_1, \ldots, d_s \in F$ such that $[c_1', d_1'] \perp \cdots \perp [c_s', d_s'] \simeq [c_1, d_1] \perp \cdots \perp [c_s, d_s]$. Hence the claim. $\square$

Corollary 2.8. *Let* $d \in F^*$*,* R, R' *nonsingular* F*-quadratic forms and* Q *a totally singular* F*-quadratic form. Suppose*

$$(R \perp Q \perp dQ)_L \sim (R' \perp Q \perp dQ)_L,$$

where $L = F$ *or* L *is a multiquadratic purely inseparable extension of* F*. Then, there exists a nonsingular* F*-quadratic form* λ *such that:*

$$(R \perp Q)_L \sim (R' \perp \langle 1, d \rangle_b \otimes \lambda \perp Q)_L.$$

Proof. Let us write $Q = \langle a_1, \cdots, a_s \rangle$ for $a_1, \cdots, a_s \in F$. Applying Lemma 2.6 to the equivalence

$$(R \perp Q \perp dQ)_L \sim (R' \perp Q \perp dQ)_L,$$

we get

$$(R \perp Q)_L \perp \sum_{i=1}^{s} [0, da_i] \sim (R' \perp Q)_L \perp \sum_{i=1}^{s} [b_i, da_i] \tag{2.2}$$

for suitable $b_1, \ldots, b_s \in L$. Since $\langle a_i \rangle \sim \langle a_i \rangle \perp [a_i, db_i]$ and $\langle a_i \rangle \subset Q$, we may write (2.2) as follows

$$(R \perp Q)_L \sim R'_L \perp \langle 1, d \rangle_b \otimes \lambda \perp Q_L$$

where $\lambda = \sum_{i=1}^{s} [db_i, a_i]$ for all $1 \leq i \leq s$. Moreover, when $a_i \neq 0$ we have $[db_i, a_i] \simeq a_i[1, a_i db_i] \simeq a_i[1, (a_i db_i)^2]$ is defined over F since $L^2 \subset F$, as desired. □

Corollary 2.9. *Let R, R' be nonsingular F-quadratic forms, and $Q = \langle c_1, \cdots, c_s \rangle$, Q' totally singular F-quadratic forms. Let $d \in F^*$ be such that*

$$R \perp Q \perp dQ \perp Q' \sim R' \perp Q \perp dQ \perp Q'.$$

Then, for any $x_1, \cdots, x_s \in F$, there exist λ and R'' nonsingular F-quadratic forms such that $\dim R'' = 2s$ and

$$R \perp \sum_{i=1}^{s} [c_i, x_i] \perp Q' \sim R' \perp \langle 1, d \rangle_b \otimes \lambda \perp R'' \perp Q'.$$

Proof. We apply Lemma 2.6 to the equivalence $R \perp Q \perp dQ \perp Q' \sim R' \perp Q \perp dQ \perp Q'$, we get

$$R \perp \sum_{i=1}^{s} ([c_i, x_i] \perp [dc_i, 0]) \perp Q' \sim R' \perp \sum_{i=1}^{s} ([c_i, y_i] \perp [dc_i, z_i]) \perp Q'.$$

for suitable $y_i, z_i \in F$ for $1 \leq i \leq s$. Moreover, using $[c_i, y_i] \perp [dc_i, z_i] \sim \langle 1, d \rangle_b \otimes [c_i, y_i] \perp [dc_i, d^{-1}y_i + z_i]$, we obtain

$$R \perp \sum_{i=1}^{s} [c_i, x_i] \perp Q' \sim R' \perp \langle 1, d \rangle_b \otimes \lambda \perp R'' \perp Q',$$

where $\lambda = \sum_{i=1}^{s} [c_i, y_i]$ and $R'' = \sum_{i=1}^{s} [dc_i, d^{-1}y_i + z_i]$ of dimension $2s$, as desired. □

2.4. **A result on excellence.** A field extension L/F is called excellent for quadratic forms if for any F-quadratic form φ, there exists an F-quadratic form ψ such that $(\varphi_L)_{an} \simeq \psi_L$. The excellence property holds for totally singular quadratic forms over any field extension [12, Lem. 2.1]. More precisely, we have:

Lemma 2.10. *Let L/F be a field extension and φ a totally singular F-quadratic form. Then, there exists ψ a subform of φ such that $(\varphi_L)_{an} \simeq \psi_L$.*

Hence, using the uniqueness of the quasilinear part, we get the following lemma.

Corollary 2.11. *(*[6, Lemma 2.2]*) Let φ be a quadratic form over F, and L any field extension of F. Then, there exists ψ a subform of $\mathrm{ql}(\varphi)$ such that $(\mathrm{ql}(\varphi_L))_{an} \simeq \psi_L$.*

A classical example of excellent extensions is given by quadratic extensions of F (separable or inseparable). More generally, for multiquadratic purely inseparable extensions we recall the following important result:

Theorem 2.12. *(*[5, Main theorem]*) If L/F is a multiquadratic purely inseparable extension of F, then L/F is excellent for quadratic forms.*

3. ON QUASI-HYPERBOLICITY OVER PURELY INSEPARABLE MODULAR EXTENSIONS

Let K be a purely inseparable extension of F. We say that K/F is modular if there exist $K_1, \cdots, K_s$ simple purely inseparable extensions of F such that $K \simeq K_1 \otimes_F \cdots \otimes_F K_s$. In other words, K/F is a purely inseparable modular extension if there exist $d_1, \cdots, d_s \in F$ and $n_1, \cdots, n_s \in \mathbb{N}_0$ such that:

$$K = F(\sqrt[2^{n_1}]{d_1}, \dots, \sqrt[2^{n_s}]{d_s}) \text{ and } [K : F] = 2^{n_1+\cdots+n_s}. \tag{3.1}$$

In this case, the scalars $d_1, \cdots, d_s$ are 2-independent, and thus the quasi-Pfister form $\pi := \langle\langle d_1, \cdots, d_s\rangle\rangle$ is anisotropic. The exponent e of K is just the biggest integer among $n_1, \cdots, n_s$. It is the smallest integer e satisfying $K^{2^e} \subset F$.

We recall a result on the isotropy of totally singular forms over purely inseparable extensions which are modular.

Theorem 3.1. ([4, Theorem 5.9]) *Let K be as in (3.1) and $\pi = \langle\langle d_1, \cdots, d_s\rangle\rangle$. Let φ be an anisotropic totally singular F-quadratic form. Then, the following statements are equivalent:*
(1) φ_K is isotropic.
(2) $\pi \otimes \varphi$ is isotropic.

We use Theorem 3.1 to give criterion for quasi-hyperbolicity of totally singular quadratic forms over K.

Corollary 3.2. *Let K be as in (3.1) and $\pi = \langle\langle d_1, \cdots, d_s\rangle\rangle$. Let φ be an anisotropic totally singular F-quadratic form. Then, φ_K is quasi-hyperbolic if and only if $\pi \otimes \varphi$ is quasi-hyperbolic.*

Proof. Since π is round, we get $(\pi \otimes \varphi)_{an} \simeq \pi \otimes \varphi'$ for a suitable totally singular quadratic form. Let us write $\pi \otimes \varphi \simeq \epsilon \times \langle 0\rangle \perp \pi \otimes \varphi'$. Extending $\pi \otimes \varphi$ to K, and using the fact that $\pi_K \simeq \langle 1\rangle \perp (2^s - 1) \times \langle 0\rangle$, we get

$$\varphi_K \perp (2^s - 1) \dim \varphi \times \langle 0\rangle \simeq \epsilon \times \langle 0\rangle \perp \varphi'_K \perp (2^s - 1) \dim \varphi' \times \langle 0\rangle .$$

The form φ'_K is anisotropic because $\pi \otimes \varphi'$ is anisotropic (Theorem 3.1). By the uniqueness of anisotropic part, it follows from the previous isometry that $(\varphi_K)_{an} \simeq \varphi'_K$.

Obviously, $\dim(\pi \otimes \varphi') \leq \frac{\dim(\pi\otimes\varphi)}{2}$ iff $\dim \varphi' \leq \frac{\dim \varphi}{2}$, and thus $\pi \otimes \varphi$ is quasi-hyperbolic iff φ_K is quasi-hyperbolic. $\square$

In the case of simple purely inseparable extensions, the quasi-hyperbolicity criterion for totally singular forms has an equivalence formulation as given in the following proposition:

Proposition 3.3. ([4, Th. 7.7]) *Let $d \in F \setminus F^2$ and $K = F(\sqrt[2^n]{d})$ ($n \geq 1$). Let φ be an anisotropic totally singular F-quadratic form. Then, φ becomes quasi-hyperbolic over K iff $\varphi \simeq \langle 1, d\rangle \otimes \rho$ for suitable totally singular F-quadratic form ρ.*

Note that the form ρ in Proposition 3.3 is anisotropic over $F(\sqrt[2^n]{d})$ as $\varphi \simeq \langle 1, d\rangle \otimes \rho$ is anisotropic (Theorem 3.1).

Corollary 3.4. *Let $d \in F \setminus F^2$ and $K = F(\sqrt[2^n]{d})$ ($n \geq 1$). Let φ be an anisotropic totally singular F-quadratic form. Then , we have:*

(1) φ_K is quasi-hyperbolic iff $\langle 1, d\rangle \otimes \varphi$ is quasi-hyperbolic iff there exists a totally singular form ρ such that $\varphi \simeq \langle 1, d\rangle \otimes \rho$.
(2) If one of the equivalent conditions in (1) is satisfied, then $D_F(\langle 1, d\rangle \otimes \varphi) = D_F(\varphi)$.

Proof. (1) This is a direct consequence of Corollary 3.2 and Proposition 3.3.

(2) Suppose that one of the equivalent conditions in (1) is satisfied. Then, there exists a totally singular form ρ such that $\varphi \simeq \langle 1, d\rangle \otimes \rho$. Hence, we get the following equivalences:

$$\begin{aligned} u \in D_F(\langle 1, d\rangle \otimes \varphi) &\iff u \in D_F(\langle 1, d\rangle \otimes \langle 1, d\rangle \otimes \rho) \\ &\iff u \in D_F(\langle 1, d, 0, 0\rangle \otimes \rho) \\ &\iff u \in D_F(\langle 1, d\rangle \otimes \rho \perp 2\dim\rho \times \langle 0\rangle) \\ &\iff u \in D_F(\varphi). \end{aligned}$$

□

We recall the norm theorem that will be used repeatedly in our proofs.

Theorem 3.5. ([15, Theorem 1.1]) *Let φ be a nondefective semisingular quadratic form of dimension ≥ 3 over F, and let $p \in F[x_1, x_2, \dots, x_n]$ be a normed irreducible polynomial and $L = F(x_1, x_2, \dots, x_n)$. Then, the following two conditions are equivalent:*
(1) φ is quasi-hyperbolic over $F(p)$.
(2) p is a norm of φ_L.

We keep $K = F(\sqrt[2^{n_1}]{d_1}, \cdots, \sqrt[2^{n_s}]{d_s})$ and $\pi = \langle\langle d_1, \cdots, d_s\rangle\rangle$ as in the beginning of this section, and we let e the exponent of K. In the case when $e > 1$, we attach to F the following 2-fold quadratic Pfister forms:

$$\left\langle\left\langle u, u^{2^t} d_1^{k(t,1)} \cdots d_s^{k(t,s)}\right]\right] \tag{3.2}$$

such that $u \in F^*$, $1 \leq t < e$, $0 \leq k(t, l) < 2^t$ and $\max\{1, 2^{t-n_l+1}\} \mid k(t, l)$ for all $1 \leq l \leq s$. These forms will play a crucial role in the classifications given in Theorems 7.1 and 5.1, and appear in a recent paper of Sobiech [17, Th. 5.3] where the (graded-)Witt kernel of an arbitrary purely inseparable extension of F is given.

Notations 3.6. *Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be a singular quadratic form over F and $a \in D_F(R)$. We fix the following notations:*
(1) $P_F(n_1, d_1, \cdots, n_s, d_s)$ is the set of 2-fold quadratic Pfister forms as given in equation (3.2).
(2) $P_F^a(n_1, d_1, \cdots, n_s, d_s; \varphi)$ is the set of 2-fold quadratic Pfister forms as given in equation (3.2) and satisfying the additional condition $u \in D_F(a\pi \otimes \mathrm{ql}(\varphi))$.

The following remark is a direct consequence of Corollary 3.4 and will appear for the case of simple purely inseparable extensions:

Remark 3.7. *When $K = F(\sqrt[2^{n_1}]{d_1})$, $n_1 \geq 2$, is a simple purely inseparable extension, then n_1 is the exponent of K and we have:*
(1) $P_F(n_1, d_1) = \{\langle\langle u, u^{2^t} d_1^k]] \mid u \in F^*,\ 1 \leq t < n_1$ and $0 \leq k < 2^t\}$.
(2) If $\varphi = R \perp \mathrm{ql}(\varphi)$ is an anisotropic singular form that becomes quasi-hyperbolic over K and $a \in D_F(R)$, then:

(a) *There exists a totally singular form* ρ *such that* $\mathrm{ql}(\varphi) \simeq \langle 1, d_1\rangle \otimes \rho$.
(b) $P^a_F(n_1, d_1;\varphi) = \big\{\langle\langle u, u^{2^t} d_1^k]] \mid u \in D_F(a\mathrm{ql}(\varphi)),\ 1 \leq t < n_1 \text{ and } 0 \leq k < 2^t\big\}$.

To state our results in a simple way, we introduce the following notations:

Notations 3.8. *(1)* $\mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ *is the* $W(F)$*-submodule of* $W_q(F)$ *generated by* $P_F(n_1, d_1, \cdots, n_s, d_s)$.
(2) $\mathcal{N}_F(d_1, \cdots, d_s)$ *is the submodule* $\sum_{i=1}^s \langle 1, d_i\rangle_b \otimes W_q(F)$ *of* $W_q(F)$.
(3) $\mathcal{M}^a_F(n_1, d_1, \cdots, n_s, d_s; \varphi)$ *is the subgroup of* $W_q(F)$ *additively generated by* $P^a_F(n_1, d_1, \cdots, n_s, d_s; \varphi)$.

The following remark is obvious but useful for our proofs:

Remark 3.9. *Let* $\varphi = R \perp \mathrm{ql}(\varphi)$ *be a singular quadratic form over* F *and* $a \in D_F(R)$*. Then,* $\mathcal{M}^a_F(n_1, d_1, \cdots, n_s, d_s; \varphi) = \mathcal{M}^1_F(n_1, d_1, \cdots, n_s, d_s; a\varphi)$.

In our proofs of Theorems 7.1 and 5.1, we will first treat the case of semisingular quadratic forms of type $(1, s)$. This is the reason why we need to consider the sets $\mathcal{M}^a_F(n_1, d_1, \cdots, n_s, d_s; \varphi)$ for scalars $a \in F^*$ represented by the nonsingular part of φ.

Recall that any 2-fold quadratic Pfister form as in (3.2) is hyperbolic over K [17, Theorem 5.3], and thus any form in $\mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ becomes hyperbolic over K. Obviously, any form in $\mathcal{N}_F(d_1, \cdots, d_s)$ is hyperbolic over K as well.

4. Some useful results

4.1. **Representation results.** Let φ be an F-quadratic form of underlying vector space V, and $F[x]$ the ring of polynomials in the indeterminate x. We denote by $V(x)$ and $V[x]$ the spaces $V \otimes_F F(x)$ and $V \otimes_F F[x]$, respectively. We recall the Cassels-Pfister theorem:

Theorem 4.1. ([3, Th. 17.3]) *We keep the same notations and hypotheses as before. Let* $v \in V(x)$ *be such that* $\varphi(v) \in F[x]$*. Then, there exists a vector* $w \in V[x]$ *such that* $\varphi(v) = \varphi(w)$.

We also need the following result which is a stronger version of Cassels-Pfister theorem.

Proposition 4.2. ([3, Prop. 17.9]) *We keep the same notations and hypotheses as before. Suppose that* φ *is anisotropic and let* $s \in V$, $v \in V(x)$ *be such that* $\varphi(v) \in F[x]$ *and* $B_\varphi(s, v) \in F[x]$*. Then, there exists* $w \in V[x]$ *such that* $\varphi(v) = \varphi(w)$ *and* $B_\varphi(s, v) = B_\varphi(s, w)$.

Another result that will play a crucial role for the quasi-hyperbolicity over the function fields of certain inseparable polynomials is the so-called "Witt Extension Theorem" that states the following:

Theorem 4.3. ([3, Theorem 8.3]) *Let* φ *and* ψ *be isometric* F*-quadratic forms whose underlying vector spaces are* U *and* V*, respectively. Let* $U' \subset U$ *and* $V' \subset V$ *be subspaces such that* $U' \cap \mathrm{Rad}(\varphi) = 0$ *and* $V' \cap \mathrm{Rad}(\psi) = 0$*. Suppose there is an isometry* $\alpha : \varphi_{|U'} \longrightarrow \psi_{|V'}$*. Then, there is an isometry* $\alpha' : \varphi \longrightarrow \psi$ *such that* $\alpha'(U') = V'$ *and* $\alpha'_{|U'} = \alpha$.

The following corollary is the starting point for the quasi-hyperbolicity of semisingular quadratic forms over inseparable quadratic extensions.

Corollary 4.4. *Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be an anisotropic semisingular F-quadratic form. Let $d \in F \setminus F^2$ and suppose that φ is quasi-hyperbolic over $F(\sqrt{d})$. Then,* $\dim R \geq 4$*, and for any scalar $\alpha \in D_F(R)$, there exist a nonsingular form R' of dimension* $\dim R - 2$ *and $a \in F$ such that*

$$\varphi \sim \alpha \langle 1, d\rangle_b \otimes [1, a] \perp R' \perp \mathrm{ql}(\varphi). \tag{4.1}$$

Proof. Let $L = F(\sqrt{d})$ and V the underlying vector space of φ. We may suppose $\alpha = 1$. Since φ_L is quasi-hyperbolic, then $\mathrm{ql}(\varphi)_L$ is quasi-hyperbolic, and thus $\mathrm{ql}(\varphi) \simeq \langle 1, d\rangle \otimes \rho$ for some totally singular form ρ. Moreover, $x^2 + d$ is a norm of $\varphi_{F(x)}$ (Theorem 3.5). Hence, $\varphi_{F(x)}(v) = x^2 + d$ for some vector $v \in V(x)$. We may suppose $v \in V[x]$ (Theorem 4.1). Since φ is anisotropic, we may write $v = v_0 + xv_1$ for $v_0, v_1 \in V$. Hence, the condition $\varphi_{F(x)}(v) = x^2 + d$ implies the following relations: $\varphi(v_0) = d$, $\varphi(v_1) = 1$ and $B_\varphi(v_0, v_1) = 0$, meaning that $\langle 1, d\rangle \prec \varphi$. It follows from Corollary 2.7 that $\dim R \geq 4$, and we have the following isometry:

$$\varphi \simeq [1, a] \perp d[1, b] \perp \theta \perp \mathrm{ql}(\varphi),$$

where $a, b \in F$ and θ is a nonsingular quadratic form of dimension $\dim R - 4$. The previous isometry can be re-written as follows:

$$\varphi \sim \langle 1, d\rangle_b \otimes [1, a] \perp R' \perp \mathrm{ql}(\varphi),$$

where $R' = d[1, a + b] \perp \theta$ is of dimension $\dim R - 2$. Hence the corollary. □

Remark 4.5. *Note that in Corollary 4.4, the hypothesis of quasi-hyperbolicity is necessary to make the scalar α appear in the right hand side of equation (4.1).*

Corollary 4.6. *Let $d \in F \setminus F^2$ and φ an anisotropic semisingular F-quadratic form. Then, φ becomes quasi-hyperbolic over $F(\sqrt{d})$ iff we have the two conditions:*

(1) $\varphi \sim \varphi_1 \perp \mathrm{ql}(\varphi)$ for some $\varphi_1 \in \mathcal{N}_F(d)$.
(2) $\mathrm{ql}(\varphi) \simeq \langle 1, d\rangle \otimes \rho$ for some totally singular F-quadratic form ρ.

Proof. Let $L = F(\sqrt{d})$. Since φ_L is quasi-hyperbolic, then $\mathrm{ql}(\varphi) \simeq \langle 1, d\rangle \otimes \rho$ for some totally singular form ρ. The rest of the corollary follows from Corollary 4.4 using an induction on the dimension of the nonsingular part of φ. □

4.2. **Certain calculations on generator quadratic forms.** Our criterion for (strict) quasi-hyperbolicity over purely inseparable modular extensions is based on the 2-fold Pfister forms given in equation (3.2). Our aim in this section is to prove some results helping us to expand these forms when they are given over a quadratic inseparable extension of the base field. This is due to the fact that in our proofs, we will proceed by induction on the degree of the extension, and thus we are brought to consider the generators first on a quadratic inseparable extension. To this end we recall a lemma that will play a key role in this process.

Lemma 4.7. *Let $a_1, \ldots, a_n \in F^*$ ($n \geq 2$) and $c \in F$ be such that $a := \sum_{i=1}^n a_i \neq 0$. Then, we have*

$$\langle 1, a\rangle_b \otimes [1, c] \sim \sum_{i=1}^{n} \langle 1, a_i\rangle_b \otimes \left[1, \frac{a_i c}{a}\right].$$

This lemma is due to Aravire and Baeza for $n = 2$ [1, Lemma 2.1], but their proof easily extends to any integer $n \geq 2$.

For the rest of this subsection we fix an inseparable quadratic extension $F(\sqrt{d})$ of F. As a direct consequence of Lemma 4.7, we have the following result.

Corollary 4.8. *Let $s = s_0 + s_1\sqrt{d} \in F(\sqrt{d})$ with $s_0, s_1 \in F^*$. Then, we have the following equivalence over $F(\sqrt{d})$:*

$$\langle 1, s\rangle_b \otimes [1, s^{2^t}(\sqrt{d})^k] \sim \langle 1, s_0\rangle_b \otimes \left[1, s_0 s^{2^t-1}(\sqrt{d})^k\right] + \langle 1, s_1\sqrt{d}\rangle_b \otimes \left[1, s_1 s^{2^t-1}(\sqrt{d})^{k+1}\right].$$

We mention an easy lemma but useful for the sequel.

Lemma 4.9. *Let $s = s_0 + s_1\sqrt{d} \in F(\sqrt{d})$ with $s_0, s_1 \in F$. Then, we have:*

$$s^{2^t-1} = (s_0 + s_1\sqrt{d})^{2^t-1} = \sum_{j=0}^{2^t-1} s_0^j (s_1\sqrt{d})^{2^t-1-j}.$$

Proof. We already know $s^{2^t-1} = (s_0 + s_1\sqrt{d})^{2^t-1} = \sum_{j=0}^{2^t-1} \binom{2^t-1}{j} s_0^j (s_1\sqrt{d})^{2^t-1-j}$. Since $\binom{2^t-1}{j} \equiv 1 \pmod 2$ for all $0 \leq j \leq 2^t - 1$, and we are working over a field of characteristic 2, we get the desired conclusion. □

We will often use the following lemma.

Lemma 4.10. *Let ρ be a totally singular F-quadratic form such that $s \in D_{F(\sqrt{d})}(\rho \perp \sqrt{d}\rho)$, where $s = s_0 + s_1\sqrt{d}$ for some $s_0, s_1 \in F$. Then, for each $i = 0, 1$, when $s_i \neq 0$ we obtain: $s_i \in D_F(\rho \perp d\rho)$, and $s_i, s_i\sqrt{d} \in D_{F(\sqrt{d})}(\rho \perp \sqrt{d}\rho)$.*

Proof. Put $L = F(\sqrt{d})$. Since $s \in D_{F(\sqrt{d})}(\rho \perp \sqrt{d}\rho)$, there exist v_0, v_1, w_0, w_1 vectors in the underlying F-vector space of ρ such that

$$s = \rho(v_0 + \sqrt{d}v_1) + \sqrt{d}\rho(w_0 + \sqrt{d}w_1) = \rho(v_0) + d\rho(v_1) + \sqrt{d}(\rho(w_0) + d\rho(w_1)).$$

Hence, $s_0 = \rho(v_0) + d\rho(v_1)$ and $s_1 = \rho(w_0) + d\rho(w_1)$. So when $s_i \neq 0$, we get $s_i \in D_F(\rho \perp d\rho)$.

Moreover, when $s_i \neq 0$, the condition $s_i \in D_F(\rho \perp d\rho)$ implies $s_i \in D_L(\rho \perp \dim\rho \times \langle 0\rangle)$, and thus $s_i \in D_L(\rho)$. In particular, $s_i \in D_L(\rho \perp \sqrt{d}\rho)$ and $s_i\sqrt{d} \in D_L(\rho \perp \sqrt{d}\rho)$. □

Further joining the Lemma 4.10 with Corollary 4.8 gives us the following corollary:

Corollary 4.11. *Let ρ, s, s_o, s_1 be as in Lemma 4.10, and $\beta \in F$. Then, we have the following equivalence over $F(\sqrt{d})$:*

$$\left\langle\left\langle s, s^{2^t}(\sqrt{d})^k\beta\right]\right] \perp \rho \perp \sqrt{d}\rho \sim [1, s_0^{2^{t+1}} d^k \beta^2] \perp [1, s_1^{2^{t+1}} d^{2^t+k}\beta^2] \perp \rho \perp \sqrt{d}\rho. \tag{4.2}$$

Proof. Put $Q = \rho \perp \sqrt{d}\rho$. If $s_0 = 0$ or $s_1 = 0$, then the corollary is obviously satisfied. So we deal with the case $s_0, s_1 \in F^*$. With the help of Corollary 4.8 we have the following equivalence:

(4.3)
$$\left\langle\left\langle s, s^{2^t}(\sqrt{d})^k\beta\right]\right] \perp Q \sim \langle 1, s_0\rangle_b \otimes [1, s_0 s^{2^t-1}(\sqrt{d})^k\beta] \perp \langle 1, s_1\sqrt{d}\rangle_b \otimes [1, s_1 s^{2^t-1}(\sqrt{d})^{k+1}\beta] \perp Q.$$

Now using the expansion $s^{2^t-1} = \sum_{j=0}^{2^t-1} s_0^j(s_1\sqrt{d})^{2^t-1-j}$ (Lemma 4.9), and repeatedly use the equivalence $[1, a+b] \sim [1, a] \perp [1, b]$ to get the following equation:

$$\begin{aligned}\left\langle\left\langle s, s^{2^t}(\sqrt{d})^k\beta\right]\right] \perp Q \;\sim\; & \sum_{j=0}^{2^t-1} \langle 1, s_0\rangle_b \otimes [1, s_0^{j+1} s_1^{2^t-1-j}(\sqrt{d})^{2^t-1-j+k}\beta] \\ & \perp \sum_{l=0}^{2^t-1} \left\langle 1, s_1\sqrt{d}\right\rangle_b \otimes [1, s_0^l s_1^{2^t-l}(\sqrt{d})^{2^t-l+k}\beta] \perp Q.\end{aligned}$$

Hence, we get

(4.4)
$$\begin{aligned}\left\langle\left\langle s, s^{2^t}(\sqrt{d})^k\beta\right]\right] \perp Q \sim & \sum_{j=0}^{2^t-2} \langle 1, s_0\rangle_b \otimes [1, s_0^{j+1} s_1^{2^t-1-j}(\sqrt{d})^{2^t-1-j+k}\beta] \perp \langle 1, s_0\rangle_b \otimes [1, s_0^{2^t}(\sqrt{d})^k\beta] \\ & \perp \left\langle 1, s_1\sqrt{d}\right\rangle_b \otimes [1, s_1^{2^t}(\sqrt{d})^{2^t+k}\beta] \perp \sum_{l=1}^{2^t-1} \left\langle 1, s_1\sqrt{d}\right\rangle_b \otimes [1, s_0^l s_1^{2^t-l}(\sqrt{d})^{2^t-l+k}\beta] \perp Q.\end{aligned}$$

Note that the terms $s_0^{j+1}s_1^{2^t-1-j}(\sqrt{d})^{2^t-1-j+k}$ and $s_0^l s_1^{2^t-l}(\sqrt{d})^{2^t-l+k}$ are on the respective range of j and l. Hence, we can re-write equation (4.4) as:

(4.5)
$$\begin{aligned}\left\langle\left\langle s, s^{2^t}(\sqrt{d})^k\beta\right]\right] \perp Q \sim & \sum_{j=0}^{2^t-2} s_0[1, s_0^{j+1} s_1^{2^t-1-j}(\sqrt{d})^{2^t-1-j+k}\beta] \perp [1, s_0^{2^t}(\sqrt{d})^k\beta] \perp s_0[1, s_0^{2^t}(\sqrt{d})^k\beta] \\ & \perp [1, s_1^{2^t}(\sqrt{d})^{2^t+k}\beta] \perp s_1\sqrt{d}[1, s_1^{2^t}(\sqrt{d})^{2^t+k}\beta] \perp \sum_{l=1}^{2^t-1} s_1\sqrt{d}[1, s_0^l s_1^{2^t-l}(\sqrt{d})^{2^t-l+k}\beta] \perp Q.\end{aligned}$$

Since $s_0, s_1\sqrt{d} \in D_{F(\sqrt{d})}(Q)$, we know:

$$Q \sim s_0[1, u] \perp s_1\sqrt{d}[1, v] \perp Q,$$

for any $u, v \in F(\sqrt{d})$. Thus, we can express equation (4.5) as follows:

$$\left\langle\left\langle s, s^{2^t}(\sqrt{d})^k\beta\right]\right] \perp Q \sim [1, s_0^{2^t}(\sqrt{d})^k\beta] \perp [1, s_1^{2^t}(\sqrt{d})^{2^t+k}\beta] \perp Q. \tag{4.6}$$

Now over equation (4.6), we use the isometry $[1, u] \simeq [1, u^2]$ to get equation (4.2). □

5. The case of simple purely inseparable extensions

Throughout this section, we fix $d \in F \setminus F^2$ and $K = F(\sqrt[2^n]{d})$, $n \geq 2$, a simple purely inseparable extension of F. Our aim is to prove Theorem 5.1 below that deals with the quasi-hyperbolicity over K of semisingular quadratic forms. Although the proof is also covered in the general case coming up in the next section, we mention the proof here to form a clear idea for the reader, and because the strategy of proof dealing with the quasilinear part for a general extension is a little bit different than that used for simple purely inseparable extensions.

Theorem 5.1. *Let $K = F(\sqrt[2^n]{d})$, $n \geq 2$, be as before and φ an anisotropic semisingular F-quadratic form. Then, φ is quasi-hyperbolic over K iff we have the two conditions:*

(1) $\varphi \sim \varphi_1 \perp \varphi_2 \perp \mathrm{ql}(\varphi)$ for suitable $\varphi_1 \in \mathcal{M}_F(n, d)$ and $\varphi_2 \in \mathcal{N}_F(d)$,

(2) $\mathrm{ql}(\varphi) \simeq \langle 1, d\rangle \otimes \rho$ for a suitable totally singular form ρ.

We first prove Proposition 5.2 which deals with the quasi-hyperbolicity over K of semisingular quadratic forms φ of type $(1, s)$. The classification given in this proposition is more precise than that given in Theorem 5.1 because we take a form in $\mathcal{M}^a_F(n, d; \varphi)$ multiplied by a scalar represented by the nonsingular part of φ. This is crucial to prove the proposition using an induction on the integer n. Based on this proposition and an induction on the dimension of the nonsingular part, we will prove Theorem 5.1.

Proposition 5.2. *Let $K = F(\sqrt[2^n]{d})$, $n \geq 2$, be as before and $\varphi = a[1, b] \perp \mathrm{ql}(\varphi)$ an anisotropic semisingular F-quadratic form. Then, φ is quasi-hyperbolic over K iff we have the two conditions:*

(1) $\varphi \sim a\varphi_1 \perp \mathrm{ql}(\varphi)$ for suitable $\varphi_1 \in \mathcal{M}^a_F(n, d; \varphi)$,

(2) $\mathrm{ql}(\varphi) \simeq \rho \perp d\rho$ for a suitable totally singular form ρ.

Proof. Working with the form $a\varphi$ and because $\mathcal{M}^a_F(n, d; \varphi) = \mathcal{M}^1_F(n, d; a\varphi)$ (Remark 3.9), we may suppose $a = 1$. Let V be the underlying vector space of φ. The conditions (1) and (2) imply that φ_K is quasi-hyperbolic. Now suppose that φ is quasi-hyperbolic over K. It follows from Proposition 2.3 that $i_W(\varphi_{F(\sqrt[2^n]{d})}) = 1$ and $\mathrm{ql}(\varphi_{F(\sqrt[2^n]{d})})$ is quasi-hyperbolic. With the help of Proposition 3.3, we have $\mathrm{ql}(\varphi) \simeq \rho \perp d\rho$ for some totally singular F-quadratic form ρ. We proceed by induction on n to prove the first statement of the proposition.

Step 1. Suppose $n = 2$. By Theorem 3.5 we have:

$$\varphi \simeq (x^4 + d)\varphi.$$

Since the nonsingular part of φ represents 1, there exists $v \in V(x)$ such that $\varphi(v) = x^4 + d$. We may assume $v \in V[x]$ by Theorem 4.1. Since φ is anisotropic, the polynomial vector v has degree 2. Let $v = v_0 + v_1x + v_2x^2$ be such that $v_0, v_1, v_2 \in V$. Then, we get

$$\varphi(v) = x^4 + d = \varphi(v_0) + B_\varphi(v_0, v_1)x + \varphi(v_1)x^2 + B_\varphi(v_0, v_2)x^2 + B_\varphi(v_1, v_2)x^3 + \varphi(v_2)x^4.$$

So we have the following relations:

- $\varphi(v_0) = d$,

- $\varphi(v_2) = 1$,
- $\varphi(v_1) = B_\varphi(v_0, v_2)$,
- $B_\varphi(v_0, v_1) = B_\varphi(v_1, v_2) = 0$.

Case 1. Suppose $\varphi(v_1) = 0$. Since φ is anisotropic, it follows that $v_1 = 0$, and thus we reduce to the relations: $\varphi(v_0) = d$, $\varphi(v_2) = 1$ and $B_\varphi(v_0, v_2) = 0$, meaning that $\langle 1, d\rangle \prec \varphi$. This is not possible by Corollary 2.7 because φ has a nonsingular part of dimension 2.

Case 2. Suppose $\alpha := \varphi(v_1) \in F^*$. Then, the four previous relations mean $[1; \alpha; d] \perp \langle \alpha\rangle \prec \varphi$. Using the isometry $[1; \alpha; d] \simeq [1, d\alpha^{-2}]$ and the fact that the nonsingular part of φ is of dimension 2, we get

$$\varphi \simeq [1, d\alpha^{-2}] \perp \langle \alpha\rangle \perp Q,$$

for a suitable totally singular form Q satisfying $\mathrm{ql}(\varphi) \simeq \langle \alpha\rangle \perp Q$. Adding a hyperbolic plane to both sides we can write:

$$\begin{aligned}\varphi \perp \mathbb{H} &\simeq \langle 1, \alpha\rangle_b \otimes [1, d\alpha^{-2}] \perp \mathrm{ql}(\varphi),\\ &\simeq \left\langle 1, \alpha^{-1}\right\rangle_b \otimes [1, d\alpha^{-2}] \perp \mathrm{ql}(\varphi),\\ &\simeq \left\langle\langle \alpha^{-1}, d\alpha^{-2}\right]] \perp \mathrm{ql}(\varphi).\end{aligned}$$

Note that $\alpha^{-1} \in D_F(\mathrm{ql}(\varphi)) = D_F(\langle 1, d\rangle \otimes \mathrm{ql}(\varphi))$ (Corollary 3.4), and thus the form $\varphi_1 = \langle\langle \alpha^{-1}, d\alpha^{-2}]] \in \mathcal{M}^1_F(n, d; \varphi)$. Hence, the proposition is satisfied for $n = 2$.

Step 2. Suppose $n > 2$. Extending φ to $F(\sqrt{d})$, we get:

$$\varphi_{F(\sqrt{d})} \simeq ([1, b] \perp \rho)_{F(\sqrt{d})} \perp \dim\rho \times \langle 0\rangle\,.$$

Because $i_W(\varphi_{F(\sqrt[2^n]{d})}) = 1$, we get:

$$i_W(([1, b] \perp \rho)_{F(\sqrt{d})})_{F(\sqrt[2^{n-1}]{\sqrt{d}})} = 1.$$

Thus, the $F(\sqrt{d})$-form $\widetilde{\varphi} := [1, b] \perp \rho \perp \sqrt{d}\rho$ is quasi-hyperbolic over $F(\sqrt[2^{n-1}]{\sqrt{d}})$. This enables us to apply our induction hypothesis to the form $\widetilde{\varphi}$ provided it is anisotropic over $F(\sqrt{d})$.

(a) Suppose that $\widetilde{\varphi}$ is isotropic over $F(\sqrt{d})$. Since ρ_K is anisotropic, it follows that $\rho_{F(\sqrt[4]{d})}$ is anisotropic as well. Theorem 3.1 implies that $\rho \perp \sqrt{d}\rho$ is anisotropic over $F(\sqrt{d})$. Hence, the isotropy of $\widetilde{\varphi}$ implies

$$\widetilde{\varphi} \simeq \mathbb{H} \perp \rho \perp \sqrt{d}\rho \tag{5.1}$$

Extending equation (5.1) to $F(\sqrt[4]{d})$ yields:

$$\widetilde{\varphi}_{F(\sqrt[4]{d})} \simeq ([1, b] \perp \rho)_{F(\sqrt[4]{d})} \perp \dim\rho \times \langle 0\rangle \simeq \mathbb{H} \perp \rho_{F(\sqrt[4]{d})} \perp \dim\rho \times \langle 0\rangle\,.$$

Clearly, we can re-write the above equation as follows:

$$([1, b] \perp \rho \perp d\rho)_{F(\sqrt[4]{d})} \simeq \mathbb{H} \perp \rho_{F(\sqrt[4]{d})} \perp \dim\rho \times \langle 0\rangle\,.$$

Hence, φ is quasi-hyperbolic over $F(\sqrt[4]{d})$, which reduces us to **Step 1**.

(b) Suppose that $\widetilde{\varphi}$ is anisotropic over $F(\sqrt{d})$. Since $\widetilde{\varphi}$ is quasi-hyperbolic over $F(\sqrt[2^{n-1}]{\sqrt{d}})$, we deduce by induction that there exists a finite set I such that:

$$\widetilde{\varphi} \sim \sum_{i\in I} G_i \perp \rho \perp \sqrt{d}\rho, \tag{5.2}$$

where $G_i = \left\langle\left\langle s_i, s_i^{2^t}(\sqrt{d})^k\right]\right]$ such that $1 \le t \le n-2$, $1 \le k \le 2^t - 1$ and $s_i \in D_{F(\sqrt{d})}(\rho \perp \sqrt{d}\rho)$ for all $i \in I$ (Corollary 3.4).

We write $s_i = s_{i0} + s_{i1}\sqrt{d}$ for $s_{i0}, s_{i1} \in F$. Using Corollary 4.11 we re-write equation 5.2 as follows:

$$[1, b] \perp \rho \perp \sqrt{d}\rho \sim \sum_{i\in I}\left([1, s_{i0}^{2^{t_0}} d^{k_0}] \perp [1, s_{i1}^{2^{t_1}} d^{k_1}]\right) \perp \rho \perp \sqrt{d}\rho, \tag{5.3}$$

where $1 \le t_0, t_1 \le n-1$, $1 \le k_0 \le 2^{t_0} - 1$, $1 \le k_1 \le 2^{t_1} - 1$, and each scalar s_{i0}, s_{i1} belongs to $D_F(\rho \perp d\rho)$ when it is nonzero.

Moreover, using the equivalence $[1, x] \perp [1, y] \sim [1, x+y]$, we deduce from equation (5.3) the following:

$$[1, b + \sum_{i\in I}(s_{i0}^{2^{t_0}} d^{k_0} + s_{i1}^{2^{t_1}} d^{k_1})] \perp \rho \perp \sqrt{d}\rho \sim \rho \perp \sqrt{d}\rho. \tag{5.4}$$

Let $b' = b + \sum_{i\in I}(s_{i0}^{2^{t_0}} d^{k_0} + s_{i1}^{2^{t_1}} d^{k_1})$. We now extend equation (5.4) to $F(\sqrt[4]{d})$ getting:

$$[1, b'] \perp \rho \perp \dim\rho \times \langle 0\rangle \sim \rho \perp \dim\rho \times \langle 0\rangle\,.$$

Equivalently, we can write

$$([1, b'] \perp \rho \perp d\rho)_{F(\sqrt[4]{d})} \sim \rho \perp \dim\rho \times \langle 0\rangle\,. \tag{5.5}$$

Thus, the F-quadratic form $[1, b'] \perp \rho \perp d\rho$ is quasi-hyperbolic over $F(\sqrt[4]{d})$. We treat two cases:

(b.1) Suppose that $[1, b'] \perp \rho \perp d\rho$ is isotropic over F. Then, since $\rho \perp d\rho$ is anisotropic over F, we have:

$$[1, b'] \perp \rho \perp d\rho \sim \rho \perp d\rho.$$

Consequently, we get

$$\varphi \sim \sum_{i\in I}\left([1, s_{i0}^{2^{t_0}} d^{k_0}] \perp [1, s_{i1}^{2^{t_1}} d^{k_1}]\right) \perp \rho \perp d\rho. \tag{5.6}$$

Since $s_{i0}, s_{i1} \in D_F(\rho \perp d\rho)$, we deduce from equation (5.6) the following:

$$\varphi \sim \varphi_1 \perp \rho \perp d\rho,$$

where $\varphi_1 = \sum_{i\in I}\left(\langle 1, s_{i0}\rangle_b \otimes [1, s_{i0}^{2^{t_0}} d^{k_0}] \perp \langle 1, s_{i1}\rangle_b \otimes [1, s_{i1}^{2^{t_1}} d^{k_1}]\right) \in \mathcal{M}_F^1(n, d; \varphi)$.

(b.2) Suppose that $[1, b'] \perp \rho \perp d\rho$ is anisotropic. Since it is quasi-hyperbolic over $F(\sqrt[4]{d})$, we deduce from **Step 1.** the following:

$$[1, b'] \perp \rho \perp d\rho \sim \langle\langle r, r^2 d]] \perp \rho \perp d\rho$$

for a scalar $r \in D_F(\rho \perp d\rho)$. Hence, as in case (b.1), we get

$$\varphi \sim \varphi_1 \perp \rho \perp d\rho,$$

where $\varphi_1 = \langle\langle r, r^2 d]] \perp \sum_{i\in I} \big(\langle 1, s_{i0}\rangle_b \otimes [1, s_{i0}^{2^{t_0}} d^{k_0}] \perp \langle 1, s_{i1}\rangle_b \otimes [1, s_{i1}^{2^{t_1}} d^{k_1}]\big)$ belongs to $\mathcal{M}_F^1(n, d; \varphi)$. This proves the proposition. $\square$

Now we are able to prove Theorem 5.1.

Proof of Theorem 5.1. The two conditions imply that φ_K is quasi-hyperbolic. Conversely, suppose that φ_K is quasi-hyperbolic. Then, we know that $\mathrm{ql}(\varphi) \simeq \rho \perp d\rho$ for some totally singular form ρ which is anisotropic over K. Let $R = [a_1, b_1] \perp \ldots \perp [a_r, b_r]$ be such that $\varphi = R \perp \mathrm{ql}(\varphi)$. We prove the statement (1) by induction on r.

The case $r = 1$ is given in Proposition 5.2. So we suppose $r \geq 2$. By the quasi-hyperbolicity of φ_K, we get

$$R_K \perp \rho_K \perp \dim\rho \times \langle 0\rangle \simeq r \times \mathbb{H} \perp \rho_K \perp \dim\rho \times \langle 0\rangle\,.$$

Using Witt cancellation (Proposition 2.4), we deduce

$$R_K \perp \rho_K \simeq r \times \mathbb{H} \perp \rho_K.$$

Now adding $\langle a_2, a_3, \ldots, a_r\rangle_K$ in both sides of the previous equation, we get

$$R_K \perp \rho_K \perp \langle a_2, a_3, \ldots, a_r\rangle_K \simeq r \times \mathbb{H} \perp \rho_K \perp \langle a_2, a_3, \ldots, a_r\rangle_K\,.$$

Note that $[a_i, b_i] \perp \langle a_i\rangle \simeq \mathbb{H} \perp \langle a_i\rangle$. Thus, after canceling hyperbolic planes, we have the following isometry:

$$[a_1, b_1]_K \perp \rho_K \perp \langle a_2, a_3, \ldots, a_r\rangle_K \simeq \mathbb{H} \perp \rho_K \perp \langle a_2, a_3, \ldots, a_r\rangle_K\,. \tag{5.7}$$

Moreover, by Lemma 2.10, there exists ρ_1 a subform of $\rho \perp \langle a_2, a_3, \ldots, a_r\rangle$ such that

$$((\rho \perp \langle a_2, a_3, \ldots, a_r\rangle)_K)_{an} \simeq (\rho_1)_K.$$

Let ρ_2 be a totally singular form such that $\rho \perp \langle a_2, a_3, \ldots, a_r\rangle \simeq \rho_1 \perp \rho_2$. Canceling the zero form in (5.7) yields the following

$$[a_1, b_1]_K \perp (\rho_1)_K \simeq \mathbb{H} \perp (\rho_1)_K. \tag{5.8}$$

Let $\widetilde{\varphi} = [a_1, b_1] \perp \rho_1 \perp d\rho_1$. The form $(\rho_1)_K$ is anisotropic, and thus $(\rho_1)_{F(\sqrt{d})}$ is also anisotropic. Consequently, $\rho_1 \perp d\rho_1$ is anisotropic (Theorem 3.1). We distinguish two cases:

Case 1. Suppose $i_W(\widetilde{\varphi}) = 0$. Hence, $\widetilde{\varphi}$ is anisotropic. Clearly, equation (5.8) implies that $\widetilde{\varphi}_K$ is quasi-hyperbolic. We now apply Proposition 5.2 to get

$$[a_1, b_1] \perp \rho_1 \perp d\rho_1 \sim a_1\varphi_1 \perp \rho_1 \perp d\rho_1, \tag{5.9}$$

where $\varphi_1 \in \mathcal{M}_F^{a_1}(n, d; \varphi)$. Adding in both sides of equation (5.9) the form $\rho_2 \perp d\rho_2$, we get

$$[a_1, b_1] \perp \mathrm{ql}(\varphi) \perp \langle 1, d\rangle \otimes \langle a_2, a_3, \ldots, a_r\rangle \sim a_1\varphi_1 \perp \mathrm{ql}(\varphi) \perp \langle 1, d\rangle \otimes \langle a_2, a_3, \ldots, a_r\rangle\,.$$

Now we apply Corollary 2.9 (for $Q = \langle a_2, a_3, \ldots, a_r\rangle$, $Q' = \mathrm{ql}(\varphi)$ and $x_i = b_i$ for all $2 \leq i \leq r$), we obtain

$$\varphi \sim a_1\varphi_1 \perp \varphi_2 \perp R'' \perp \mathrm{ql}(\varphi) \tag{5.10}$$

for $\varphi_2 \in \mathcal{N}_F(d)$ and R'' a nonsingular form of dimension $2(r-1)$. Since φ_K is quasi-hyperbolic, the form $R'' \perp \mathrm{ql}(\varphi)$ is also quasi-hyperbolic over K. Hence, we conclude by induction.

Case 2. Suppose $i_W(\widetilde{\varphi}) = 1$. Then, $\widetilde{\varphi} \sim \rho_1 \perp d\rho_1$. Adding in both sides the form $\rho_2 \perp d\rho_2$, we get

$$[a_1, b_1] \perp \mathrm{ql}(\varphi) \perp \langle 1, d\rangle \otimes \langle a_2, a_3, \ldots, a_r\rangle \sim \mathrm{ql}(\varphi) \perp \langle 1, d\rangle \otimes \langle a_2, a_3, \ldots, a_r\rangle .$$

Hence, we conclude as in **Case 1**. This proves the theorem.

6. THE CASE OF MULTIQUADRATIC PURELY INSEPARABLE EXTENSIONS

The case of purely inseparable modular extensions of exponent > 1 is treated in Theorem 7.1. To complete the picture, we treat the case of multiquadratic purely inseparable extensions.

Theorem 6.1. *Let $d_1, \cdots, d_s \in F$ and $K = F(\sqrt{d_1}, \ldots, \sqrt{d_s})$ be such that $[L : F] = 2^s$. Let φ be an anisotropic semisingular F-quadratic form. Then, φ is strictly quasi-hyperbolic over K iff the two conditions hold:*
(1) $\varphi \sim \varphi_1 \perp \mathrm{ql}(\varphi)$ for a suitable form $\varphi_1 \in \mathcal{N}_F(d_1, \cdots, d_s)$,
(2) $\langle\langle d_1, \cdots, d_s\rangle\rangle \otimes \mathrm{ql}(\varphi)$ is quasi-hyperbolic.

Proof. We write $\varphi = R \perp \mathrm{ql}(\varphi)$ for some nonsingular form R of dimension $2r$.

Clearly, the conditions (1) and (2) imply that φ_K is strictly quasi-hyperbolic since the condition (2) implies that $\mathrm{ql}(\varphi)_K$ is quasi-hyperbolic (Corollary 3.1).

Conversely, suppose that φ_K is strictly quasi-hyperbolic. Then, $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over K, which implies that $\langle\langle d_1, \cdots, d_s\rangle\rangle \otimes \mathrm{ql}(\varphi)$ is quasi-hyperbolic (Corollary 3.1). To prove the statement (1), we proceed by induction on s. The case $s = 1$ was already treated in Corollary 4.6. So suppose $s \geq 2$.

Let $L = F(\sqrt{d_2}, \ldots, \sqrt{d_s})$, so $K = L(\sqrt{d_1})$. By Corollary 2.11, there exists η a subform of $\mathrm{ql}(\varphi)$ such that $(\mathrm{ql}(\varphi)_K)_{an} \simeq \eta_K$. Let η' be such that $\mathrm{ql}(\varphi) \simeq \eta \perp \eta'$.

Since η is anisotropic over $K = L(\sqrt{d_1})$, it follows that $\eta \perp d_1\eta$ is anisotropic over L (Theorem 3.1). Moreover, since L/F is excellent (Theorem 2.12), there exists a nonsingular F-quadratic form R_1 such that $\psi := ((R \perp \eta \perp d_1\eta)_L)_{an} \simeq (R_1 \perp \eta \perp d_1\eta)_L$.

Case 1. Suppose $\dim R_1 > 0$. Since

$$\varphi_K \simeq r \times \mathbb{H} \perp \eta_K \perp s \times \langle 0\rangle \simeq (R \perp \eta)_K \perp s \times \langle 0\rangle ,$$

for some integer $s \geq 0$ and η_K is anisotropic, it follows from Witt cancellation (Proposition 2.4) that

$$r \times \mathbb{H} \perp \eta_K \simeq (R \perp \eta)_K.$$

In particular, we have

$$\begin{aligned} r \times \mathbb{H} \perp \eta_K \perp \dim\eta \times \langle 0\rangle &\simeq (R \perp \eta)_K \perp \dim\eta \times \langle 0\rangle \\ &\simeq (R \perp \eta \perp d_1\eta)_K. \end{aligned} \tag{6.1}$$

Now since $(R \perp \eta \perp d_1\eta)_L \sim (R_1 \perp \eta \perp d_1\eta)_L$, equation (6.1) ensures that $(R_1 \perp \eta \perp d_1\eta)_L$ becomes quasi-hyperbolic over $K = L(\sqrt{d_1})$. It follows from Corollary 4.4 that

$$(R_1 \perp \eta \perp d_1\eta)_L \sim \alpha \langle 1, d_1\rangle_b \otimes [1, \beta] \perp R_2 \perp (\eta \perp d_1\eta)_L, \tag{6.2}$$

where α is an arbitrary scalar in $D_F(R_1)$, $\beta \in L$ and R_2 a nonsingular form over L of dimension $\dim R_1 - 2$. We may suppose $\beta \in F$ because $[1, \beta] \simeq [1, \beta^2]$ and $L^2 \subset F$. Hence, by the excellence of L/F we may suppose R_2 defined over F.

Since $(R_1 \perp \eta \perp d_1\eta)_L$ is quasi-hyperbolic over K, the form $(R_2 \perp \eta \perp d_1\eta)_L$ is also quasi-hyperbolic over K. We repeat a number of times the same arguments as we did for $R_1 \perp \eta \perp d_1\eta$, to prove

$$(R_1 \perp \eta \perp d_1\eta)_L \sim \psi_1 \perp (\eta \perp d_1\eta)_L$$

for some $\psi_1 \in \mathcal{N}_F(d_1)$. In particular

$$(R \perp \eta \perp d_1\eta)_L \sim \psi_1 \perp (\eta \perp d_1\eta)_L.$$

By Corollary 2.8 (applied to $Q = \eta$), there exists $\psi_2 \in \mathcal{N}_F(d_1)$ such that

$$(R \perp \psi_2 \perp \eta)_L \sim \eta_L. \tag{6.3}$$

Now consider the F-quadratic form $\theta := R \perp \psi_2 \perp \langle 1, d_2\rangle \otimes \eta$. The form $\eta \perp d_2\eta$ is anisotropic over F because $\eta_{F(\sqrt{d_2})}$ is also anisotropic. Hence, we conclude from equation (6.3) that θ is strictly quasi-hyperbolic over L. The induction hypothesis implies

$$\theta \sim \psi_3 \perp \eta \perp d_2\eta,$$

where $\psi_3 \in \mathcal{N}_F(d_2, \cdots, d_s)$. Again using Corollary 2.8 (applied for $Q = \eta$), we deduce

$$R \perp \psi_2 \perp \eta \sim \psi_3' \perp \eta, \tag{6.4}$$

for some $\psi_3' \in \mathcal{N}_F(d_2, \ldots, d_s)$. To conclude we add in both sides of (6.4) the form η' to get

$$\varphi \sim \varphi_1 \perp \mathrm{ql}(\varphi),$$

where $\varphi_1 = \psi_2 \perp \psi_3' \in \mathcal{N}_F(d_1, \cdots, d_s)$.

Case 2. Suppose $\dim R_1 = 0$. Then, $(R \perp \eta \perp d_1\eta)_L \sim (\eta \perp d_1\eta)_L$. By Corollary 2.8 (applied to $Q = \eta$), we get

$$(R \perp \delta \perp \eta)_L \sim \eta_L, \tag{6.5}$$

where $\delta \in \mathcal{N}_F(d_1)$. Hence, we are in the conditions of equation (6.3), and thus we finish the proof as in the previous case. □

7. The case of purely inseparable modular extensions

Throughout this section we fix the same notations as in Section 3, namely: A purely inseparable modular extension $K = F(\sqrt[2^{n_1}]{d_1}, \cdots, \sqrt[2^{n_s}]{d_s})$, the quasi-Pfister form $\pi = \langle\langle d_1, \cdots, d_s\rangle\rangle$ and the exponent e of K which is nothing but $\max\{n_1, \cdots, n_s\}$. We suppose $e > 1$. The strict quasi-hyperbolicity over K is given by the following theorem:

Theorem 7.1. *Let K be as before and φ an anisotropic semisingular F-quadratic form. Then, φ is strictly quasi-hyperbolic over K iff we have the two conditions:*

(1) $\varphi \sim \varphi_1 \perp \varphi_2 \perp \mathrm{ql}(\varphi)$ *for suitable* $\varphi_1 \in \mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ *and* $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$*,*

(2) $\pi \otimes \mathrm{ql}(\varphi)$ *is quasi-hyperbolic.*

We will proceed in the same manner as we did in Section 5 for simple purely inseparable extensions. We first prove Proposition 7.2 that treats the case of semisingular quadratic forms of type $(1, s)$, and then proceed by induction on the dimension of nonsingular part to prove Theorem 7.1.

Proposition 7.2. *Let* K *be as before and* $\varphi = a[1, b] \perp \mathrm{ql}(\varphi)$ *an anisotropic semisingular quadratic form over* F*. Then,* φ *is strictly quasi-hyperbolic over* K *iff the following conditions are satisfied:*

(1) $\varphi \sim a\varphi_1 \perp \varphi_2 \perp \mathrm{ql}(\varphi)$ *for suitable* $\varphi_1 \in \mathcal{M}^a_F(n_1, d_1, \cdots, n_s, d_s; \varphi)$ *and* $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$*.*

(2) $\pi \otimes \mathrm{ql}(\varphi)$ *is quasi-hyperbolic.*

Proof. The conditions described in the proposition are sufficient to get the strict quasi-hyperbolicity. Conversely, suppose that φ is strictly quasi-hyperbolic over K. Since $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over K, it follows that $\pi \otimes \mathrm{ql}(\varphi)$ is quasi-hyperbolic (Corollary 3.2). For the rest of the proof we proceed by induction on $n_1 + \cdots + n_s$. The case $n_1 = n_2 = \cdots = n_s = 1$ is excluded since $e > 1$. So without loss of generality, we may suppose $n_1 \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_s$. Hence $e = n_1$. The case $n_1 = 2, n_2 = \cdots = n_s = 0$ is covered by Proposition 5.2.

Now we will discuss the case when $[K : F] > 4$, and because $\mathcal{M}^a_F(n_1, d_1, \cdots, n_s, d_s; \varphi) = \mathcal{M}^1_F(n_1, d_1, \cdots, n_s, d_s; a\varphi)$ we may suppose $a = 1$. Let η be a subform of $\mathrm{ql}(\varphi)$ such that $(\mathrm{ql}(\varphi)_K)_{an} \simeq \eta_K$ and let η' be a form such that $\mathrm{ql}(\varphi) \simeq \eta \perp \eta'$. Consider the quadratic form $\psi := [1, b] \perp \eta \perp \sqrt{d_1}\eta$ over $L := F(\sqrt{d_1})$.

Observe that $\eta \perp d_1\eta$ is anisotropic over F (*resp.* $\eta \perp \sqrt{d_1}\eta$ is anisotropic over L) because $\eta_{F(\sqrt{d_1})}$ and $\eta_{F(\sqrt[4]{d_1})}$ are anisotropic as η_K is also anisotropic. Moreover, the form ψ is strictly quasi-hyperbolic over K because $([1, b] \perp \eta)_K \simeq \mathbb{H} \perp \eta_K$.

(A) Suppose that ψ is isotropic over L. Since $\eta \perp \sqrt{d_1}\eta$ is anisotropic over L, we have the following isometry:

$$[1, b] \perp \eta \perp \sqrt{d_1}\eta \simeq \mathbb{H} \perp \eta \perp \sqrt{d_1}\eta.$$

We extend ψ to $F(\sqrt[4]{d_1})$ to get:

$$([1, b] \perp \eta)_{F(\sqrt[4]{d_1})} \perp \dim \eta \times \langle 0 \rangle \simeq \mathbb{H} \perp \eta \perp \dim \eta \times \langle 0 \rangle\,,$$

or equivalently, $\theta := [1, b] \perp \eta \perp d_1\eta$ is quasi-hyperbolic over $F(\sqrt[4]{d_1})$.

(1) If θ is anisotropic over F, we deduce from Proposition 5.2 that

$$[1, b] \perp \eta \perp d_1\eta \sim \psi_1 \perp \eta \perp d_1\eta, \tag{7.1}$$

where $\psi_1 \in \mathcal{M}^1_F(2, d_1; \theta)$. Corollary 2.8, applied to equation (7.1), implies

$$[1, b] \perp \eta \sim \psi_1 \perp \psi_2 \perp \eta,$$

for some $\psi_2 \in \mathcal{N}_F(d_1)$. We add in both sides of the previous equivalence the form η' to get $\mathrm{ql}(\varphi)$. Clearly, $\mathcal{M}_F^1(2, d_1; \theta) \subset \mathcal{M}_F^1(n_1, d_1, \cdots, n_s, d_s; \varphi)$ and $\mathcal{N}_F(d_1) \subset \mathcal{N}_F(d_1, \cdots, d_s)$. Hence, the proposition is satisfied.

(2) If θ is isotropic, then there exist $x_1 \in D_F(\eta)$ such that $d_1 x_1 \in D_F([1,b] \perp \eta)$. Since $\eta \perp d_1\eta$ is anisotropic, we get the following isometry over F:

$$[1,b] \perp \eta \simeq d_1x_1[1,v] \perp \eta,$$

for some $v \in F$. Further, since $x_1 \in D_F(\eta)$ we have the following equivalence over F:

$$\begin{aligned}[1,b] \perp \eta &\sim d_1x_1[1,v] \perp x_1[1,v] \perp \eta,\\ &\sim \langle 1, d_1\rangle_b \otimes (x_1[1,v]) \perp \eta.\end{aligned}$$

In particular, $[1,b] \perp \mathrm{ql}(\varphi) \sim \psi_3 \perp \mathrm{ql}(\varphi)$, where $\psi_3 = \langle 1, d_1\rangle_b \otimes (x_1[1,v]) \in \mathcal{N}_F(d_1)$. Hence, the proposition is satisfied.

(B) Suppose that ψ is anisotropic over L. Since ψ_K is strictly quasi-hyperbolic and $[K:L] < [K:F]$, we get by induction the following equivalence over L:

$$\begin{aligned}[1,b] \perp \eta \perp \sqrt{d_1}\eta &\sim \sum_{i\in I} \left\langle\!\left\langle u_i, u_i^{2^t}(\sqrt{d_1})^{k(t,1)} d_2^{k(t,2)} \cdots d_s^{k(t,s)}\right]\!\right] \perp \left\langle 1, \sqrt{d_1}\right\rangle_b \otimes \rho_1\\ &\quad \perp \sum_{j=2}^{s} \langle 1, d_j\rangle_b \otimes \rho_j \perp \eta \perp \sqrt{d_1}\eta,\end{aligned} \tag{7.2}$$

where $u_i \in D_L\left(\langle\langle\sqrt{d_1}, d_2, \ldots, d_s\rangle\rangle \otimes (\langle\langle\sqrt{d_1}\rangle\rangle \otimes \eta)\right) = D_L(\langle\langle\sqrt{d_1}, d_2, \ldots, d_s\rangle\rangle \otimes \eta)$, $\rho_j \in W_q(L)$ and

- $1 \le t < e'$,
- $0 \le k(t,l) < 2^t$,
- $\max\{1, 2^{t-n_l+1}\} \mid k(t,l)$ for $l \neq 1$, and $\max\{1, 2^{t-(n_1-1)+1}\} \mid k(t,1)$,

where e' is the exponent of the extension K/L. Observe that $e' \in \{e-1, e\}$. Completing the quaslinear part in both sides of equation (7.2) yields:

$$\begin{aligned}[1,b] \perp \left\langle\!\left\langle \sqrt{d_1}, d_2, \ldots, d_s\right\rangle\!\right\rangle \otimes \eta &\sim \sum_{i\in I} \left\langle\!\left\langle u_i, u_i^{2^t}(\sqrt{d_1})^{k(t,1)} d_2^{k(t,2)} \cdots d_s^{k(t,s)}\right]\!\right]\\ &\quad \perp \left\langle 1, \sqrt{d_1}\right\rangle_b \otimes \rho_1 \perp \sum_{j=2}^{s} \langle 1, d_j\rangle_b \otimes \rho_j \perp \left\langle\!\left\langle \sqrt{d_1}, d_2, \ldots, d_s\right\rangle\!\right\rangle \otimes \eta,\end{aligned} \tag{7.3}$$

For all $i \in I$, let us write $u_i = u_{i0} + u_{i1}\sqrt{d_1}$ such that $u_{i0}, u_{i1} \in F$. Recall (Lemma 4.10):

- $u_{i0}, u_{i1} \in D_F(\pi \otimes \eta) \subset D_F(\pi \otimes \mathrm{ql}(\varphi))$,
- $u_{i0}, u_{i1}\sqrt{d_1} \in D_L(\langle\langle\sqrt{d_1}, d_2, \ldots, d_s\rangle\rangle \otimes \eta)$.

Put $d_2^{k(t,2)} \cdots d_s^{k(t,s)} = \beta \in F$. Applying Corollary 4.11 to equation (7.3) yields:

$$\begin{aligned}[1,b] \perp \left\langle\!\left\langle \sqrt{d_1}, d_2, \ldots, d_s\right\rangle\!\right\rangle \otimes \eta &\sim \sum_{i\in I} \left([1, u_{i0}^{2^{t+1}} d_1^{k(t,1)}\beta^2] \perp [1, u_{i1}^{2^{t+1}} d_1^{k(t,1)+2^t}\beta^2]\right)\\ &\quad \perp \left\langle 1, \sqrt{d_1}\right\rangle_b \otimes \rho_1 \perp \sum_{j=2}^{s} \langle 1, d_j\rangle_b \otimes \rho_j \perp \left\langle\!\left\langle \sqrt{d_1}, d_2, \ldots, d_s\right\rangle\!\right\rangle \otimes \eta.\end{aligned} \tag{7.4}$$

Using the isometry $[1,x] \perp [1,y] \sim [1,x+y]$, we deduce from equation (7.4) the following:

$$[1,b'] \perp \left\langle\left\langle \sqrt{d_1}, d_2, \ldots, d_s \right\rangle\right\rangle \otimes \eta \sim \left\langle 1, \sqrt{d_1} \right\rangle_b \otimes \rho_1 \perp \sum_{j=2}^{s} \langle 1, d_j\rangle_b \otimes \rho_j \perp \left\langle\left\langle \sqrt{d_1}, d_2, \ldots, d_s \right\rangle\right\rangle \otimes \eta, \tag{7.5}$$

where $b' = b + \sum_{i\in I} \left(u_{i0}^{2^{t+1}} d_1^{k(t,1)} \beta^2 + u_{i1}^{2^{t+1}} d_1^{k(t,1)+2^t} \beta^2 \right)$.

Let $M = F(\sqrt[4]{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_s})$ and $S = F(\sqrt{d_2}, \ldots, \sqrt{d_s})$. We extend equation (7.5) to M getting:

$$([1,b'] \perp \eta)_M \perp (2^s - 1)\dim\eta \times \langle 0\rangle \sim \eta \perp (2^s-1)\dim\eta \times \langle 0 \rangle.$$

Equivalently, we can cancel $(2^s-1)\dim\eta \times \langle 0\rangle$ and write:

$$([1,b'] \perp \eta \perp d_1\eta)_M \sim \eta \perp \dim\eta \times \langle 0\rangle.$$

Thus, the F-quadratic form $[1,b'] \perp \eta \perp d_1\eta$ is strictly quasi-hyperbolic over M.

(1) Suppose that $[1,b'] \perp \eta \perp d_1\eta$ is isotropic over F. Then, its Witt index is 1 because $\eta \perp d_1\eta$ is anisotropic over F, and thus

$$[1,b'] \perp \eta \perp d_1\eta \sim \eta \perp d_1\eta. \tag{7.6}$$

(2) Suppose that $[1,b'] \perp \eta \perp d_1\eta$ is anisotropic over F. Since η_K is anisotropic, the form $\eta_{S(\sqrt{d_1})}$ is anisotropic as well, and thus $(\eta \perp d_1\eta)_S$ is anisotropic.

Case 1. Suppose $i_W(([1,b'] \perp \eta \perp d_1\eta)_S) = 1$. Then, $[1,b'] \perp \eta \perp d_1\eta$ is strictly quasi-hyperbolic over $S(\sqrt{d_1}) = F(\sqrt{d_1}, \ldots, \sqrt{d_s})$. Hence, we get by Theorem 6.1

$$[1,b'] \perp \eta \perp d_1\eta \sim \sum_{l=1}^{s} \langle 1, d_l\rangle_b \otimes \gamma_l \perp \eta \perp d_1\eta$$

for suitable $\gamma_1, \cdots, \gamma_s \in W_q(F)$. In particular, we have

$$([1,b'] \perp \eta \perp d_1\eta)_S \sim \langle 1, d_1\rangle_b \otimes \gamma_1 \perp \eta \perp d_1\eta \tag{7.7}$$

Case 2. Suppose $i_W(([1,b'] \perp \eta \perp d_1\eta)_S) = 0$. Then, $[1,b'] \perp \eta \perp d_1\eta$ is anisotropic over S and becomes quasi-hyperbolic over M. Hence, by the quartic case (Step 1 in the proof of Proposition 5.2), we get

$$([1,b'] \perp \eta \perp d_1\eta)_S \sim \langle\langle \alpha, \alpha^2 d_1]] \perp \eta \perp d_1\eta, \tag{7.8}$$

where $\alpha \in D_S(\eta \perp d_1\eta)$. Obviously, $\alpha \in D_S(\eta \perp d_1\eta)$ implies $\alpha \in D_F(\pi \otimes \eta)$.

To summarize, in every situation (equations (7.6), (7.7) and (7.8)), we have:

$$([1,b'] \perp \eta \perp d_1\eta)_S \sim (G \perp \eta \perp d_1\eta)_S, \tag{7.9}$$

where $G \in \mathcal{N}_F(d_1)$ or $G = \langle\langle \alpha, \alpha^2 d_1]]$ such that $\alpha \in D_F(\pi \otimes \eta) \subset D_F(\pi \otimes \mathrm{ql}(\varphi))$.

We add to equation (7.9) the form $\sum_{i\in I}\left([1,u_{i0}^{2^{t+1}}d_1^{k(t,1)}\beta^2]\perp[1,u_{i1}^{2^{t+1}}d_1^{k(t,1)+2^t}\beta^2]\right)$, and complete the quasilinear part to get:

$$(7.10)\quad ([1,b]\perp\pi\otimes\eta)_S\sim\left(\sum_{i\in I}[1,u_{i0}^{2^{t+1}}d_1^{k(t,1)}\beta^2]\perp[1,u_{i1}^{2^{t+1}}d_1^{k(t,1)+2^t}\beta^2]\perp G\perp\pi\otimes\eta\right)_S.$$

Recall that $u_{i0},u_{i1}\in D_F(\pi\otimes\eta)$ and the exponent e' of K/L is equal to $e-1$ or e. Now we discuss on each integer t that appears in equation (7.10) to get the desired generators over F:

(a) Suppose $t+1<e$. This happens when $t\le e'-2$ or $e'=e-1$. Obviously, for all $l=2,\dots,s$, we have $0\le 2k(t,l)<2.2^t=2^{t+1}$ and the condition $\max\{1,2^{t-n_l+1}\}$ divides $k(t,l)$ implies that $\max\{1,2^{(t+1)-n_l+1}\}$ divides $2.k(t,l)$. Moreover, $k(t,1)<2^t<2^{t+1}$ and $k(t,1)+2^t<2^t+2^t=2^{t+1}$. So in this case, we add to (7.10) the form $u_{i0}[1,u_{i0}^{2^{t+1}}d_1^{k(t,1)}\beta^2]\perp u_{i1}[1,u_{i1}^{2^{t+1}}d_1^{k(t,1)+2^t}\beta^2]$ to get the generators $\left\langle\left\langle u_{i0},u_{i0}^{2^{t+1}}d_1^{k(t,1)}\beta^2\right]\right]$ and $\left\langle\left\langle u_{i1},u_{i1}^{2^{t+1}}d_1^{k(t,1)+2^t}\beta^2\right]\right]$.

(b) Suppose $t+1=e$. Recall that $e=n_1$. Since by induction hypothesis $\max\{1,2^{t-(n_1-1)+1}\}$ divides $k(t,1)$, then $k(t,1)$ is even. Hence, we get the following isometries over F:

$$[1,u_{i0}^{2^{t+1}}d_1^{k(t,1)}\beta^2]\simeq[1,u_{i0}^{2^t}d_1^{k(t,1)/2}\beta],$$
$$[1,u_{i1}^{2^{t+1}}d_1^{k(t,1)+2^t}\beta^2]\simeq[1,u_{i1}^{2^t}d_1^{(k(t,1)/2)+2^{t-1}}\beta].$$

We have by induction hypothesis $0\le k(t,l)<2^t$ and $\max\{1,2^{t-n_l+1}\}$ divides $k(t,l)$ for all $2\le l\le s$. Obviously, we have $\max\{1,2^{t-n_1+1}\}$ divides $k(t,1)/2$ and $0\le k(t,1)/2+2^{t-1}<2^t$. So in this case, we add to (7.10) the form $u_{i0}[1,u_{i0}^{2^t}d_1^{k(t,1)/2}\beta]\perp u_{i1}[1,u_{i1}^{2^t}d_1^{(k(t,1)/2)+2^{t-1}}\beta]$ to get the generators $\left\langle\left\langle u_{i0},u_{i0}^{2^t}d_1^{k(t,1)/2}\beta\right]\right]$ and $\left\langle\left\langle u_{i1},u_{i1}^{2^t}d_1^{k(t,1)/2+2^{t-1}}\beta\right]\right]$.

Moreover, $\pi\otimes\eta$ is quasi-hyperbolic over S because $i_d(\pi_S)=2^s-2$. Hence, after recovering the generators as explained in (a) and (b), we apply Theorem 6.1 to equation (7.10) to get:

$$(7.11)\qquad [1,b]\perp\pi\otimes\eta\sim\varphi_1\perp\varphi_2\perp\pi\otimes\eta,$$

for some $\varphi_1\in\mathcal{M}_F^1(n_1,d_1,\cdots,n_s,d_s;\varphi)$ and $\varphi_2\in\mathcal{N}_F(d_1,\cdots,d_s)$.

We apply Corollary 2.8 to (7.11) to eliminate the form $d_1\langle\langle d_2,\cdots,d_s\rangle\rangle\otimes\eta$ in both sides, and get in the right hand side a form in $\mathcal{N}_F(d_1)$. So the new quasilinear part is $\langle\langle d_2,\cdots,d_s\rangle\rangle\otimes\eta$. Again we eliminate the form $d_2\langle\langle d_3,\cdots,d_s\rangle\rangle\otimes\eta$ and get in the right hand side a form in $\mathcal{N}_F(d_2)$. Step by step we continue this process until we reduce the quasilinear part to η. Now adding in both sides of (7.11) the form η', we recover the quasilinear part $\mathrm{ql}(\varphi)$. This ends the proof of the proposition. $\square$

Now we give the proof of Theorem 7.1 which is an adaptation of that of Theorem 5.1.

Proof of Theorem 7.1. Let $\varphi=R\perp\mathrm{ql}(\varphi)$ be an anisotropic semisingular quadratic form over F. The two conditions given in Theorem 7.1 imply that φ is strictly quasi-hyperbolic over K.

Conversely, suppose that φ_K is strictly quasi-hyperbolic. Since $\mathrm{ql}(\varphi)_K$ is quasi-hyperbolic, it follows from Corollary 3.2 that $\pi\otimes\mathrm{ql}(\varphi)$ is quasi-hyperbolic. Let us write $R=[a_1,b_1]\perp$

$\ldots \perp [a_r, b_r]$ for some $a_i, b_i \in F^*$. We proceed by induction on $\dim R$. Since $i_W(\varphi)_K = r$, we have

$$\varphi_K \simeq r \times \mathbb{H} \perp \mathrm{ql}(\varphi)_K.$$

Adding $\langle a_2, a_3, \ldots, a_r\rangle$ to this equation, and canceling the hyperbolic planes, we get:

$$[a_1, b_1] \perp \langle a_2, a_3, \ldots, a_r\rangle_K \perp \mathrm{ql}(\varphi)_K \simeq \mathbb{H} \perp \langle a_2, a_3, \ldots, a_r\rangle_K \perp \mathrm{ql}(\varphi)_K. \tag{7.12}$$

Let δ be a subform of $\langle a_2, a_3, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi)$ such that

$$\delta_K \simeq (\langle a_2, a_3, \ldots, a_r\rangle_K \perp \mathrm{ql}(\varphi)_K)_{an}.$$

Let δ' be a form such that $\langle a_2, a_3, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi) \simeq \delta \perp \delta'$. Canceling the zero form in (7.12), we obtain $([a_1, b_1] \perp \delta)_K \simeq \mathbb{H} \perp \delta_K$. This implies that $\widetilde{\varphi} := [a_1, b_1] \perp \delta \perp d_1\delta$ is strictly quasi-hyperbolic over K.

Case 1. If $\widetilde{\varphi}$ is anisotropic, then applying Proposition 7.2 yields:

$$\widetilde{\varphi} \sim a_1\varphi_1 \perp \varphi_2 \perp \langle 1, d_1\rangle \otimes \delta, \tag{7.13}$$

where $\varphi_1 \in \mathcal{M}_F^{a_1}(n_1, d_1, \cdots, n_s, d_s)$ and $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$. Since $\langle 1, d_1\rangle \otimes \delta \subseteq \langle 1, d_1\rangle \otimes (\langle a_2, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi))$, we obtain

$$[a_1, b_1] \perp \langle 1, d_1\rangle \otimes (\langle a_2, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi)) \sim a_1\varphi_1 \perp \varphi_2 \perp \langle 1, d_1\rangle \otimes (\langle a_2, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi)). \tag{7.14}$$

We apply Corollary 2.8 to eliminate the form $d_1(\langle a_2, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi))$ in both sides of (7.14), and get a new form $\varphi_2' \in \mathcal{N}_F(d_1, \cdots, d_s)$ such that

$$[a_1, b_1] \perp \langle a_2, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi) \sim a_1\varphi_1 \perp \varphi_2' \perp \langle a_2, \ldots, a_r\rangle \perp \mathrm{ql}(\varphi). \tag{7.15}$$

We use Lemma 2.6 on equation (7.15) by completing $\langle a_2, a_3, \ldots, a_r\rangle$ on left side with $[a_2, b_2] \perp \ldots \perp [a_n, b_n]$, we get

$$\varphi \sim a_1\varphi_1 \perp \varphi_2' \perp [a_2, x_2] \perp \ldots \perp [a_r, x_r] \perp \mathrm{ql}(\varphi), \tag{7.16}$$

for suitable scalars $x_2, \ldots, x_r \in F$. Since φ_K is strictly quasi-hyperbolic, it follows from equation (7.16) that $\varphi' := [a_2, x_2] \perp \ldots \perp [a_r, x_r] \perp \mathrm{ql}(\varphi)$ is also strictly quasi-hyperbolic over K. We conclude by induction because the nonsingular part of φ'_{an} is of dimension $< 2r$.

Case 2. If $\widetilde{\varphi}$ is isotropic, then $i_W(\widetilde{\varphi}) = 1$ because $\langle 1, d_1\rangle \otimes \delta$ is anisotropic over F as $\delta_{F(\sqrt{d_1})}$ is anisotropic. Then, we have

$$[a_1, b_1] \perp \langle 1, d_1\rangle \otimes \delta \simeq \mathbb{H} \perp \langle 1, d_1\rangle \otimes \delta.$$

Hence, we are in the condition of equation (7.13) with φ_1 and φ_2 hyperbolic, and thus we conclude as in the first case. This proves the theorem.

8. Isotropy over purely inseparable modular extensions

Throughout this section, we let $K = F(\sqrt[2^{n_1}]{d_1}, \cdots, \sqrt[2^{n_s}]{d_s})$ a purely inseparable modular extension of F as in Section 3. Let π be the anisotropic quasi-Pfister form $\langle\langle d_1, \cdots, d_s\rangle\rangle$ attached to K, and let e be the exponent of K.

Our aim is to apply Theorem 7.1 to classify semisingular F-quadratic forms that become isotropic or more generally having a given Witt index over K.

8.1. **Maximality of Witt index over** K**.** We first give a preliminary lemma.

Lemma 8.1. *Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be a semingular F-quadratic form. Suppose that φ has maximal Witt index over K. Then, the form $(R \perp \pi \otimes \mathrm{ql}(\varphi))_{an}$ is strictly quasi-hyperbolic over K.*

Proof. By the roundness of π, there exists a totally singular form S over F such that $(\pi \otimes \mathrm{ql}(\varphi))_{an} \simeq \pi \otimes S$. Note that S_K is anisotropic because $\pi \otimes S$ is anisotropic (Theorem 3.1).

Let $l = i_d(\pi \otimes \mathrm{ql}(\varphi))$ and $\dim R = 2r$. By the uniqueness of the quasilinear part, we have

$$(R \perp \pi \otimes \mathrm{ql}(\varphi))_{an} \simeq R' \perp \pi \otimes S$$

for some nonsingular form R' of dimension $2r' \leq 2r$. Hence

$$R \perp \pi \otimes \mathrm{ql}(\varphi) \sim R' \perp \pi \otimes S \perp l \times \langle 0\rangle \,. \tag{8.1}$$

Moreover, we have

$$(R \perp \mathrm{ql}(\varphi))_K \sim \mathrm{ql}(\varphi)_K,$$

and

$$(\pi \otimes \mathrm{ql}(\varphi))_K \simeq S_K \perp (l + (2^s - 1)\dim S) \times \langle 0\rangle \,.$$

Thus, extending equation (8.1) to K yields

$$S_K \perp (l + (2^s - 1)\dim S) \times \langle 0\rangle \sim (R' \perp S)_K \perp (l + (2^s - 1)\dim S) \times \langle 0\rangle \,. \tag{8.2}$$

Since S_K is anisotropic, the form $(R' \perp S)_K$ is nondefective. Using Witt cancellation (Proposition 2.4), we deduce from equation (8.2) the equivalence $(R' \perp S)_K \sim S_K$. Hence

$$(R' \perp \pi \otimes S)_K \sim S_K \perp (2^s - 1)\dim S \times \langle 0\rangle \,.$$

Consequently, $(R' \perp \pi \otimes S)_K$ is strictly quasi-hyperbolic, as desired. □

The following corollary classifies semisingular quadratic forms that have maximal Witt index over K.

Corollary 8.2. *Let φ be an anisotropic semisingular F-quadratic form. Suppose that φ has maximal Witt index over K. Then,*

$$\varphi \sim \varphi_1 \perp \varphi_2 \perp \mathrm{ql}(\varphi),$$

where $\varphi_1 \in \mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ and $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$.

Proof. Let us write $\varphi = R \perp \mathrm{ql}(\varphi)$. Since φ_K has maximal Witt index, it follows from Lemma 8.1 that $(R \perp \pi \otimes \mathrm{ql}(\varphi))_{an}$ is strictly quasi-hyperbolic over K. Theorem 7.1 implies

$$R \perp \pi \otimes \mathrm{ql}(\varphi) \sim \varphi_1 \perp \varphi_2 \perp \pi \otimes \mathrm{ql}(\varphi), \tag{8.3}$$

where $\varphi_1 \in \mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ and $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$.

By Corollary 2.8, we successively eliminate in equation (8.3) the forms $d_1 \langle\langle d_2, \cdots, d_s\rangle\rangle \otimes \mathrm{ql}(\varphi)$, $d_2 \langle\langle d_3, \cdots, d_s\rangle\rangle \otimes \mathrm{ql}(\varphi), \cdots, d_s\mathrm{ql}(\varphi)$, and we obtain in the right hand side forms in $\mathcal{N}_F(d_1), \mathcal{N}_F(d_2), \cdots, \mathcal{N}_F(d_s)$. Hence, after this process, we recover the initial quasilinear part $\mathrm{ql}(\varphi)$. This proves the corollary. $\square$

8.2. **On the isotropy over** K**.** Let $L = F(\sqrt[2^{n_1-1}]{d_1}, \cdots, \sqrt[2^{n_s-1}]{d_s})$ and $\alpha_i = \sqrt[2^{n_i-1}]{d_i}$ for all $1 \leq i \leq s$ (note that $L = F$ when $e = 1$). Let $J = \{(\epsilon_1, \cdots, \epsilon_s) \in \mathbb{N}^s \mid 0 \leq \epsilon_i \leq 2^{n_i-1} - 1\}$. For any $\epsilon = (\epsilon_1, \cdots, \epsilon_s) \in J$, let $\alpha^\epsilon = \alpha_1^{\epsilon_1} \cdots \alpha_s^{\epsilon_s}$.

Notation. To any $a, b \in L$ such that $0 \neq a = \sum_{\epsilon \in J} x_\epsilon \alpha^\epsilon$ and $x_\epsilon \in F$, we attach the quadratic form $G(a, b)$ in $I_q^2(F)$ given as follows:

$$G(a,b) = \sum_{\epsilon \in J}{}' \langle 1, x_\epsilon\rangle_b \otimes \left[1, \left(\frac{bx_\epsilon\alpha^\epsilon}{a}\right)^{2^{e-1}}\right].$$

Here $\sum'$ means that the sum is taken over all ϵ for which x_ϵ is nonzero. Clearly, the form $G(a, b)$ is defined over F because $L^{2^{e-1}} \subset F$. A property satisfied by the forms $G(a, b)$ is given by the following lemma:

Lemma 8.3. *We keep the same notations as before. Then, we have the following equivalence:*

$$G(a, b)_K \perp [1, b]_K \sim a[1, b]_K.$$

Proof. We have $\alpha^\epsilon \in K^2$ for all $\epsilon \in J$ because $\alpha_i \in K^2$ for all $1 \leq i \leq s$. Moreover, $[1, x] \simeq [1, x^2]$ for any $x \in K$. Hence, we get

$$\begin{aligned} G(a,b)_K &= \left(\sum_{\epsilon \in J}{}' \langle 1, x_\epsilon\rangle_b \otimes \left[1, \left(\frac{bx_\epsilon\alpha^\epsilon}{a}\right)^{2^{e-1}}\right]\right)_K \\ &\simeq \left(\sum_{\epsilon \in J}{}' \langle 1, x_\epsilon\alpha^\epsilon\rangle_b \otimes \left[1, \frac{bx_\epsilon\alpha^\epsilon}{a}\right]\right)_K. \end{aligned}$$

It follows from Lemma 4.7 that $G(a, b)_K \sim \langle 1, a\rangle_b \otimes [1, b]_K$. Hence, $G(a, b)_K \perp [1, b]_K \sim a[1, b]_K$. $\square$

Our result on the isotropy of semisingular F-quadratic form over K is the following:

Theorem 8.4. *Let K and L as before, and φ a semisingular quadratic form over F whose nonsingular part is of dimension $2r$. Then, the following statements are equivalent:*

(1) $i_W(\varphi_K) \geq m$.
(2) There exist $a_1, \cdots, a_{r-m}, b_1, \cdots, b_{r-m} \in L$ *such that* $a_j \neq 0$ *for all* $1 \leq j \leq r-m$ *and*

$$\varphi \sim \varphi_1 \perp \varphi_2 \perp \sum_{j=1}^{r-m} ([1, b_j^{2^{e-1}}] \perp G(a_j, b_j)) \perp \mathrm{ql}(\varphi),$$

where $\varphi_1 \in \mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ *and* $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$.

Proof. Suppose that $i_W(\varphi_K) \geq m$. Then, we have

$$\varphi_K \simeq m \times \mathbb{H} \perp a_1[1, b_1] \perp \ldots \perp a_{r-m}[1, b_{r-m}] \perp \mathrm{ql}(\varphi)_K, \tag{8.4}$$

where $a_j, b_j \in K$ and $a_j \neq 0$. Since K/L is excellent (Theorem 2.12), we may suppose $a_j, b_j \in L$ for all $1 \leq j \leq r-m$. Adding $\sum_{j=1}^{r-m}[1, b_j]$ to both sides of equation (8.4), we get:

$$\varphi_K \perp \sum_{j=1}^{r-m}[1, b_j]_K \simeq m \times \mathbb{H} \perp \sum_{j=1}^{r-m} \langle 1, a_j \rangle_b \otimes [1, b_j]_K \perp \mathrm{ql}(\varphi)_K.$$

We write $a_j = \sum_{\epsilon \in J} x_{j,\epsilon} \alpha^\epsilon$ for $x_{j,\epsilon} \in F$, $0 \leq j \leq r-m$. We expand the form $\langle 1, a_j \rangle_b \otimes [1, b_j]$ according to Lemma 4.7 to get:

$$\varphi_K \perp \sum_{j=1}^{r-m}[1, b_j]_K \sim \sum_{j=1}^{r-m} \sideset{}{'}\sum_{\epsilon \in J} \langle 1, x_{j,\epsilon}\alpha^\epsilon \rangle_b \otimes \left[1, \frac{b_j x_{j,\epsilon}\alpha^\epsilon}{a_j}\right]_K \perp \mathrm{ql}(\varphi)_K, \tag{8.5}$$

where the notation $\sum'$ in the double sum means that the sum is taken over all j and ϵ for which $x_{j,\epsilon}$ is nonzero. Since $\alpha \in K^2$ and $[1, x] \simeq [1, x^2]$, we can express equation (8.5) as follows:

$$\left(\varphi \perp \sum_{j=1}^{r-m} \left([1, b_j^{2^{e-1}}] \perp G(a_j, b_j)\right)\right)_K \sim \mathrm{ql}(\varphi)_K.$$

Notice that $b_j^{2^{e-1}} \in F$ since $L^{2^{e-1}} \subset F$. Hence, the F-quadratic form $\varphi \perp \sum_{j=1}^{r-m}[1, b_j^{2^{e-1}}] \perp G(a_j, b_j)$ has maximal Witt index over K. Using Corollary 8.2, we get

$$\varphi \perp \sum_{j=1}^{r-m} \left([1, b_j^{2^{e-1}}] \perp G(a_j, b_j)\right) \sim \varphi_1 \perp \varphi_2 \perp \mathrm{ql}(\varphi) \tag{8.6}$$

for $\varphi_1 \in \mathcal{M}_F(n_1, d_1, \cdots, n_s, d_s)$ and $\varphi_2 \in \mathcal{N}_F(d_1, \cdots, d_s)$.

Conversely, let φ be a semisingular quadratic form as in equation (8.6) whose nonsingular part is of dimension $2r$. Extending scalars to K, and using Lemma (8.3) with the isometry $[1, b_j^{2^{e-1}}] \simeq [1, b_j]$, we get $\varphi_K \sim \sum_{j=1}^{r-m} a_j[1, b_j] \perp \mathrm{ql}(\varphi)$. Hence, $i_W(\varphi_K) \geq m$. $\square$

9. Similar results on function fields of some irreducible polynomials

In this section, we are interested in studying the quasi-hyperbolicity of semisingular quadratic forms over different field extensions.

9.1. **On quasi-hyperbolicity over function fields of quadrics.** Our aim is to extend some results from [11] to the setting of semisingular quadratic forms. Let us first fix few terminologies to be useful for our results.

We define $SQ(F)$ to be the set of isometry classes of semisingular quadratic forms over F. This is a semi-group with respect to the orthogonal sum. To any field extension K of F, we attach its Witt kernel $W_q(K/F)$, i.e. the group of nonsingular F-quadratic forms that become hyperbolic over K. Similarly, let $SQ(K/F)$ be the set of semisingular F-quadratic forms that become strictly quasi-hyperbolic over K.

Definition 9.1. *Let K/F be a field extension, and I a nonempty subset of $\mathbb{N}$.*

(1) $SQ(K/F)$ is called an I-Pfister set if any quadratic form in $SQ(K/F)$ is Witt-equivalent to a quadratic form whose nonsingular part is a sum of forms in $GP_{n_i}(F) \cap W_q(K/F)$ for suitable $n_i \in I$.

(2) $SQ(K/F)$ is called a strong I-Pfister set if any anisotropic quadratic form in $SQ(K/F)$ is isometric to a quadratic form whose nonsingular part is a sum of forms in $GP_{n_i}(F) \cap W_q(K/F)$ for suitable $n_i \in I$.

Note that when $K = F(\psi)$ for an anisotropic quadratic form ψ and $SQ(K/F) \neq \emptyset$, then ψ is totally singular (Proposition 2.3). Moreover, for simplicity, we talk about n-Pfister set (strong n-Pfister set) in the particular case $I = \{n\}$.

Our aim is to extend some results of the first author from the setting of nonsingular quadratic forms to semisingular quadratic forms. The proofs follow on the same lines as in [11, Proposition 3.9, Theorem 1.5]. Our main result in this sense is the following theorem:

Theorem 9.2. *Let ψ be an anisotropic totally singular quadratic form over F of dimension greater than 3, and let ψ' be a form dominated by ψ such that $\dim\psi = \dim\psi' + 1$ and $SQ(F(\psi')/F)$ is a strong m-Pfister set. Then, $SQ(F(\psi)/F)$ is an $\{m, m+1\}$-Pfister set.*

To prove this result we need some facts. Let ψ be a nonzero totally singular F-quadratic form. The norm field of ψ is the field $N_F(\psi) := F^2(\alpha\beta \mid \alpha, \beta \in D_F(\psi))$. We have $[N_F(\psi) : F^2] = 2^d$ for some integer $d \geq 1$, and thus there exist $x_1, \cdots, x_d \in F$ such that $N_F(\psi) = F^2(x_1, \cdots, x_d)$. The quasi-Pfister form $\pi_\psi := \langle\langle x_1, \ldots, x_d\rangle\rangle$ is anisotropic and uniquely determined by ψ. Moreover, we have $\alpha\psi \subset \pi_\psi$ for any scalar $\alpha \in D_F(\psi)$, and π_ψ is the quasi-Pfister form of smallest dimension satisfying this property (see [6, Section 8] for more details). The following result deals with the quasi-hyperbolicity over function fields of totally singular quadratic forms.

Theorem 9.3. *([10, Theorem 1.5]) Let φ and ψ be anisotropic totally singular quadratic forms over F. Then, φ is quasi-hyperbolic over $F(\psi)$ iff $\varphi \simeq \pi_\psi \otimes \rho$ for some totally singular F-quadratic form ρ.*

Moreover, we need a generalization of the domination theorem to the setting of semisingular quadratic forms.

Proposition 9.4. *(*[14, Prop. 1.3]*) Let $\varphi = R' \perp \mathrm{ql}(\varphi)$ be an anisotropic semisingular F-quadratic form. Let ψ be an anisotropic totally singular quadratic form such that $\varphi_{F(\psi)}$ is quasi-hyperbolic. Then, for any $\alpha \in D_F(\psi)$, $\beta \in D_F(R')$ and $\gamma \in D_F(\mathrm{ql}(\varphi))$, there exists a nonsingular quadratic form R such that $\varphi \simeq R \perp \mathrm{ql}(\varphi)$, $\psi \prec \alpha\beta R$ and $\psi \prec \alpha\gamma\mathrm{ql}(\varphi)$.*

Recall that if ψ and ψ' are two anisotropic quadratic forms over F such that $\psi_{F(\psi')}$ is isotropic, then $W_q(F(\psi)/F) \subset W_q(F(\psi')/F)$ [11, Prop. 3.9]. This result remains true for the semi-group $SQ(F)$ as the following proposition shows.

Proposition 9.5. *Let ψ and ψ' be anisotropic totally singular forms over F such that $\psi_{F(\psi')}$ is isotropic. Then $SQ(F(\psi)/F) \subset SQ(F(\psi')/F)$.*

Proof. Suppose $SQ(F(\psi)/F) \neq 0$ and let $\eta \in SQ(F(\psi)/F)$ be anisotropic. We will prove that η belongs to $SQ(F(\psi')/F)$ using an induction on $\dim \eta$. By Proposition 9.4, there exists a nonsingular F-quadratic form R such that $\eta \simeq R \perp \mathrm{ql}(\eta)$ and ψ is weakly dominated by R and $\mathrm{ql}(\eta)$. Since $\psi_{F(\psi')}$ is isotropic, it follows that $R_{F(\psi')}$ is also isotropic. In particular, we have $i_W(\eta_{F(\psi')}) \geq 1$. Hence, the extension $F(\psi')(\eta)/F(\psi')$ is purely transcendental [3, Proposition 22.9]. We discuss two cases:

Case 1. Suppose $i_W(\eta_{F(\eta)}) = \frac{\dim \eta}{2}$, i.e., $\eta_{F(\eta)} = \frac{\dim R}{2} \times \mathbb{H} \perp \mathrm{ql}(\eta)_{F(\eta)}$. Extending this isometry to $F(\eta)(\psi')$, we get:

$$\eta_{F(\eta)(\psi')} = \frac{\dim R}{2} \times \mathbb{H} \perp \mathrm{ql}(\eta)_{F(\eta)(\psi')}. \tag{9.1}$$

We claim that that $\mathrm{ql}(\eta)_{F(\eta)(\psi')}$ is quasi-hyperbolic. In fact, since $\eta_{F(\psi)}$ is quasi-hyperbolic, the form $\mathrm{ql}(\eta)_{F(\psi)}$ is quasi-hyperbolic. By Theorem 9.3, we get $\mathrm{ql}(\eta) \simeq \pi_\psi \otimes \rho$, where π_ψ is the quasi-Pfister form associated to ψ, and ρ is a suitable totally singular form. Since, $\psi_{F(\psi')}$ is isotropic and ψ is similar to a subform of π_ψ, it follows that $(\pi_\psi)_{F(\psi')}$ is also isotropic. Thus, $(\pi_\psi)_{F(\psi')}$ is quasi-hyperbolic [6, Corollary 8.14]. Consequently, $\mathrm{ql}(\eta)$ is quasi-hyperbolic over $F(\psi')$, in particular $\mathrm{ql}(\eta)$ is quasi-hyperbolic over $F(\eta)(\psi')$. Hence, equation (9.1) yields the quasi-hyperbolicity of $\eta_{F(\eta)(\psi')}$. Now since $F(\psi')(\eta)/F(\psi')$ is a purely transcendental extension, the form η is quasi-hyperbolic over $F(\psi')$.

Case 2. Suppose $i_W(\eta_{F(\eta)}) < \frac{\dim \eta}{2}$. Let $\eta_1 := (\eta_{F(\eta)})_{an}$. The form $\psi_{F(\eta)}$ is anisotropic because ψ is totally singular and η is not totally singular [13, Cor. 3.1]. Moreover, the form $(\eta_1)_{F(\eta)(\psi)}$ is quasi-hyperbolic because $\eta_1 \sim \eta_{F(\eta)}$ and $\eta_{F(\psi)}$ is quasi-hyperbolic. Since $\psi_{F(\eta)(\psi')}$ is isotropic, we deduce by induction that $(\eta_1)_{F(\eta)(\psi')}$ is quasi-hyperbolic. Since $F(\eta)(\psi') = F(\psi')(\eta)$ and the extension $F(\psi')(\eta)/F(\psi')$ is purely transcendental extension, the form $\eta_{F(\psi')}$ is quasi-hyperbolic. Hence, in both cases we have $\eta \in SQ(F(\psi')/F)$, as desired. □

As an immediate consequence we get the following corollary:

Corollary 9.6. *Let π be an anisotropic quasi-Pfister form and ψ a quasi-Pfister neighbor of π. Then, $SQ(F(\pi)/F) = SQ(F(\psi)/F)$.*

Proof. Since ψ is a quasi-Pfister form of π, it follows that $\pi_{F(\psi)}$ and $\psi_{F(\pi)}$ are isotropic [6, Prop. 8.9(iii)]. Then, the corollary follows from Proposition 9.5. □

Now we are able to prove Theorem 9.2.

Proof of Theorem 9.2. Let ψ and ψ' be anisotropic totally singular quadratic form over F such that $\dim\psi \geq 3$, ψ' is dominated ψ and $\dim\psi = \dim\psi' + 1$. Suppose that $SQ(F(\psi')/F)$ is a strong m-Pfister set. We have to prove that $SQ(F(\psi)/F)$ is an $\{m, m+1\}$-Pfister set.

Let $\eta \in SQ(F(\psi)/F)$ be anisotropic and write $\eta = R \perp \mathrm{ql}(\eta)$. We proceed by induction on $\dim R$. By Proposition 9.5, $\eta \in SQ(F(\psi')/F)$ and by assumption $\eta \simeq \perp_{i=1}^{r} \eta_i \perp \mathrm{ql}(\eta)$, where $\eta_i \in GP_m(F) \cap W_q(F(\psi')/F)$ for all $1 \leq i \leq r$. After scaling, we may assume that ψ' and η_1 represent 1, and thus $\psi' \prec \eta_1$. Put $\gamma = \perp_{i=2}^{r} \eta_i \perp \mathrm{ql}(\eta)$.

If $\eta_1 \in W_q(F(\psi)/F)$, then $\gamma_{F(\psi)}$ is quasi-hyperbolic and we are done by induction. So we assume that $\eta_1 \notin W_q(F(\psi)/F)$, which is equivalent to saying that η_1 is anisotropic over $F(\psi)$, and thus in particular η_1 does not dominate ψ.

Since η_1 dominates ψ' but not ψ, it follows that $i_t(\psi \perp \eta_1) = \dim\psi'$ [6, Corollary 3.13]. Note that $i_t(\psi \perp \eta_1) = i_W(\psi \perp \eta_1)$ because η_1 is nonsingular and ψ is anisotropic. If σ denotes the anisotropic part of $\psi \perp \eta_1$, then

$$\psi \perp \eta_1 \simeq \sigma \perp u \times \mathbb{H}$$

with $u = \dim\psi'$.

By our choice, η is quasi-hyperbolic over $F(\psi)$, and since both η and ψ represent 1, we have η dominates ψ by Proposition 9.4. Hence, $i_t(\psi \perp \eta) = \dim\psi = \dim\psi' + 1$. Therefore,

$$\psi \perp \eta \simeq \sigma \perp \gamma \perp u \times \mathbb{H}.$$

Comparing the total Witt index on both sides shows that $\sigma \perp \gamma$ is isotropic, so there exists $x \in D_F(\sigma) \cap D_F(\gamma)$ since σ and γ are anisotropic.

Now consider $\pi \simeq \eta_1 \perp x\eta_1$. This form is anisotropic because $\eta_1 \perp \langle x \rangle$, which is a Pfister neighbor of π, is dominated by η. Then,

$$\psi \perp \pi \simeq \sigma \perp x\rho_1 \perp u \times \mathbb{H}.$$

Note that $\sigma \perp x\rho_1$ is isotropic. Hence, $i_t(\psi \perp \pi) \geq u + 1 = \dim\psi$, and thus π dominates ψ [6, Corollary 3.13]. Consequently, $\pi \in P_{m+1}(F) \cap W_q(F(\psi)/F)$.

Now let η' be the anisotropic part of $\eta \perp \pi$. We have both η and π dominate $\rho_1 \perp \langle x \rangle$, so $i_t(\eta \perp \pi) \geq \dim(\rho_1 \perp \langle x \rangle) = 2^m + 1$. Since $\mathrm{ql}(\eta)$ is anisotropic, we have $i_t(\eta \perp \pi) = i_W(\eta \perp \pi)$. Thus, $\dim\eta' \leq \dim\eta + \dim\pi - 2(2^m + 1) < \dim\eta$. Moreover $\eta' \in SQ(F(\psi)/F)$, thus we conclude by induction. $\square$

We finish this section by some computations of $SQ(F(\psi)/F)$. The first one concerns the case where ψ is a quasi-Pfister neighbor, it extends [11, Theorem 1.4].

Proposition 9.7. *Let ψ be an anisotropic quasi-Pfister neighbor of a quasi-Pfister form $\langle\langle a_1, \cdots, a_n \rangle\rangle$ over F. Then, any anisotropic form in $SQ(F(\psi)/F)$ is isometric to $B \otimes R \perp \pi \otimes \rho$ for suitable nonsingular form R and a totally singular form ρ, where $B = \langle\langle a_1, \cdots, a_n \rangle\rangle_b$. In particular, $SQ(F(\psi)/F)$ is a strong $(n+1)$-Pfister set.*

Proof. Let $\pi = \langle\langle a_1, \cdots, a_n \rangle\rangle$ and $B = \langle\langle a_1, \cdots, a_n \rangle\rangle_b$. Let J be the set of n-tuples $(\epsilon_1, \cdots, \epsilon_n)$ such that $\epsilon_i \in \{0, 1\}$ for all $1 \leq i \leq n$. For any $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in J$, we let $a_\epsilon = a_1^{\epsilon_1} \ldots a_n^{\epsilon_n}$. Then, $\pi = \perp_{\epsilon \in J} \langle a_\epsilon \rangle$.

We may suppose $1 \in D_F(R)$. Since $\varphi_{F(\pi)}$ is quasi-hyperbolic, the form $\mathrm{ql}(\varphi)_{F(\pi)}$ is also quasi-hyperbolic. It follows from Theorem 9.3 that $\mathrm{ql}(\varphi) \simeq \pi \otimes \rho$ for some totally singular form ρ. Moreover, by [14, Proposition 1.3], we get

$$\varphi \simeq \perp_{\epsilon \in J} a_\epsilon[1, b_\epsilon] \perp R' \perp \mathrm{ql}(\varphi), \tag{9.2}$$

for some scalars $b_\epsilon \in F^*$ for all $\epsilon \in J$, and a nonsingular form R'. Consequently, equation (9.2) can be re-written as follows

$$\varphi \perp (2^n - 1) \times \mathbb{H} \simeq B \otimes [1, b_{\mathbf{0}}] \perp R'' \perp \mathrm{ql}(\varphi),$$

where $R'' = R' \perp (\perp_{\epsilon \in J \setminus \{\mathbf{0}\}} a_\epsilon[1, b_{\mathbf{0}} + b_\epsilon])$ and $\mathbf{0} = (0, 0, \cdots, 0)$ is the zero tuple. Since $\dim R'' = \dim R - 2$ and the form $R'' \perp \mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(\pi)$, we conclude by induction on the nonsingular part that

$$\varphi \sim B \otimes \gamma \perp \mathrm{ql}(\varphi)$$

for some nonsingular form γ.

Moreover, by the roundness of B (see the subsection 2.2), we may suppose that $B \otimes \gamma$ is anisotropic. We also choose γ of minimal dimension for the property that $\varphi \sim B \otimes \gamma \perp \mathrm{ql}(\varphi)$. Then, we get $\varphi \simeq B \otimes \gamma \perp \mathrm{ql}(\varphi)$, otherwise there would exist $x \in D_F(B \otimes \gamma) \cap D_F(\mathrm{ql}(\varphi))$. Using the roundness of B and π, we would get $B \otimes \gamma \simeq B \otimes ([x, y] \perp \gamma')$ and $\mathrm{ql}(\varphi) \simeq \pi \otimes (\langle x \rangle \perp \rho')$ for some $y \in F$ and forms γ' and ρ'. Hence, $\varphi \sim B \otimes \gamma' \perp \mathrm{ql}(\varphi)$ since $B \otimes [x, y] \perp \pi \otimes \langle x \rangle \sim \pi \otimes \langle x \rangle$, a contradiction to the choice of γ. $\square$

As a corollary we get an analogue of [11, Cor. 6.3] for semisingular quadratic forms:

Corollary 9.8. *Let ψ be an anisotropic totally singular form of dimension n such that $2 \leq n \leq 4$. Then, we have:*
(1) $SQ(F(\psi)/F)$ is a strong 2-Pfister set for $n = 2$.
(2) $SQ(F(\psi)/F)$ is a strong 3-Pfister set for $n = 3$ or ($n = 4$ and ψ is similar to a quasi-Pfister form).
(3) $SQ(F(\psi)/F)$ is a $\{3, 4\}$-Pfister set for $n = 4$ but ψ not similar to a quasi-Pfister form.

Proof. The statements (1) and (2) are a consequence of Proposition 9.7. For (3), we take ψ' a subform of ψ of dimension 3. Since $SQ(F(\psi')/F)$ is a strong 3-Pfister set, then $SQ(F(\psi)/F)$ is a $\{3, 4\}$-Pfister set by Theorem 9.2. $\square$

9.2. **Quasi-hyperbolicity over other field extensions.** We now consider the problem that consists in giving conditions under which an anisotropic semisingular quadratic form φ represents an inseparable polynomial $p \in F[x_1, \ldots, x_n]$. The case of p given by a totally singular quadratic form is answered by Cassels-Pfister subform theorem. Namely, if an anisotropic quadratic form φ represents an irreducible polynomial $a_1x_1^2 + \cdots + a_nx_n^2 \in F[x_1, \cdots, x_n]$, then $\langle a_1, \cdots, a_n \rangle$ is dominated by φ. We wish to give a similar criteria for certain polynomials of total degree 4. The first case we study concerns the polynomial $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4$ for which we have the following characterization:

Proposition 9.9. *Let φ be an anisotropic semisingular F-quadratic form that represents the polynomial $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4$ and the form $\langle 1, a_1, \ldots, a_n\rangle$ is anisotropic. Then, we have:*

$$\langle 1, a_1, a_2, \ldots, a_{n-1}, a_n\rangle \prec \varphi.$$

Proof. We proceed by induction on n.

Step 1. Suppose $n = 1$. The form φ represents $a_1x_1^2 + x^4$, i.e., there exists $v \in V(x, x_1)$ such that $\varphi(v) = a_1x_1^2 + x^4$. By Theorem 4.1 on $V(x)$, we may assume $v \in V(x)[x_1]$. Since φ is anisotropic, we may write $v = v_0 + v_1x_1$ such that $v_0, v_1 \in V(x)$. Then, $\varphi(v) = \varphi(v_0) + B_\varphi(v_0, v_1)x_1 + \varphi(v_1)x_1^2$. So we have the following relations:

- $\varphi(v_0) = x^4$,
- $\varphi(v_1) = a_1$,
- $B_\varphi(v_0, v_1) = 0$.

We may suppose $v_1 \in V$, in fact we have two cases:
(i) If $a_1 \in D_F(\mathrm{ql}(\varphi))$, then there exists $v_1' \in \mathrm{Rad}(\varphi)$ such that $\varphi(v_1') = a_1$. Since the condition $B_\varphi(v_0, v_1') = 0$ is satisfied, we may replace v_1 by v_1'.
(ii) If $a_1 \notin D_F(\mathrm{ql}(\varphi))$. Then, the condition $a_1 = \varphi(v_1) \in D_{F(x)}(\varphi)$ implies the existence of a vector $v_1'' \in V$ such that $\varphi(v_1'') = a_1$. Since $v_1, v_1'' \notin \mathrm{Rad}(\varphi)$, we apply the Witt extension theorem (Theorem 4.3) to the spaces $W := F(x)v_1$ and $W' := F(x)v_1''$ and the isometry $\alpha : W \longrightarrow W'$ given by $\alpha(v_1) = v_1''$, to get an isometry of φ sending v_1 to v_1''. Hence, after replacing if necessary v_1 by v_1'', we may suppose $v_1 \in V$.
Now by Proposition 4.2, we may suppose $v_0 \in V[x]$. Since φ is anisotropic, we write $v_0 = w_0 + w_1x + w_2x^2$ for $w_0, w_1, w_2 \in V$. Then, $w_i \perp v_1$ for all $i \in \{1, 2, 3\}$. Moreover, we have

$$\begin{aligned} \varphi(v_0) &= x^4 \\ &= \varphi(w_0) + B_\varphi(w_0, w_1)x + \varphi(w_1)x^2 + B_\varphi(w_0, w_2)x^2 + B_\varphi(w_1, w_2)x^3 + \varphi(w_2)x^4. \end{aligned}$$

This gives us the following relations:

- $\varphi(w_2) = 1$,
- $\varphi(w_1) + B_\varphi(w_0, w_2) = 0$,
- $\varphi(w_0) = B_\varphi(w_0, w_1) = B_\varphi(w_1, w_2) = 0$.

Here, $\varphi(w_0) = 0$ implies $w_0 = 0$ since φ is anisotropic. Thus, $B_\varphi(w_0, w_2) = \varphi(w_1) = 0$. Again, $w_1 = 0$ because φ is anisotropic. So, we have $\langle 1, a_1\rangle \prec \varphi$.

Step 2. Suppose $n \geq 2$ and φ represents $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4$. By Theorem 4.1, there exist $v \in V(x, x_1, \ldots, x_{n-1})[x_n]$ such that

$$\varphi(v) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4.$$

Let us write $v = v_0 + v_nx_n$, where $v_0, v_n \in V(x, x_1, \ldots, x_{n-1})$. Then, $\varphi(v) = \varphi(v_0) + B_\varphi(v_0, v_n)x_n + \varphi(v_n)x_n^2$. So we have the following relations over $L := F(x, x_1, \cdots, x_{n-1})$:

- $\varphi(v_0) = a_1x_1^2 + a_2x_2^2 + \ldots + a_{n-1}x_{n-1}^2 + x^4$,
- $\varphi(v_n) = a_n$,
- $B_\varphi(v_0, v_n) = 0$.

As explained in **Step 1** we may reduce to the case where $v_n \in V$. Let $(Fv_n)^\perp$ be the orthogonal of the space Fv_n with respect to the polar form B_φ of φ. Moreover, the previous condition $B_\varphi(v_0, v_n) = 0$ implies that $v_0 \in (Fv_n)^\perp \otimes_F L$. Now working with the form $\varphi|_{(Fv_n)^\perp}$, we get by induction from the relation $\varphi(v_0) = a_1x_1^2 + a_2x_2^2 + \ldots + a_{n-1}x_{n-1}^2 + x^4$ that $\langle 1, a_1, a_2, \ldots, a_{n-1}\rangle \prec \varphi|_{(Fv_n)^\perp}$. We already have $\langle a_n\rangle \prec \varphi|_{Fv_n}$. Since the form $\langle 1, a_1, \ldots, a_n\rangle$ is anisotropic we get $\langle 1, a_1, a_2, \ldots, a_{n-1}, a_n\rangle \prec \varphi$. □

Likewise we give similar criteria for the polynomials $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4 + ax^2 + b$ (*resp.* $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4 + b$).

Proposition 9.10. *Let φ be an anisotropic semisingular F-quadratic form that represents the polynomial $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4 + ax^2 + b$ and the form $\langle 1, a_1, \ldots, a_n, a, b\rangle$ is anisotropic. Then, one of the following quadratic forms is dominated by φ:*

(1) $\langle 1, a_1, a_2, \ldots, a_n, a, b\rangle$.
(2) $[1, ba^{-2}] \perp \langle a_1, a_2, \ldots, a_n\rangle$.
(3) $[1; a+\alpha; b] \perp \langle a_1, a_2, \ldots, a_n\rangle \perp \langle \alpha\rangle$, *where $\alpha \in F^*$ such that $a + \alpha \neq 0$.*
(4) $[1; a + a_1p_1^2 + a_2p_2^2 + \ldots + a_np_n^2; b] \perp \langle a_1, a_2, \ldots, a_n\rangle$, *where $p_i \in F$ such that $a + a_1p_1^2 + a_2p_2^2 + \ldots + a_np_n^2 \neq 0$.*

Proof. We proceed by induction on n.

Step 1. Suppose $n = 1$. The form φ represents $a_1x_1^2 + x^4 + ax^2 + b$, i.e., there exists $v \in V(x, x_1)$ such that $\varphi(v) = a_1x_1^2 + x^4 + ax^2 + b$. By Theorem 4.1 on $V(x)$, we may assume $v \in V(x)[x_1]$. Let $v = v_0 + v_1x_1$, where $v_0, v_1 \in V(x)$. Then, $\varphi(v) = \varphi(v_0) + B_\varphi(v_0, v_1)x_1 + \varphi(v_1)x_1^2$. So we have the following relations:

- $\varphi(v_0) = x^4 + ax^2 + b$,
- $\varphi(v_1) = a_1$,
- $B_\varphi(v_0, v_1) = 0$.

We may suppose $v_1 \in V$, as explained in in **Step 1** of the proof of Proposition 9.9. Now, by Proposition 4.2, we may suppose $v_0 \in V[x]$. Since φ is anisotropic, we write $v_0 = w_0 + w_1x + w_2x^2$ for $w_0, w_1, w_2 \in V$. Then, $w_i \perp v_1$ for all $i \in \{1, 2, 3\}$. Moreover, we have

$$\begin{aligned}\varphi(v_0) &= x^4 + ax^2 + b \\ &= \varphi(w_0) + B_\varphi(w_0, w_1)x + \varphi(w_1)x^2 + B_\varphi(w_0, w_2)x^2 + B_\varphi(w_1, w_2)x^3 + \varphi(w_2)x^4.\end{aligned}$$

This gives us the following relations:

- $\varphi(w_2) = 1$,
- $\varphi(w_0) = b$,
- $\varphi(w_1) + B_\varphi(w_0, w_2) = a$,
- $B_\varphi(w_0, w_1) = B_\varphi(w_1, w_2) = 0$.

Case 1. Suppose $B_\varphi(w_0, w_2) = 0$. Thus, we have $\langle 1, a_1, a, b\rangle \prec \varphi$.
Case 2. Suppose $B_\varphi(w_0, w_2) \neq 0$ and $\varphi(w_1) = 0$. Then, $w_1 = 0$ because φ is anisotropic. Consequently, we are reduced to the relations $\varphi(w_2) = 1$, $\varphi(w_0) = b$ and $B_\varphi(w_0, w_2) = a$. It follows that $[1; a; b] \perp \langle a_1\rangle \simeq [1, ba^{-2}] \perp \langle a_1\rangle \prec \varphi$.

Case 3. Suppose $B_\varphi(w_0, w_2) \neq 0$ and $\varphi(w_1) \neq 0$. Let $\varphi(w_1) = \alpha$. We might have $a_1 = \alpha \mod F^{*2}$, in which case we have the following:

$$[1; a + a_1 p_1^2; b] \perp \langle a_1 \rangle \prec \varphi,$$

for some $p_1 \in F^*$. If $a_1 \neq \alpha \mod F^{*2}$, then we have the following:

$$[1; a + \alpha; b] \perp \langle \alpha, a_1 \rangle \prec \varphi.$$

Thus, all the cases are in accordance with our claim.

Step 2. Suppose $n \geq 2$ and φ represents $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4 + ax^2 + b$. By Theorem 4.1, there exist $v \in V(x, x_1, \ldots, x_{n-1})[x_n]$ such that

$$\varphi(v) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4 + ax^2 + b.$$

Let us write $v = v_0 + v_nx_n$, where $v_0, v_n \in V(x, x_1, \ldots, x_{n-1})$. Then, $\varphi(v) = \varphi(v_0) + B_\varphi(v_0, v_n)x_n + \varphi(v_n)x_n^2$. So we have the following relations over $L := F(x, x_1, \cdots, x_{n-1})$:

- $\varphi(v_0) = a_1x_1^2 + a_2x_2^2 + \ldots + a_{n-1}x_{n-1}^2 + x^4 + ax^2 + b$,
- $\varphi(v_n) = a_n$,
- $B_\varphi(v_0, v_n) = 0$.

As explained in **Step 1** of the proof of Proposition 9.9, we may reduce to the case where $v_n \in V$. Let $(Fv_n)^\perp$ be the orthogonal of the space Fv_n with respect to the polar form B_φ of φ. Moreover, the previous condition $B_\varphi(v_0, v_n) = 0$ implies that $v_0 \in (Fv_n)^\perp \otimes_F L$. Now working with the form $\varphi|_{(Fv_n)^\perp}$, we get by induction from the relation $\varphi(v_0) = a_1x_1^2 + a_2x_2^2 + \ldots + a_{n-1}x_{n-1}^2 + x^4 + ax^2 + b$ that one of the following conditions is satisfied:

(a) $\langle 1, a_1, a_2, \ldots, a_{n-1}, a, b \rangle \prec \varphi|_{(Fv_n)^\perp}$.
(b) $[1, ba^{-2}] \perp \langle a_1, a_2, \ldots, a_{n-1} \rangle \prec \varphi|_{(Fv_n)^\perp}$.
(c) $[1; a+\alpha; b] \perp \langle a_1, a_2, \ldots, a_{n-1} \rangle \perp \langle \alpha \rangle \prec \varphi|_{(Fv_n)^\perp}$, where $\alpha \in F^*$ such that $a + \alpha \neq 0$.
(d) $[1; a + a_1p_1^2 + a_2p_2^2 + \ldots + a_{n-1}p_{n-1}^2; b] \perp \langle a_1, a_2, \ldots, a_{n-1} \rangle \prec \varphi|_{(Fv_n)^\perp}$, where $p_i \in F$ such that $a + a_1p_1^2 + a_2p_2^2 + \ldots + a_{n-1}p_{n-1}^2 \neq 0$.

We already have $\langle a_n \rangle \prec \varphi|_{Fv_n}$. If we are in the case (a), (b) or (d), then we get the case (1), (2) or (4) of the proposition since the form $\langle 1, a_1, \ldots, a_n, a, b \rangle$ is anisotropic. If we are in the case (c) and the form $\langle a_1, a_2, \ldots, a_n \rangle \perp \langle \alpha \rangle$ is anisotropic, then we get the case (3) of the proposition, if not then $\alpha \in D_F(\langle a_1, a_2, \ldots, a_{n-1}, a_n \rangle)$, and thus we get the case (4) of the proposition. □

Following the same proof as that of Proposition 9.10, we get the following result:

Proposition 9.11. *Let φ be an anisotropic semisingular F-quadratic form that represents the polynomial $a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 + x^4 + b$ and the form $\langle 1, a_1, \ldots, a_n, b \rangle$ is anisotropic. Then, one of the following quadratic forms is dominated by φ:*

(1) $\langle 1, a_1, a_2, \ldots, a_n, b \rangle$.
(2) $[1; \alpha; b] \perp \langle a_1, a_2, \ldots, a_n \rangle \perp \langle \alpha \rangle$, *where* $\alpha \in F^*$.
(3) $[1; a_1p_1^2 + a_2p_2^2 + \ldots + a_np_n^2; b] \perp \langle a_1, a_2, \ldots, a_n \rangle$, *where* $p_i \in F$ *such that* $a_1p_1^2 + a_2p_2^2 + \ldots + a_np_n^2 \neq 0$.

Before we state other results, we recall some facts: Let $p \in F[x_1, \cdots, x_n]$ be a nonzero polynomial. We write $p = \sum_{i=1}^{n} a_i m_i$ such that $a_i \in F^*$ and m_i is a monomial for all $1 \leq i \leq n$. We suppose $m_1 > m_2 > \cdots > m_n$ with respect to the lexicographical ordering. The coefficient a_1 is called the leading coefficient of p, and we say that p is normed when $a_1 = 1$. Let π_p be the quasi-Pfister form associated to the field $F^2(a_1, \ldots, a_n)$, i.e., $\pi_p = \langle\langle a_1, \cdots, a_n \rangle\rangle_{an}$. With these notations, we recall a result by Hoffmann which deals with the quasi-hyperbolciity of totally singular quadratic form over the function field of p.

Theorem 9.12. *(*[4, Theorem 6.10]*) Let Q be an anisotropic totally singular form over F, and $p \in F[x_1, \ldots, x_n]$ be a normed irreducible polynomial. Then, the following statements are equivalent:*

(1) p is a norm of Q,
(2) Q is quasi-hyperbolic over $F(p)$,
(3) $p \in F[x_1^2, \cdots, x_n^2]$ and Q is divisible by π_p, i.e., there exists a totally singular form γ such that $Q \simeq \pi_p \otimes \gamma$.

This result is more general that Proposition 3.3 that concerns the inseparable polynomials $x_1^{2^n} + d \in F[x_1]$.

Our next goal is to discuss the quasi-hyperbolicity of semisingular quadratic forms over the function field of some irreducible polynomials and find the generator quadratic forms related to these polynomials. The first that we state concerns the polynomial $x^4 + ax^2 + b$ such that $\langle 1, a, b \rangle$ is anisotropic. This case was studied by Hoffmann and Sobiech in [7] for nonsingular forms. We adapt here their proof to the setting of semisingular forms with a slightly different argument, especially the use of the norm theorem. Moreover, reproducing this proof will help us to treat the case of the polynomial $x^4 + ay^2 + b$ (Proposition 9.14).

Proposition 9.13. *Let φ be an anisotropic semisingular F-quadratic form and $p = x^4 + ax^2 + b \in F[x]$ an irreducible polynomial such that $\langle 1, a, b \rangle$ is anisotropic. If φ is quasi-hyperbolic over $F(p)$, then $\mathrm{ql}(\varphi)$ is divisible by $\langle\langle a, b \rangle\rangle$, and φ is Witt equivalent to a semisingular quadratic form whose nonsingular part belongs to the $W(F)$-module generated by the forms:*

(F1) $[1, ba^{-2}]$,
(F2) $\langle\langle \alpha, b(\alpha + a)^{-2}]]$ for some scalar $\alpha \in F^*$ such that $a + \alpha \neq 0$,
(F3) $\langle\langle a, b, c]]$ for some scalar $c \in F$.

In particular, $SQ(F(p)/F)$ is an $\{1, 2, 3\}$-Pfister set.

Proof. Let us write $\varphi = R \perp \mathrm{ql}(\varphi)$ for a nonsingular form R. Suppose that φ is quasi-hyperbolic over $F(p)$, where $p = x^4 + ax^2 + b \in F[x]$ is an irreducible polynomial. Since $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(p)$, it follows from Theorem 9.12 that $\mathrm{ql}(\varphi) \simeq \langle\langle a, b \rangle\rangle \otimes \gamma$ for a suitable totally singular form γ.

Without loss of generality, we may suppose $1 \in D_F(R)$. We will proceed by induction on $\dim R$. Since φ is quasi-hyperbolic over $F(p)$, we get by Theorem 3.5

$$\varphi \simeq (x^4 + ax^2 + b)\varphi.$$

Hence, φ represents the polynomial $x^4 + ax^2 + b$. Using the same argument an in Step 1 of the proof of Proposition 9.10, we conclude that one of the following forms is dominated by φ:

- $[1, ba^{-2}]$.
- $[1; a+\alpha; b] \perp \langle\alpha\rangle$, where $\alpha \in F^*$ such that $a + \alpha \neq 0$.
- $\langle 1, a, b\rangle$.

Case 1. Suppose $[1, ba^{-2}] \prec \varphi$. Then ,we get

$$\varphi \simeq [1, ba^{-2}] \perp R_1 \perp \mathrm{ql}(\varphi),$$

for a nonsingular form R_1 defined over F of dimension $\dim R - 2$. In this case, we get the generator *(F1)* and we conclude by induction.

Case 2. Suppose $[1, a+\alpha, b] \perp \langle\alpha\rangle \prec \varphi$. This means that we have one of the following cases:

$$\begin{aligned}\varphi &\simeq [1; a+\alpha; b] \perp R_2 \perp \langle\alpha\rangle \perp Q_2\\ &\sim [1, b(\alpha+a)^{-2}] \perp \alpha[1, b(\alpha+a)^{-2}] \perp R_2 \perp \mathrm{ql}(\varphi),\end{aligned}$$

or

$$\begin{aligned}\varphi &\simeq [1; a+\alpha; b] \perp \alpha[1, \beta] \perp R_3 \perp \mathrm{ql}(\varphi)\\ &\sim [1, b(\alpha+a)^{-2}] \perp \alpha[1, b(\alpha+a)^{-2}] \perp \alpha[1, b(\alpha+a)^{-2} + \beta] \perp R_3 \perp \mathrm{ql}(\varphi),\end{aligned}$$

for suitable nonsingular forms R_2, R_3, and a totally singular form Q_2 such that $\mathrm{ql}(\varphi) \simeq \langle\alpha\rangle \perp Q_2$. Thus, in both cases we are reduced to the following equation:

$$\varphi \sim \langle\langle \alpha, b(\alpha+a)^{-2}]] \perp R' \perp \mathrm{ql}(\varphi),$$

for a nonsingular form R' of dimension $\dim R - 2$. In this case, we get the generator *(F2)* and we conclude by induction.

Case 3. Suppose that $\langle 1, a, b\rangle \prec \varphi$. Then, Corollary 2.7 implies that $\langle 1, a, b\rangle$ is necessary dominated by the nonsingular part of φ because $\mathrm{ql}(\varphi)$ is divisible by $\langle\langle a, b\rangle\rangle$.

Let α be a root of p in an algebraic extension of F and $\beta = \alpha^2$. Thus, β is a root of the polynomial $x^2 + ax + b$. Consider $L := F(\beta) = F(a^{-1}\beta) = F(\wp^{-1}(a^{-2}b))$. If φ_L is isotropic, then $e[1, ba^{-2}] \subset \varphi$ for some scalar $e \in F^*$ [2, Th. 4.2, page 121]. Thus, in this situation we are reduced to **Case 1**.

Hence, we may suppose that φ_L is anisotropic. We have $b = \beta^2 + a\beta$ over L, so $\langle 1, a, b\rangle_L \simeq \langle 1, a, a\beta\rangle_L \prec \varphi_L$. Hence, $\langle 1, a, \beta\rangle_L \prec a\varphi_L$. Since $\varphi_{L(\sqrt{\beta})}$ is quasi-hyperbolic (recall that $F(p) = L(\sqrt{\beta})$), it follows from Corollary 4.6 that φ_L is divisible by $\langle 1, \beta\rangle$. In particular, $a\varphi_L$ is also divisible by $\langle 1, \beta\rangle$. Now we are equipped to apply [7, Lemma 2.4][1], which gives us $\langle 1, a, \beta, a\beta\rangle \prec a\varphi_L$. In particular, $\langle 1, a, \beta, a\beta\rangle \prec \varphi_L$. Moreover, we have $\langle\langle a, \beta\rangle\rangle_L \simeq \langle\langle a, b\rangle\rangle_L$.

In conclusion, we have $\langle\langle a, b\rangle\rangle_L \prec \varphi_L$, $\langle 1, a, b\rangle \prec \varphi$ and φ_L is anisotropic.

For the rest of the proof, we discuss two cases:

(3.a) Suppose $\langle\langle a, b\rangle\rangle \prec \varphi$. Then, by Corollary 2.7, $\langle\langle a, b\rangle\rangle$ is dominated by the nonsingular part of φ. Hence, we can write:

$$\varphi \perp 3\mathbb{H} \simeq \langle\langle a, b, c]] \perp R' \perp \mathrm{ql}(\varphi),$$

[1]This result remains true for semisingular quadratic forms as we can easily check.

where $c \in F$ and R' a nonsingular form of dimension $\dim R - 2$. In this case we get the generator *(F3)* and we conclude by induction.

(3.b) Suppose $\langle\langle a, b\rangle\rangle \not\prec \varphi$. Since $\langle 1, a, b\rangle \prec \varphi$ we have

$$\varphi \perp \langle\langle a, b\rangle\rangle \simeq 3\mathbb{H} \perp \varphi_0, \tag{9.3}$$

where $\varphi_0 := R_0 \perp \langle\langle a, b\rangle\rangle \perp \mathrm{ql}(\varphi)$ is anisotropic. Recall that $\langle\langle a, b\rangle\rangle_L \prec \varphi_L$, thus extending (9.3) to L implies that $(\varphi_0)_L$ is isotropic. Thus, $\varphi_0 \simeq \lambda[1, ba^{-2}] \perp \varphi_1$ for a suitable anisotropic singular form φ_1 and a scalar $\lambda \in F^*$. We can re-write equation (9.3) as follows:

$$\varphi \perp \lambda[1, ba^{-2}] \perp \langle\langle a, b\rangle\rangle \simeq 5\mathbb{H} \perp \varphi_1.$$

This implies that $i_W(\varphi \perp \lambda[1, ba^{-2}]) \geq 1$, but since φ_L is anisotropic $i_W(\varphi \perp \lambda[1, ba^{-2}]) = 1$. Let $\psi = (\varphi \perp \lambda[1, ba^{-2}])_{an}$. Note that ψ is quasi-hyperbolic over $F(p)$ and we also have the following isometry:

$$\psi \perp \mathbb{H} \perp \langle\langle a, b\rangle\rangle \simeq 5\mathbb{H} \perp \varphi_1,$$

i.e., $\langle\langle a, b\rangle\rangle \prec \psi$. Note that the nonsingular part of ψ is of dimension $\dim R$. In this case, we are reduced to one of the previous cases and we get the desired result by induction. $\square$

Now we give a similar criteria for irreducible polynomials $x^4 + ay^2 + b \in F[x, y]$.

Proposition 9.14. *Let φ be an anisotropic semisingular F-quadratic form and $p = x^4 + ay^2 + b \in F[x, y]$ an irreducible polynomial such that $\langle 1, a, b\rangle$ is anisotropic. If φ is quasi-hyperbolic over $F(p)$, then $\mathrm{ql}(\varphi)$ is divisible by $\langle\langle a, b\rangle\rangle$, and φ is Witt equivalent to a semisingular quadratic form whose nonsingular part belongs to the $W(F)$-module generated by the forms:*

- *(G1)* $\langle\langle a, b, \beta]]$ *for some scalar $\beta \in F$,*
- *(G2)* $\langle\langle a, b(a\gamma^2)^{-2}]]$ *for some $\gamma \in F^*$.*
- *(G3)* $\langle\langle a, \alpha, b(\alpha)^{-2}]]$ *for some scalar $\alpha \in F^*$.*
- *(G4)* $\langle\langle a, \alpha + a\beta^2, b\alpha^{-2}]]$ *for some $\alpha, \beta \in F$ such that $\alpha \neq 0$ and $\alpha + a\beta^2 \neq 0$.*

In particular, $SQ(F(p)/F)$ is an $\{2, 3\}$-Pfister set.

Proof. Let us write $\varphi = R \perp \mathrm{ql}(\varphi)$ for a nonsingular form R. Let V be the underlying vector space of φ. Since $\mathrm{ql}(\varphi)$ is quasi-hyperbolic over $F(p)$, it follows from Theorem 9.12 that $\mathrm{ql}(\varphi) \simeq \langle\langle a, b\rangle\rangle \otimes \rho$ for a totally singular form ρ.

We may suppose that $1 \in D_F(R)$. We will proceed by induction on $\dim R$. Since φ is quasi-hyperbolic over $F(p)$, it follows from Theorem 3.5 that

$$\varphi \simeq (x^4 + ay^2 + b)\varphi.$$

Hence, there exists $v \in V(x, y)$ such that $\varphi(v) = x^4 + ay^2 + b$. With the help of Proposition 4.1, we can assume $v \in V(x)[y]$. Since φ is anisotropic, we may write $v = v_0 + v_1 y$ such that $v_0, v_1 \in V(x)$. Then, we get $\varphi(v) = x^4 + ay^2 + b = \varphi(v_0) + B_\varphi(v_0, v_1)y + \varphi(v_1)y^2$. So we have the following relations:

- $\varphi(v_0) = x^4 + b$,
- $\varphi(v_1) = a$,
- $B_\varphi(v_0, v_1) = 0$.

Using the Witt extension theorem (Theorem 4.3) and the same argument as in Step 1 of the proof of Proposition 9.9, we may suppose $v_1 \in V$. With the help of Theorem 4.1, we can assume $v_0 \in V[x]$. Let $w_0, w_1, w_2 \in V$ be such that $v_0 = w_0 + w_1 x + w_2 x^2$. Then

$$\begin{aligned}\varphi(v_0) &= x^4 + b \\ &= \varphi(w_0) + B_\varphi(w_0, w_1)x + \varphi(w_1)x^2 + B_\varphi(w_0, w_2)x^2 + B_\varphi(w_1, w_2)x^3 + \varphi(w_2)x^4\end{aligned}$$

So we have the following relations:

- $\varphi(w_0) = b$,
- $\varphi(w_2) = 1$,
- $\varphi(w_1) = B_\varphi(w_0, w_2)$,
- $B_\varphi(w_0, w_1) = B_\varphi(w_1, w_2) = 0$.

Moreover the condition $B_\varphi(v_0, v_1) = 0$ gives us $B_\varphi(w_i, v_1) = 0$ for all $i \in \{0, 1, 2\}$.

Case 1. Suppose $w_0 \perp w_2$, then $\varphi(w_1) = 0$ and thus $w_1 = 0$ because φ is anisotropic. In this case, we are just reduced to the relations: $\varphi(w_0) = b$, $\varphi(w_2) = 1$, $\varphi(v_1) = a$ and the vectors v_1, w_0, w_2 are pairwise orthogonal. Hence, $\langle 1, a, b\rangle \prec \varphi$. Moreover, by Corollary 2.7, the form $\langle 1, a, b\rangle$ is necessary dominated by the nonsingular part of φ.

We have $F(p) = F(u)(\sqrt{a^{-1}(u^4 + b)})$ such that u is an indeterminate over F. Hence, $\varphi_{F(u)}$ is anisotropic. Let $v = \sqrt{a^{-1}(u^4 + b)}$. Clearly, $\langle 1, a, av^2\rangle \simeq \langle 1, a, b\rangle_{F(u)}$, and thus $\langle 1, a, av^2\rangle \prec \varphi_{F(u)}$. Moreover, since p is a norm of $\varphi_{F(x,y)}$, we get by a specialization argument that $\varphi_{F(u)} \simeq a\varphi_{F(u)}$. Hence, $\langle 1, a, v^2\rangle \prec \varphi_{F(u)}$. Moreover, the quasi-hyperbolicity of $\varphi_{F(p)}$ implies that $\varphi_{F(u)}$ is divisible by $\langle 1, v^2\rangle$ (Corollary 4.6). Now using [7, Lemma 2.4], we get $\langle\langle a, v^2\rangle\rangle \prec \varphi_{F(u)}$. Moreover, $\langle\langle a, b\rangle\rangle_{F(u)} \simeq \langle\langle a, v^2\rangle\rangle$. Hence, $\langle\langle a, b\rangle\rangle_{F(u)} \prec \varphi_{F(u)}$. Since $F(u)/F$ is purely transcendental, it follows that $\langle\langle a, b\rangle\rangle \prec \varphi$. As in the subcase (3.a) in the proof of Proposition 9.13, we obtain

$$\varphi \sim \langle\langle a, b, \beta]] \perp R_1 \perp \mathrm{ql}(\varphi),$$

for some scalar $\beta \in F$ and a nonsingular form R_1 of dimension $< \dim R$. Thus, we get the generator *(G1)* and we complete the proof by induction.

Case 2. Suppose $w_0 \not\perp w_2$, and let $\alpha := B_\varphi(w_0, w_2) = \varphi(w_1)$. In this case, we are just reduced to the relations: $\varphi(w_0) = b$, $\varphi(w_2) = 1$, $B_\varphi(w_0, w_2) = \alpha = \varphi(w_1)$, $\varphi(v_1) = a$ and $B_\varphi(w_i, v_1) = 0$ for all $i \in \{0, 1, 2\}$. We then have two scenarios:

(1) Suppose $\alpha = a \mod F^{*2}$. Then, there exists $\gamma \in F^{*2}$ such that $\alpha = a\gamma^2$, and the vectors v_1 and w_1 are linearly dependent. Let W be the subspace of V generated by w_0, w_1 and w_2. Hence, $\varphi|_W \simeq [1; a\gamma^2; b] \perp \langle a\rangle \prec \varphi$. Observe that $a \notin D_F(\mathrm{ql}(\varphi))$ by Corollary 2.7, and thus

$$\varphi \simeq [1, a\gamma^2, b] \perp a[1, \delta] \perp R_2 \perp \mathrm{ql}(\varphi),$$

for some scalar $\delta \in F$ and a nonsingular form R_2 of dimension $\dim R - 4$. Then

$$\varphi \sim \langle\langle a, b(a\gamma^2)^{-2}]] \perp R_2' \perp \mathrm{ql}(\varphi),$$

where $R_2' = a[1, \delta + b(a\gamma^2)^{-2}] \perp R_2$. Hence, we get the generator *(G2)*. Since $\dim R_2' < \dim R$, we conclude by induction.

(2) Suppose $\alpha \neq a \mod F^{*2}$. Then, w_0, w_1, w_2 and v_1 are linearly independent. In this case, we have $[1, \alpha, b] \perp \langle \alpha, a \rangle \prec \varphi$. We consider two cases:

(2.a) Suppose $D_F^0(\langle a, \alpha \rangle) \cap D_F^0(\mathrm{ql}(\varphi)) = \{0\}$. Then, the condition $[1; \alpha; b] \perp \langle \alpha, a \rangle \prec \varphi$ implies the following

$$\varphi \simeq [1, b\alpha^{-2}] \perp a[1, k] \perp \alpha[1, l] \perp R_1 \perp \mathrm{ql}(\varphi), \tag{9.4}$$

for suitable $k, l \in F$ and a nonsingular form R_1 of dimension $\dim R - 6$. We add a hyperbolic plane to equation (9.4) to get:

$$\varphi \perp \mathbb{H} \simeq \langle 1, a \rangle_b \otimes [1, b\alpha^{-2}] \perp a[1, k + b\alpha^{-2}] \perp \alpha[1, l] \perp R_1 \perp \mathrm{ql}(\varphi). \tag{9.5}$$

Recall that φ has $x^4 + ay^2 + b$ as a norm over $F(x, y)$. In particular, substituting x to 0, we get $\varphi \simeq (ay^2 + b)\varphi$ over $F(y)$ [15, Proposition 5.3]. Moreover, the polynomial $ay^2 + b$ is a norm of $\langle 1, a \rangle_b \otimes [1, b\alpha^{-2}]$ as it is represented by this form. Hence, $ay^2 + b$ is also a norm of

$$\varphi' := (a[1, k + b\alpha^{-2}] \perp \alpha[1, l] \perp R_1 \perp \mathrm{ql}(\varphi))_{an}.$$

In particular, $\alpha(ay^2+b)$ is represented by φ' over $F(y)$. Using Theorem 4.1 as before, we deduce that $\alpha \langle a, b \rangle \prec \varphi'$ (because φ' is anisotropic). Note that $D_F^0(\alpha \langle a, b \rangle) \cap D_F^0(\mathrm{ql}(\varphi)) = \{0\}$, otherwise α would be represented by $\mathrm{ql}(\varphi)$ as $\langle\langle a, b \rangle\rangle$ divides $\mathrm{ql}(\varphi)$, and thus φ would be isotropic. Hence, we get

$$a[1, k + b\alpha^{-2}] \perp \alpha[1, l] \perp R_1 \perp \mathrm{ql}(\varphi) \simeq a\alpha[1, s_1] \perp b\alpha[1, s_2] \perp R_2 \perp \mathrm{ql}(\varphi) \tag{9.6}$$

for some scalars $s_1, s_2 \in F$ and a nonsingular form R_2. Consequently, we combine (9.6) with (9.5) to get:

$$\varphi \perp \mathbb{H} \simeq \langle 1, a \rangle_b \otimes [1, b\alpha^{-2}] \perp a\alpha[1, s_1] \perp b\alpha[1, s_2] \perp R_2 \perp \mathrm{ql}(\varphi), \tag{9.7}$$

Adding $2\mathbb{H}$ to equation (9.7) yields:

$$\begin{aligned} \varphi \perp 3\mathbb{H} \simeq \langle 1, a \rangle_b \otimes [1, b\alpha^{-2}] \perp a\alpha[1, b\alpha^{-2}] \perp b\alpha[1, b\alpha^{-2}] \\ \perp a\alpha[1, s_1 + b\alpha^{-2}] \perp b\alpha[1, s_2 + b\alpha^{-2}] \perp R_2 \perp \mathrm{ql}(\varphi). \end{aligned}$$

Since $b\alpha[1, b\alpha^{-2}] \simeq \alpha[1, b\alpha^{-2}]$, we get

$$\varphi \perp 3\mathbb{H} \simeq \langle\langle a, \alpha, b\alpha^{-2}]] \perp R_2' \perp \mathrm{ql}(\varphi),$$

where $R_2' = a\alpha[1, s_1 + b\alpha^{-2}] \perp b\alpha[1, s_2 + b\alpha^{-2}] \perp R_2$. In this case we get the generator *(G3)* and we conclude by induction as $\dim R_2' = \dim R - 2$.

(2.b) Suppose $D_F^0(\langle a, \alpha \rangle) \cap D_F^0(\mathrm{ql}(\varphi)) \neq \{0\}$. Hence, with the condition $[1; \alpha; b] \perp \langle \alpha, a \rangle \prec \varphi$, there exist $e, f, g \in F^*$ such that $\langle a, \alpha \rangle \simeq \langle e, f \rangle$ and

$$\varphi \simeq [1; \alpha; b] \perp [e, g] \perp R_1 \perp \mathrm{ql}(\varphi),$$

where $f \in D_F(\mathrm{ql}(\varphi))$ and R_1 is a nonsingular form of dimension $\dim R - 4$. Since $a \notin D_F(\mathrm{ql}(\varphi))$, we have $a = ex^2 + fy^2$ for suitable $x, y \in F$ and $x \neq 0$. Using the isometry $[r, s] \perp \langle t \rangle \simeq [r + t, s] \perp \langle t \rangle$, we may suppose $e = a$. Hence

$$\varphi \perp \mathbb{H} \simeq \langle 1, a \rangle_b \otimes [1, b\alpha^{-2}] \perp a[1, ag + b\alpha^{-2}] \perp R_1 \perp \mathrm{ql}(\varphi). \tag{9.8}$$

Since $\mathrm{ql}(\varphi)$ is divisible by $\langle\langle a, b\rangle\rangle$, the polynomial $ay^2 + b$ is a norm of $\mathrm{ql}(\varphi)_{F(y)}$. Consequently, $\mathrm{ql}(\varphi)_{F(y)}$ represents $(ay^2 + b)f$, and thus $\langle af, bf\rangle$ is a subform of $\mathrm{ql}(\varphi)$. Note that $bf[1, b\alpha^{-2}] \simeq f[1, b\alpha^{-2}]$. Hence, equation (9.8) can be written as follows:

$$\varphi \perp 3\mathbb{H} \simeq \langle 1, a\rangle_b \otimes [1, b\alpha^{-2}] \perp f[1, b\alpha^{-2}] \perp af[1, b\alpha^{-2}] \perp a[1, ag + b\alpha^{-2}] \perp R_1 \perp \mathrm{ql}(\varphi).$$

Consequently, we get

$$\varphi \perp 3\mathbb{H} \simeq \langle\langle a, f, b\alpha^{-2}]] \perp R_2 \perp \mathrm{ql}(\varphi),$$

where $R_2 = a[1, ag + b\alpha^{-2}] \perp R_1$. Moreover, modulo a square, we may suppose $f = \alpha + a\beta^2$ for some $\beta \in F$ (because $\langle a, \alpha\rangle \simeq \langle e, f\rangle$ and $a \notin D_F(\mathrm{ql}(\varphi))$). Hence, in this case we get the generator *(G4)*. Since R_2 is of dimension $\dim R - 2$, we conclude by induction. □

A similar characterization could also be given for quadratic forms quasi-hyperbolic over the function field of an irreducible polynomial $x^4 + ay^4 + bx^2 \in F[x, y]$ such that $\langle 1, a, b\rangle$ is anisotropic. In this case, we are reduced to Proposition 9.14, by change of indeterminates $X = xy^{-1}$ and $Y = xy^{-2}$.

Remark 9.15. *Note that in Proposition 9.13 and 9.14, we were able to take shelter of technical calculations to get the desired result since, we are dealing with polynomials of small degree. With the increase in degree of field extension, we need a more complex arguments in the induction process as we did in Theorems 7.1 and 5.1.*

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