Independence in Qualitative Uncertainty Frameworks

Nahla Benamor, Salem Benferhat, Didier Dubois, Hector Geffner, Henri Prade

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N. Ben Amor
ISG Tunis*
nahla.benamor@ihec.rnu.tn

S. Benferhat
I.R.I.T[†]
benferhat@irit.fr

D. Dubois
I.R.I.T
dubois@irit.fr

H. Geffner
USB[‡]
hector@usb.ve

H. Prade
I.R.I.T
prade@irit.fr

Abstract

The notion of independence is central in many information processing areas, such as multiple criteria decision making, databases organization, or uncertain reasoning. This is especially true in the later case where the success of Bayesian networks is basically due to the graphical representation of independence they provide. This paper first studies qualitative independence relations when uncertainty is encoded by a complete pre-order between states of the world. While a lot of work has focused on the formulation of suitable definitions of independence in different qualitative uncertainty frameworks, our interest in this paper is rather to formulate a general definition of independence based on pure ordering considerations, and that applies to all qualitative uncertainty frameworks. The second part of the paper investigates the impact of the embedding of qualitative independence relations into qualitative uncertainty frameworks such as possibility theory, or Spohn functions. The absolute scale used for grading uncertainty in these settings enforces the commensurateness between local pre-orders (since they share the same scale). This leads to an easy decomposability property of the joint distributions into more elementary relations on the basis of the independence relations.

*Institut Supérieur de Gestion Tunis.
†Institut de Recherche en Informatique de Toulouse, France.
‡Universidad Simon Bolivar, Caracas, Venezuela.

1 Introduction

Independence relations between variables play an important role in the handling of uncertain information. Indeed various notions of (in)dependence are central in multiple criteria analysis, in data decomposition, or in uncertain reasoning based on Bayesian networks or logical reasoning. There has been a considerable interest in artificial intelligence in the last few years for discussing independence in various representation frameworks.

Bayesian independence and database functional dependencies have been formally related (Wong et al. 1995). Independence in the logical setting has received a lot of attention, e.g., (Darwiche 1997), (Lakemeyer 1997), (Lang and Marquis 1998). In this paper we rather investigate possible definitions of independence in qualitative settings, using qualitative uncertainty relations, or their graded counterparts such as possibility measures for instance.

¿From an operational point of view, two forms of independence can be distinguished:

- *decomposition independence* which allows the decomposition of a joint distribution pertaining to tuples of variables into local distributions on smaller subsets of variables, in order to have a reasoning machinery working at a local level without losing any information.

- *causal independence* for expressing the absence of causality. This form of independence is always characterized in semantic terms, e.g. two variables (or sets of variables) are said to be independent if our belief in the value of one of them does not change when learning something about the value of the other.

These two kinds of independence are not necessarily mutually exclusive. Ideally, a good definition of independence both expresses the lack of causality and is useful for computations.

In Section 2, we first present a general qualitative uncertainty framework where uncertainty is represented by total pre-orders on different subsets of situations. Then, in Section 3, we study several definitions of qualitative independence (causal and decomposition ones). A noticeable causal qualitative independence is proposed. This definition extends the one proposed by Darwiche (Darwiche 1997) to qualitative uncertainty frameworks. Finally, we discuss the advantages of representing a total pre-order with ranking functions. In Section 5 after providing a background on ranking functions frameworks, we show that this is crucial for decomposing joint distributions on the basis of independence relations. Indeed, some qualitative uncertainty relations ordered with respect to well-known principles (like leximin or leximax ordering) cannot be decomposable in the qualitative setting while they become decomposable with the help of ranking functions which make these relations commensurable. Lastly, in Section 6, we provide a comparative study between existing possibilistic independence and the qualitative independence relations proposed in this paper.

2 Qualitative uncertainty framework

2.1 Notations

Let $U = \{A, B, C, ...\}$ be the set of variables. We denote by $D_A = \{a_1, ..., a_n\}$ the supposedly finite domain associated to the variable A. By a we denote any instance of A. $X, Y, Z, ...$ denote disjoint subsets of variables in U, and $D_X = \{x_1, x_2, ..., x_m\}$ represents the Cartesian product of variable domains in X. By x we denote any instance of X. Ω denotes the universe of discourse, which is the Cartesian product of all variable domains in U. Each element of Ω is called interpretation, situation or state of the world. ϕ, ψ denote the subsets of Ω, called formula (or event) and $\neg\phi$ denotes the complementary set of ϕ i.e. $\neg\phi = \Omega - \phi$.

2.2 Representation of uncertain information

In the following, we give a formal description of the qualitative representation of uncertainty we are using. The basic idea is to represent the available incomplete information of the real world by a **total pre-order** [1], on Ω. This total pre-order called a *qualitative plausibility relation*, will be denoted by $\geq_\Omega$. The relation $\omega_1 \geq_\Omega \omega_2$ means that ω_1 is more plausible than ω_2. We denote $=_\Omega$ and $>_\Omega$ respectively the equality and the strict inequality relations associated with $\geq_\Omega$.

[1] a relation $\geq_\Omega$ on Ω is a total pre-order if $\geq_\Omega$ is reflexive, transitive and complete, i.e., for all ω_1, ω_2, we have: $\omega_1 \geq_\Omega \omega_2$ or $\omega_2 \geq_\Omega \omega_1$.

Given a relation $\geq_\Omega$ on Ω, we can lift it to another plausibility relation defined on the subsets of Ω (for the sake of simplicity, we use the same notation $\geq_\Omega$) by (e.g., (Dubois 1986)): $\phi \geq_\Omega \psi$ iff $\forall\omega \in \psi, \exists\omega' \in \phi$ such that $\omega' \geq_\Omega \omega$. Namely, $\phi \geq_\Omega \psi$ holds if the best element in ϕ is preferred to the best element(s) in ψ.

This qualitative representation of uncertainty is used in several non-monotonic formalisms like Lehmann's ranked model (Lehmann 1989), plausibility relations (Halpern 1997), Spohn's ordinal conditional functions (Spohn 1988) and possibility theory (Dubois and Prade 1988).

2.3 Qualitative conditioning

Conditioning is a crucial notion in studying independence relations. In the qualitative setting, it consists in transforming a plausibility relation $\geq_\Omega$ on the basis of a new information $\phi \subseteq \Omega$ into a new plausibility relation denoted by $\geq_{\Omega|\phi}$. This new relation is obtained by applying the three following postulates

$\mathbf{A}_1$ **:** $\forall\omega_1, \omega_2 \in \phi, \omega_1 >_\Omega \omega_2$ iff $\omega_1 >_{\Omega|\phi} \omega_2$,

$\mathbf{A}_2$: $\forall\omega_1 \in \phi, \forall\omega_2 \notin \phi, \omega_1 >_{\Omega|\phi} \omega_2$,

$\mathbf{A}_3$: $\forall\omega_1, \omega_2 \notin \phi, \omega_1 =_{\Omega|\phi} \omega_2$.

$\mathbf{A}_1$ means that the new plausibility relation should not alter the initial order between the elements of ϕ. $\mathbf{A}_2$ confirms that each interpretation of ϕ should be preferred to any interpretation not belonging to ϕ. Finally, the last postulate $\mathbf{A}_3$ says that the elements not belonging to ϕ are in the same equivalence class. These three postulates determine the new plausibility relation $\geq_{\Omega|\phi}$ in a **unique** manner.

2.4 Accepted beliefs

We now introduce a further definition which will be helpful in easily defining the notion of qualitative independence, namely the notion of accepted belief, e.g., (Dubois and Prade 1995), (Dubois et al., 1996), (Halpern 1996).

Definition 1 *An acceptance function associated to a plausibility relation* $\geq_\Omega$ *is a function, denoted by*

$\mathbf{Acc}_{\geq_\Omega}$*(.), which assigns to each* ϕ *a value in* $\{-1, 0, 1\}$ *in the following way:*

$$\mathbf{Acc}_{\geq_\Omega}(\phi) = \begin{cases} 1 & if\ \phi >_\Omega \neg\phi \\ 0 & if\ \phi =_\Omega \neg\phi \\ -1 & if\ \neg\phi >_\Omega \phi. \end{cases}$$

When $\mathbf{Acc}_{\geq_\Omega}(\phi) = 1$ (resp. $\mathbf{Acc}_{\geq_\Omega}(\phi) = -1$) we say that ϕ is accepted (resp. rejected). $\mathbf{Acc}_{\geq_\Omega}(\phi) = \mathbf{Acc}_{\geq_\Omega}(\neg\phi) = 0$, corresponds to the situation of total ignorance concerning ϕ, i.e., ϕ and $\neg\phi$ are equally plausible.

The function $\mathbf{Acc}_{\geq_\Omega}$ can be extended in order to take into account a given context. Then a conditional belief measure denoted by $\mathbf{Acc}_{\geq_\Omega}(.|.)$ is defined by

$$\mathbf{Acc}_{\geq_\Omega}(\phi \mid \psi) = \begin{cases} 1 & if\ \phi \cap \psi >_\Omega \neg\phi \cap \psi \\ 0 & if\ \phi \cap \psi =_\Omega \neg\phi \cap \psi \\ -1 & if\ \neg\phi \cap \psi >_\Omega \phi \cap \psi. \end{cases}$$

Remarks:

- The plausibility relation $\geq_\Omega$ determines in a unique manner $\mathbf{Acc}_{\geq_\Omega}$. The converse is not true. Namely, many plausibility relations can generate a same set of **plain beliefs**, i.e, can have the same $\mathbf{Acc}_{\geq_\Omega}$ on all formulas(including the interpretations).

  Counter-example : Let us consider the following values of $\mathbf{Acc}_{\geq_\Omega}$ relative to the two binary variables A and B:
  $\mathbf{Acc}_{\geq_\Omega}(a_1) = \mathbf{Acc}_{\geq_\Omega}(b_1) = 1$,
  $\mathbf{Acc}_{\geq_\Omega}(a_2) = \mathbf{Acc}_{\geq_\Omega}(b_2) = -1$,
  $\mathbf{Acc}_{\geq_\Omega}(a_1 \vee b_1) = 1, \mathbf{Acc}_{\geq_\Omega}(a_2 \vee b_1) = 1$,
  $\mathbf{Acc}_{\geq_\Omega}(a_1 \vee b_2) = 1, \mathbf{Acc}_{\geq_\Omega}(a_2 \vee b_2) = -1$,
  $\mathbf{Acc}_{\geq_\Omega}(a_2 \wedge b_1) = -1, \mathbf{Acc}_{\geq_\Omega}(a_2 \wedge b_2) = -1$,
  $\mathbf{Acc}_{\geq_\Omega}(a_1 \wedge b_2) = -1, \mathbf{Acc}_{\geq_\Omega}(a_1 \wedge b_1) = 1$.

  We can check that the two following plausibility relations:
  $a_1 \wedge b_1 >_{\Omega_1} a_2 \wedge b_1 >_{\Omega_1} a_1 \wedge b_2 =_{\Omega_1} a_2 \wedge b_2$, and
  $a_1 \wedge b_1 >_{\Omega_2} a_2 \wedge b_1 >_{\Omega_2} a_1 \wedge b_2 >_{\Omega_2} a_2 \wedge b_2$
  generate the same information on the accepted beliefs then those given above i.e:

  $$\mathbf{Acc}_{\geq_{\Omega_1}} = \mathbf{Acc}_{\geq_{\Omega_2}} = \mathbf{Acc}_{\geq_\Omega}.$$

- The set of all **conditional beliefs** determines in a unique manner a plausibility relation on Ω constructed in this way:

  $$\omega_1 >_\Omega \omega_2 \text{ iff } \mathbf{Acc}_{\geq_\Omega}(\{\omega_1\} \mid \{\omega_1, \omega_2\}) = 1.$$

- The measures $\mathbf{Acc}_{\geq_\Omega}(.)$ and $\mathbf{Acc}_{\geq_\Omega}(.|.)$ are linked by the following equation

  $$\mathbf{Acc}_{\geq_\Omega}(\phi \wedge \psi) = min(\mathbf{Acc}_{\geq_\Omega}(\phi \mid \psi), \mathbf{Acc}_{\geq_\Omega}(\psi))$$

  which is similar to Bayes conditioning.

- $\mathbf{Acc}_{\geq_\Omega}(. \mid \phi)$ can be defined from $\mathbf{Acc}_{\geq_{\Omega|\phi}}$ in the following way:

  $$\forall\psi, \mathbf{Acc}_{\geq_\Omega}(\psi \mid \phi) = \mathbf{Acc}_{\geq_{\Omega|\phi}}(\psi).$$

In the following, we use $\mathbf{Acc}(.)$ (resp. $\mathbf{Acc}(.|\ .)$) instead of $\mathbf{Acc}_{\geq_\Omega}(.)$ (resp. $\mathbf{Acc}_{\geq_\Omega}(.|\ .)$) when there is no ambiguity.

3 Qualitative independence

3.1 In search of causal qualitative independence

Independence can be thought of either in terms of qualitative plausibility relations or in terms of acceptance measures. The two views can be related, as shown in this section where we present three possible definitions of causal independence. Basically, two sets of variables X and Y are declared to be independent if learning any instance of Y:

- preserves the preferred (or top) instances of X, or
- preserves the accepted (resp. rejected and ignored) instances of X, or
- preserves the relative ordering between instances of X.

3.1.1 Preserving preferred instances

The first idea is to consider a variable set X as independent of Y in the context Z if for all instance z of Z, the acceptance of any instance $(x \wedge y)$ of (X, Y) is fully determined by the separate acceptance of x and y. In particular, if x and y are accepted, then $(x \wedge y)$ is accepted. One way to relate the acceptance of $(x \wedge y)$ to the acceptance of x and the acceptance of y is:

$$\mathbf{Acc}(x \wedge y \mid z) = min(\mathbf{Acc}(x \mid z), \mathbf{Acc}(y \mid z)), \forall xyz. \quad (1)$$

It can be checked that this definition only cares about the preservation of the *top elements* (i.e. best elements) in X an Y in any context z of Z. In other terms, for any plausible instance x of X (i.e., $\mathrm{Acc}(x) = 1$) and for any plausible instance y of Y, $(x \wedge y)$ should be a plausible element of (X, Y). In the following, independence relations satisfying the equation (1) are called **PT**-independence (PT for Preserving Top elements), and are denoted by I_{PT}(X,Z,Y).

This is clearly a very weak definition of independence where generally the acceptance of one instance of X or of Y is enough to conclude the independence between these two variable sets. In particular, if a plausibility relation $\geq_\Omega$ contains exactly one preferred element then all variables are pairwise PT-independent.

3.1.2 Preserving accepted beliefs

One way of making the above definition stronger is to consider conditional beliefs. Namely, a variable set X is considered as independent of Y in the context Z, if for all instance z of Z the following condition is respected: any instance x of X is accepted (resp. rejected, ignored) in the context z and remains accepted (resp. rejected, ignored) in this context after knowing any instance y of Y. In other words, believing in X does not change when Y is learned. More formally X and Y are said to be **PB**-independent (PB for Preserving Beliefs) in the context Z, denoted by I_{PB}(X,Z,Y), if:

$$(i)\ \mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z) \text{ and}$$
$$(ii)\ \mathbf{Acc}(y \mid x \wedge z) = \mathbf{Acc}(y \mid z), \forall xyz. \quad (2)$$

Note that contrarly to the situation in the probability theory, (i) and (ii) are not equivalent.

Counter-example : Consider two binary variables A and B with the following plausibility relation:

$$a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_1$$

The relation (i) is true since $\mathbf{Acc}(a \mid b) = \mathbf{Acc}(a), \forall ab$, but (ii) is false since $\mathbf{Acc}(b_1) = 1 \neq \mathbf{Acc}(b_1 \mid a_2) = -1$.

In the case when the plausibility relation only contains two equivalent classes (which can, for instance, correspond to the models and counter-models of some classical databases), then this definition is equivalent to the so-called *logical conditional independence* (LCI) proposed by Darwiche (Darwiche 1997).

This definition of independence preserves the *acceptance* of instances of X in the context of Y but does not preserve the relative *ordering* between instances of X (except when we consider binary variables).

For instance let A and B be two independent variables (according to the previous definition) such that $D_A = \{a_1, a_2, a_3\}$ and $D_B = \{b_1, b_2\}$ with the following plausibility relation: $a_1 \wedge b_1 >_\Omega a_2 \wedge b_1 >_\Omega a_3 \wedge b_1 >_\Omega a_1 \wedge b_2 >_\Omega a_2 \wedge b_2 =_\Omega a_3 \wedge b_2$

It can be checked that the relative ordering between instances of A is not preserved in context of B. Indeed, $a_1 >_\Omega a_2 >_\Omega a_3$ but $a_1 >_\Omega a_2 =_\Omega a_3$ in the context b_2.

3.1.3 Preserving the relative ordering

The definition, we propose now, simply says that two variable sets X and Y are independent in the context of Z, if for all z, the preferential ordering between the different values of X (resp. Y) is preserved after the revision by any value y of Y (resp. x of X). More formally:

Definition 2 *Let X, Y and Z be three disjoint subsets of U. X and Y are said to be PO-independent (PO for Preserving Ordering) in the context Z, denoted $I_{PO}(X,Z,Y)$, if $\forall z \in D_Z, y \in D_Y, x \in D_X$:*
(i) $\forall x_i, x_j \in D_X, x_i >_{\Omega|z} x_j$ iff $x_i >_{\Omega|y \wedge z} x_j$, and

$$(ii)\ \forall y_k, y_l \in D_Y, y_k >_{\Omega|z} y_l \text{ iff } y_k >_{\Omega|x \wedge z} y_l. \quad (3)$$

The following proposition rewrites PO-independence relations in terms of **Acc**.

Proposition 1 *X and Y are PO-independent in the context Z iff $\forall D_{X'} \subseteq D_X, \forall D_{Y'} \subseteq D_Y$ such that $D_{X'} \neq \emptyset$ and $D_{Y'} \neq \emptyset$ and $\forall xyz$:*

$$\mathbf{Acc}(x \wedge y \mid z, D_{X'}, D_{Y'}) =$$
$$min(\mathbf{Acc}(x \mid z, D_{X'}), \mathbf{Acc}(y \mid z, D_{Y'})). \quad (4)$$

As a corollary of this rewriting, we deduce:

Proposition 2 *If X and Y are PO-independent in the context Z, then they are PT-independent and PB-independent.*

Indeed equation (4) implies (1), since it is enough to let $D_{X'} = \Omega$ and $D_{Y'} = \Omega$ in (4) to obtain (1). It also implies (2) by letting $D_{X'} = \Omega$ and $D_{Y'} = \{y\}$ which leads to (i) of (2) and $D_{X'} = \{x\}$ and $D_{Y'} = \Omega$ which leads to (ii) of (2). Clearly, the converse of the above proposition, in general, does not hold. However, when we only restrict to binary variables then PO-independence and PB-independence are equivalent.

Lastly, we can check that PO-independence is equivalent to the independence relation based on *Ceteris Paribus* (all else being equal) principle used in (Boutilier et al. 1999) (Dolye and Wellman 1991). This principle is closely related to preferential independence in multiple criteria analysis, see, e.g.,(Bacchus and Grove 1996).

3.2 decomposition independence based on remarkable plausibility relations

A natural way of defining decomposition independence relations is to analyze the structure of the plausibility relation $\geq_\Omega$. A plausibility relation is said to be decomposable w.r.t. X and Y in the context Z, iff $\geq_\Omega$ is a function of the local orderings $\geq_{X \cup Z}$ and $\geq_{Y \cup Z}$.

The following introduces a well known principle, called *Pareto-principle*, which defines a partial oredering between pairs $(x_i \wedge z, y_j \wedge z)$:

Definition 3 *Let X, Y, Z three disjoint sets of U. Then the pair $(x_i \wedge z, y_k \wedge z)$ is said to be Pareto-preferred to $(x_j \wedge z, y_l \wedge z)$, denoted by $(x_i \wedge z, y_k \wedge z) \geq_P (x_j \wedge z, y_l \wedge z)$, if and only if $x_i \wedge z \geq_\Omega x_j \wedge z$ and $y_k \wedge z \geq_\Omega y_l \wedge z$*

In general $\geq_P$ is only a **partial** order. Since this paper deals with plausibility relations which are total pre-order, the following introduces a general class of plausibility relations which are compatible with the Pareto-principle:

Definition 4 *Let X, Y and Z be disjoint subsets of U. A plausibility relation $\geq_\Omega$ is said to be Pareto-compatible (or* **monotonic***) on X and Y in the context Z if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have: $(x_i \wedge z, y_k \wedge z) \geq_P (x_j \wedge z, y_l \wedge z)$ implies $(x_i \wedge y_k \wedge z) \geq_\Omega (x_j \wedge y_l \wedge z)$*

Well known orderings $>_\Omega$ used in the qualitative setting, which are Pareto-compatible, Pareto-ordering, *leximin* and *leximax* orderings that we briefly present now (Moulin, 1988).

1. A plausibility relation $\geq_\Omega$ is said to be **Pareto-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:
   $x_i \wedge y_k \wedge z \geq_\Omega x_j \wedge y_l \wedge z$ iff
   $x_i \wedge z \geq_\Omega x_j \wedge z$ and $y_k \wedge z \geq_\Omega y_l \wedge z$.

   As we will see later, this definition is very strong since it implies that one of the local distributions on X or Y should be uniform. One can weaken this definition in the following way: $\geq_\Omega$ is said to be **strict Pareto-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:
   $x_i \wedge y_k \wedge z >_\Omega x_j \wedge y_l \wedge z$ iff
   (i) $x_i \wedge z \geq_\Omega x_j \wedge z$ and $y_k \wedge z \geq_\Omega y_l \wedge z$, and
   (ii) $x_i \wedge z >_\Omega x_j \wedge z$ or $y_k \wedge z >_\Omega y_l \wedge z$.

   An example of plausibility relation which is strict Pareto-decomposable but not Pareto-decomposable is the following one defined on two binary variables A and B:
   $$a_1 b_1 >_\Omega a_1 b_2 = a_2 b_1 >_\Omega a_2 b_2.$$

2. A plausibility relation $\geq_\Omega$ is said to be **leximin-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:

   $x_i \wedge y_k \wedge z >_\Omega x_j \wedge y_l \wedge z$ iff
   (i) $min(x_i \wedge z, y_k \wedge z) >_\Omega min(x_j \wedge z, y_l \wedge z)$ or
   (ii) $min(x_i \wedge z, y_k \wedge z) =_\Omega min(x_j \wedge z, y_l \wedge z)$ and $max(x_i \wedge z, y_k \wedge z) >_\Omega max(x_j \wedge z, y_l \wedge z)$.[2]

3. A plausibility relation $\geq_\Omega$ is said to be **leximax-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:

   $x_i \wedge y_k \wedge z >_\Omega x_j \wedge y_l \wedge z$ iff
   (i) $max(x_i \wedge z, y_k \wedge z) >_\Omega max(x_j \wedge z, y_l \wedge z)$ or
   (ii) $max(x_i \wedge z, y_k \wedge z) =_\Omega max(x_j \wedge z, y_l \wedge z)$ and $min(x_i \wedge z, y_k \wedge z) >_\Omega min(x_j \wedge z, y_l \wedge z)$.

Definition 5 *X and Y are said to be* ***Pareto-independent*** *(resp.* ***leximin-independent, leximax-independent****) in the context Z if the plausibility relation $\geq_\Omega$ is Pareto-decomposable (resp. leximin-decomposable, leximax-decomposable) on X and Y in the context Z.*

Proposition 3 *If X and Y are Pareto-independent in the context Z, then they are leximin-independent and leximax-independent. Moreover, all of these relations imply the PO-independence. The converse is not true.*

As it will be discussed in Section 5.2.3, even if a plausibility relation is leximin or leximax decomposable, it cannot be decomposed without loss of information due to the absence of commensurability assumption. Such a problem can be solved by using ranking functions which are introduced now.

4 Background on ranking functions frameworks

In ranking functions frameworks, uncertainty is handled in a qualitative way, but it is encoded on some linearly ordered scale (finite or infinite). Typical examples of these frameworks are possibility theory (Dubois and Prade 1998) where uncertainty is represented in the interval $[0, 1]$ and Spohn's ordinal functions (Spohn

[2] where $max(a, b) \geq_\Omega max(c, d)$ means either $[a \geq_\Omega c$ and $a \geq_\Omega d]$ or $[b \geq_\Omega c$ and $b \geq_\Omega d]$, and $min(a, b) \leq_\Omega min(c, d)$ means either $[a \leq_\Omega c$ and $a \leq_\Omega d]$ or $[b \leq_\Omega c$ and $b \leq_\Omega d]$.

1988) which use the set of integers. In the following, we only focus on possibility theory, but results of this paper are also valid for other frameworks such as Spohn's ordinal functions, or Lehmann's ranked models, due to their close relation to possibility theory.

Let us give a brief background on possibility theory (see (Dubois and Prade 1988) for more details). The basic building block is the notion of *possibility distribution* denoted by π and corresponding to a mapping from Ω to $[0, 1]$. A possibility distribution π is said to be *normalized* if it exists at least one world ω which is totally possible, i.e. $\pi(\omega) = 1$.

Given a possibility distribution π, we can define a mapping grading the **possibility** [3] of a formula $\phi \subseteq \Omega$ by: $\Pi(\phi) = max_{\omega \in \phi} \pi(\omega)$.

Each possibility distribution π can generate a plausibility relation (qualitative ordering relation) $\geq_\Omega$ by applying:

$$\omega \geq_\Omega \omega' \text{ iff } \pi(\omega) \geq \pi(\omega'). \quad (5)$$

Similarly, if we define $\geq_\Omega$ from π then:

$$\phi \geq_\Omega \psi \text{ iff } \Pi(\phi) > \Pi(\psi).$$

5 Advantages of the ranking function setting

Let us first analyze the basic differences between ranking functions and the qualitative framework presented in Section 2, from the perspective of the study of independence relations. Basically, there are three differences:

- *Normalization*: in possibility theory, fully plausible worlds receives the grade 1 while there is no counterpart in the qualitative setting.
- *Existence of impossible worlds* graded to 0 in possibility theory, while all worlds are somewhat possible in the qualitative setting.
- *Commensurability* between uncertainty levels, where all rankings reflect grades in the same scale.

In this section, we show the advantages of working with uncertainty ranking functions in order to define more powerful notions of qualitative independence.

[3] A dual measure to the possibility is the necessity degree defined by $N(\phi) = 1 - \Pi(\neg\phi) = min_{\omega \notin \phi}(1 - \pi(\omega))$.

5.1 Consequences of the normalization and of the existence of impossible worlds

5.1.1 Definition of possibilistic conditioning

The normalization and the existence of impossible worlds enables the definition of several notions of conditioning in the graded settings, contrarily to the qualitative setting. This explains why in possibility theory there are two definitions of independence based on conditioning. The natural properties of $\pi' = \pi(. \mid \phi)$ taking into account normalization and impossible worlds become:

$\mathbf{C}_1$ **:** $\forall \omega \notin \phi, \pi'(\omega) = 0$,

$\mathbf{C}_2$ **:** $\forall \omega_1, \omega_2 \in \phi, \pi(\omega_1) > \pi(\omega_2)$ iff $\pi'(\omega_1) > \pi'(\omega_2)$,

$\mathbf{C}_3$ **:** if $\Pi(\phi) = 1$, then $\forall \omega \in \phi, \pi(\omega) = \pi'(\omega)$,

$\mathbf{C}_4$ **:** π' should be normalized,

$\mathbf{C}_5$ **:** if $\pi(\omega) = 0$ then $\pi'(\omega) = 0$.

$\mathbf{C}_1$ confirms that ϕ is a sure piece of information, $\mathbf{C}_2$ says that the new possibility distribution should not affect the possibility degrees relative to the interpretations in ϕ and $\mathbf{C}_3$ says that if ϕ is already consistent with beliefs encoded by π, then the possibility degrees of the elements in ϕ remain identical. $\mathbf{C}_5$ stipulates that impossible worlds remain impossible after conditioning.

The properties ($\mathbf{C}_1$-$\mathbf{C}_5$) do not guarantee a *unique definition of conditioning*. Indeed, the effect of the axiom $\mathbf{C}_1$ may result in a sub-normalized possibility distribution. Restoring the normalization, in order to satisfy $\mathbf{C}_4$, can be done in at least two different ways (when $\Pi(\phi) > 0$) (Dubois and Prade 1988):

- In an ordinal setting, we assign to the best elements of ϕ, the maximal possibility degree (i.e. 1), then we obtain:

  $$\pi(\omega \mid_m \phi) = \begin{cases} 1 & if\ \pi(\omega) = \Pi(\phi) \text{ and } \omega \in \phi \\ \pi(\omega) & if\ \pi(\omega) < \Pi(\phi) \text{ and } \omega \in \phi \\ 0 & otherwise \end{cases}$$

  Note that this conditioning form is equivalent to the least specific solution of the combination equation :
  $\pi(\omega \wedge \phi) = min(\pi(\omega \mid_m \phi), \Pi(\phi))$.
  proposed by Hisdal (Hisdal 1978).
- In a numerical setting, we proportionally shift up all elements of ϕ (if the definition makes sense in

the ranking scale):

$$\pi(\omega \mid_P \phi) = \begin{cases} \frac{\pi(\omega)}{\Pi(\phi)} & \text{if } \omega \in \phi \\ 0 & \textit{otherwise} \end{cases}$$

5.1.2 Definitions of possibilistic causal independence

The idea in defining possibilistic causal independence relation based on the possibilistic conditioning is that X is considered as independent from Y in the context Z if for any instance z, the possibility degree of any x remains unchanged for any value y. More formally:

$$\Pi(x \mid y \wedge z) = \Pi(x \mid z), \forall xyz. \quad (6)$$

Since possibility theory has two kinds of conditioning, this leads to two definitions of causal possibilistic independence:

- **Min-based independence relation** obtained by using $|_m$ in (6), this form of independence is not symmetric (Fonck 1994). If we enforce this property, we get a very strong relation, denoted MS-independence(M for min-based and S for symetry) since the independence between two sets of variables X and Y implies the ignorance of one of them (De Campos and Huete 1998) i.e. $\pi(x) = 1, \forall x$ or $\pi(y) = 1, \forall y$. In (Ben Amor et al., 2000), it has been shown that:

  Proposition 4 *Let π be a possibility distribution, and $\geq_\Omega$ be its associated plausibility relation. Then X and Y are MS-independent in π if and only if they are Pareto-independent in $\geq_\Omega$.*

- **Product independence relation** obtained by using $|_P$ in (6). This relation is equivalent to:

  $\Pi(x \wedge y \mid_P z) = \Pi(x \mid_P z) * \Pi(y \mid_P z), \forall xyz.$

  The product independence relation, denoted by P-independence, is equivalent to Kappa functions' independence relation based on Spohn-conditioning (Spohn 1988). Note that this relation enjoys the same properties as the independence relation proposed in the probabilistic framework. In particular, we can decompose it, i.e. we can recover $\Pi(x \wedge y \mid_P z)$ in a unique manner from $\Pi(x \mid_P z)$ and $\Pi(y \mid_P z)$ using the product operator.

The following proposition relates possibilistic causal independence to PO-independence:

Proposition 5 *If X and Y are independent in a possibility distribution π according to (6), then they are PO-independent in the plausibility relation induced by π. The converse is false.*

5.2 Effect of the commensurability

In this section we study the *decomposition* of some important independence relations. We will see that the commensurability property is crucial in the recomposition of joint distributions from marginal ones.

5.2.1 Possibilistic Non-Interactivity

In the possibilistic framework, the standard decompositional independence relation between X and Y in the context Z is the **Non-Interactivity (NI)** (Zadeh 1978) defined by:

$$\Pi(x \wedge y \mid_m z) = min(\Pi(x \mid_m z), \Pi(y \mid_m z)), \forall xyz \quad (7)$$

This relation can be defined in a purely qualitative setting by first defining $\omega \geq_\Omega \omega'$ iff $\pi(\omega) \geq \pi(\omega')$. Then X and Y are NI-independent iff:

$$x \wedge y =_{\Omega|z} x \text{ or } x \wedge y =_{\Omega|z} y, \forall xyz.$$

However, NI-independence is not interesting in a qualitative representation since it does not allow the recomposition of a unique global plausibility relation from local orders defined on independent variables (due to the non-satisfaction of the commensurability property), as shown by the example below.

Example : Given two binary variables A and B with the following local orderings:

$$\text{(i) } a_1 >_\Omega a_2$$

$$\text{(ii) } b_2 >_\Omega b_1.$$

There is no unique plausibility relation $\geq_\Omega$ satisfying (i) and (ii) such that A and B are NI-independent. Indeed, it is sufficient to consider the two plausibility relations $\geq_\Omega$ *and* $\geq_{\Omega'}$:

$$a_1 \wedge b_2 >_\Omega a_2 \wedge b_2 >_\Omega a_1 \wedge b_1 =_\Omega a_2 \wedge b_1$$

$$a_1 \wedge b_2 >_{\Omega'} a_2 \wedge b_2 =_{\Omega'} a_1 \wedge b_1 =_{\Omega'} a_2 \wedge b_1$$

However, if the local orderings are encoded in possibility theory then we will have a unique plausibility relation $\geq_\Omega$ using $\pi(a \wedge b) = min(\pi(a), \pi(b)), \forall ab$.

Note that the importance of commensurability assumption also appears in fuzzy communauty, especially in defining connectors between fuzzy sets. For instance, French (1986) comments the limitation of definition the intersection of two fuzzy sets (using the minimum operator to define the membership function associated to the intersection) when no commensurability is assumed.

5.2.2 Decomposition of PO-independence

A natural question now is to see if the encoding of a plausibility relation with a possibility distribution can be useful in the decomposition of the causal independence relation PO. Namely, if there exists a function f such that for each possibility distribution π (encoding some plausibility relation) where X and Y are PO-independent, we have $\forall x, x' \in D_X, \forall y, y' \in D_Y$:

$$\pi(x \wedge y) > \pi(x' \wedge y') \text{ iff } f(\Pi(x), \Pi(y)) > f(\Pi(x'), \Pi(y'))$$

Table 1: Possibility distributions on A and B

a	**b**	$\pi_1(a \wedge b)$	$\pi_2(a \wedge b)$
a_1	b_1	1	1
a_1	b_2	0.9	0.9
a_1	b_3	0.5	0.5
a_2	b_1	0.8	0.8
a_2	b_2	0.3	0.3
a_2	b_3	0.2	0.1
a_3	b_1	0.4	0.4
a_3	b_2	0.1	0.2
a_3	b_3	0	0

In the following, we show that this is impossible in the general case. Indeed, assume that such a function f exists, then let us consider the two possibility distributions π_1 and π_2 defined on two PO-independent variables A and B and given by Table **??**.

We have:
$\Pi_1(a_1) = \Pi_2(a_1) = 1, \Pi_1(a_2) = \Pi_2(a_2) = 0.8$,
$\Pi_1(a_3) = \Pi_2(a_3) = 0.4, \Pi_1(b_1) = \Pi_2(b_1) = 1$,
$\Pi_1(b_2) = \Pi_2(b_2) = 0.9, \Pi_1(b_3) = \Pi_2(b_3) = 0.5$. We can check that A and B are PO-independent in both π_1 and π_2, namely for all a_i, a_j, for all b_k we have: $\pi_l(a_i) > \pi_l(a_j)$ iff $\pi_l(a_i \wedge b_k) > \pi_l(a_j \wedge b_k)$, for l=1,2, and the same exchanging a and b.
Besides,π_1 induces
$\pi_1(a_2 \wedge b_3) > \pi_1(a_3 \wedge b_2)$, i.e. $f(\Pi_1(a_2), \Pi_1(b_3)) > f(\Pi_1(a_3), \Pi_1(b_2))$
while π_2 induces
$\pi_2(a_3 \wedge b_2) > \pi_2(a_2 \wedge b_3)$, i.e. $f(\Pi_2(a_3), \Pi_2(b_2)) > f(\Pi_2(a_2), \Pi_2(b_3))$,
hence a contradiction.

However, there are particular cases where decomposition can be achieved. The first particular case concerns binary variables:

Proposition 6 *If A and B are binary variables, then A and B are PO-independent iff the plausibility relation induced by π is leximin and leximax decomposable.*

The second particular case concerns binary possibility distributions.

Proposition 7 *If π is composed of two levels (namely the set of interpretations is splitted in only two classes) then X and Y are PO-independent iff the plausibility relation induced by π is leximin and leximax decomposable.*

Such decomposition can be useful for databases when the existing tuples are preferred to absent ones.

The difficulty of decomposing PO-independence relation in the general case is due to the fact that the PO-independence is a weak relation. The following proposition makes the weakness of PO-independence explicit:

Proposition 8 *X and Y are PO-independent in $\geq_\Omega$ iff $\geq_\Omega$ is Pareto-compatible (i.e., monotonic) on X and Y.*

The following subsection discusses the decomposability of Pareto, leximin and leximax relations, which are all Pareto-compatible.

5.2.3 Decomposition of Pareto, leximin and leximax independence

In this subsection we study the decomposition of Pareto, leximin and leximax decomposable relations when we use the possibilistic framework.

The decomposition of leximin and leximax-decomposable relations is immediate since we use weights represented by possibility degrees which allows the comparison of different interpretations. We illustrate it on the following example:

Example : Let us consider two variables A and B with the following plausibility relation $\geq_\Omega$ which is leximax decomposable: $a_1 \wedge b_1 >_\Omega a_1 \wedge b_2 >_\Omega a_2 \wedge b_1 >_\Omega a_1 \wedge b_3 >_\Omega a_2 \wedge b_2 >_\Omega a_2 \wedge b_3$

We can easily check that this plausibility relation cannot be recovered from the local orders on A and B induced by it:

(i) $a_1 >_\Omega a_2$

(ii) $b_1 >_\Omega b_2 >_\Omega b_3$

Indeed, it is sufficient to consider the following plausibility relation: $a_1 \wedge b_1 >_{\Omega'} a_2 \wedge b_1 >_{\Omega'} a_1 \wedge b_2 >_{\Omega'} a_1 \wedge b_3 >_{\Omega'} a_2 \wedge b_2 >_{\Omega'} a_2 \wedge b_3$

which satisfies (i) and (ii) and is also leximax decomposable.

However, if $\geq_\Omega$ is encoded by means of a possibility distribution, then the decomposition will be possible because we preserve the information concerning the relative ordering between any instance of A and any instance of B in the original relation as shown by the following example:

Example : Let π be a possibility distribution encoding the plausibility relation $\geq_\Omega$ given in the previous example (see Table **??**).

Table 2: Possibility distribution on A and B

a	**b**	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0.9
a_1	b_3	0.5
a_2	b_1	0.8
a_2	b_2	0.3
a_2	b_3	0.2

We can easily recover π from the local distributions on A and B and the numerical scale (1, .9, .8, .5, .3, .2) using the leximax ordering. Indeed, the use of the leximax on the local distributions provides the ordering relation relative to π i.e. $a_1 \wedge b_1 >_\Omega a_1 \wedge b_2 >_\Omega a_2 \wedge b_1 >_\Omega a_1 \wedge b_3 >_\Omega a_2 \wedge b_2 >_\Omega a_2 \wedge b_3$ then using the numerical scale we can recover the original distribution π.

For Pareto-independence, we also have the following result:

Proposition 9 *Let π be a possibility distribution such that X and Y are two Pareto-independent variable sets. Then, π is can be recovered from marginal probabilities $\Pi(X)$ and $\Pi(Y)$ using the product operator, namely:*

$$\pi(x \wedge y) = \Pi(x) * \Pi(y).$$

Note that other operators can also be used to recover the oredering behing π, however the advantage using the product operator is that it preserves both orderings and numerical values.

6 Comparative study

Given a joint possibility distribution π, this section provides a comparative study between the different independence relations presented in this paper. Let $\geq_\Omega$ be the plausibility relation induced from π by using equation 5.

Let us first summarize the different independence relations presented in this paper:

- *Qualitative causal independence:* three definitions have been proposed, depending if we only preserve preferred elements (i.e., I_{PT}), or we only preserve accepted beliefs (i.e., I_{PB}), or we preserve the whole relative ordering (i.e., I_{PO}).
- *Qualitative decomposition independence:* the decomposability of a plausibility relation has been based on the well-known principles: Pareto-ordering (i.e., I_{Pareto}), Leximin ordering (i.e., $I_{Leximin}$), or leximax ordering (i.e., $I_{Leximax}$).
- *Possibilistic causal independence:* two definitions have been proposed (i.e., I_{MS} and I_P) depending on which of the two definitions of conditioning in possibility theory is used.
- *Possibilistic decomposition independence:* which corresponds to the non-interactivity (i.e., I_{NI}) .

The arrows in Figure 1 show the inclusion of different independence relations. The I_{MS} and I_{Pareto} are the strongest independence relations since the MS or the Pareto independence between two sets of variables implies the ignorance of one of them. However, I_{PT} is the weakest one. Finally, note that I_{NI} is implied by I_{MS} but it is incomparable with the other independence relations.

We give now some counter-examples relative to the non-existent links.

Counter-examples : Let us consider two variables A and B with the possibility distributions given in Table **??**.

- with π_1, we can check that the relation $I_{PO}(A, \emptyset, B)$ is true while $I_{MS}(A, \emptyset, B)$, $I_P(A, \emptyset, B)$ and $I_{NI}(A, \emptyset, B)$ are false.
- with π_2 we can check that $I_{NI}(A, \emptyset, B)$ is true contrary to $I_{MS}(A, \emptyset, B)$, $I_{leximin}(A, \emptyset, B)$, $I_{leximax}(A, \emptyset, B)$ $I_P(A, \emptyset, B)$, and $I_{PO}(A, \emptyset, B)$.

Table 3: Possibility distributions on A and B

a	**b**	$\pi_1(a \wedge b)$	$\pi_2(a \wedge b)$	$\pi_3(a \wedge b)$
a_1	b_1	1	1	0.6
a_1	b_2	0.8	0.8	1
a_2	b_1	0.7	0.8	0.36
a_2	b_2	0.2	0.8	0.6

- with π_3, we can check that the relation $I_P(A, \emptyset, B)$ is true contrary to $I_{MS}(A, \emptyset, B)$ and $I_{NI}(A, \emptyset, B)$.

  Suppose now that A is a binary variable and B is a ternary variable and let us consider the possibility distributions given in Table ??.

Table 4: Possibility distributions on A and B

a	**b**	$\pi_1(a \wedge b)$	$\pi_2(a \wedge b)$
a_1	b_1	1	1
a_1	b_2	0.9	0.9
a_1	b_3	0.5	0.6
a_2	b_1	0.8	0.8
a_2	b_2	0.3	0.7
a_2	b_3	0.2	0.5

- we can check that in π_1 $I_{leximax}(A, \emptyset, B)$ is respected while $I_{NI}(A, \emptyset, B)$ is false. In addition, in π_2 $I_{leximin}(A, \emptyset, B)$ is respected contrarily to $I_{NI}(A, \emptyset, B)$.
- with π_1 we can check that A and B are PO-independent but not leximin-independent since $\pi(a_1 \wedge b_3) > \pi(a_2 \wedge b_2)$ while $min(\Pi(a_1), \Pi(b_3)) < min(\Pi(a_2), \Pi(b_2))$.
- with π_2 we can check that A and B are PO-independent but not leximax-independent since $\pi(a_2 \wedge b_2) > \pi(a_1 \wedge b_3)$ while $max(\Pi(a_2), \Pi(b_2)) < max(\Pi(a_1), \Pi(b_3))$.
- with π_1 we can check that A and B are leximax-independent but neither leximin nor Pareto independent. Moreover, with π_2 we can check that A and B are leximin-independent but neither leximax nor Pareto independent.
- with the following possibility distribution :

a	b	$\pi(a \wedge b)$
a_1	b_1	1
a_1	b_2	0.8
a_2	b_1	0.5
a_2	b_2	0.4
a_3	b_1	0.4
a_3	b_2	0.32

  we can check that I_P is satisfied while $I_{leximin}$ and $I_{leximax}$ are false.

  Note that in the binary case I_P implies $I_{leximin}$ and $I_{leximax}$.
- Lastly,let A and B be two ternary varibles. In the following possibility distributions, we can check that in π_1 $I_{leximax}$ is respected while I_P is false and that in π_2 $I_{leximin}$ is respected contrarily to I_P.

a	b	$\pi_1(a \wedge b)$	$\pi_2(a \wedge b)$
a_1	b_1	1	1
a_1	b_2	0.9	0.9
a_1	b_3	0.5	0.6
a_2	b_1	0.8	0.8
a_2	b_2	0.3	0.7
a_2	b_3	0.2	0.5

7 Concluding remarks

This paper relates the notions of independence relations defined in purely qualitative setting (when only a total pre-order is used) to the ones defined in ranking function frameworks. Two kinds of independence have been investigated : causal and decomposition ones. A first constatition is that independence relations defined on purely qualitative framework are very weak from the decomposability point of view. This paper has shown that one way to overcome this limitation is to use a ranking framework, like possibility theory, total-order. A second constatation is that the decomposition of joint distribution in possibility theory is not unique, contrary to probability theory where only the product operator is used. In possibility theory alternative operator, like leximin or leximax, can be used as well. Note that this may be worth applying to other types of numerical distributions (e.g., probability distributions) which will be leximin-independent but not independent in the usual sense of the considered uncertainty theory, in the same way as in (Wong and Butz, 1999) where weak notions of independence are exploited in the probabilistic setting.
A third constatation is that most of decomposition is independence (except the non-interactivity) are also causal. This clearly appears in Figure 1.

The notions of independence proposed in this paper extend previous works in default reasoning (Benferhat et al. 1994), and belief revision (Dubois et al. 1997) on independence between events to the case of variables which are not necessarily binary. A line for further research concerns logical counterpart of leximin and leximax independence relations in the possibilistic setting. Indeed, procedures for translating graph representations (defined from min-based and product-based conditional independence) into stratified possibilistic logic bases have been recently exhibited (Benferhat et al. 1999). This is worth doing for leximin-based independence, which is stronger than min-based independence, but still meaningful in a qualitative setting. Another line of research io to relate the results of decomposition of joint distribution defined on independent relations, to the ones provided in multi-criteria decision making for preferential independence (refxxx).

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