Compactification and decompactification by weights on Bergman spaces

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Abstract. We characterize the symbols φ for which there exists a weight w such that the weighted composition operator $M_w C_\varphi$ is compact on the weighted Bergman space $\mathfrak{B}^2_\alpha$. We also characterize the symbols for which there exists a weight w such that $M_w C_\varphi$ is bounded but not compact. We also investigate when there exists w such that $M_w C_\varphi$ is Hilbert-Schmidt on $\mathfrak{B}^2_\alpha$.

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1 Introduction

It is known (see [4] for instance) that "weightening" a composition operator C_φ on the Hardy space H^2 by some weight w, we can improve its compactness properties, and even its membership in Schatten classes S_p, or the decay of its approximation numbers ([9, Theorem 2.3], [12]), or at the opposite make a compact composition operator non compact ([12]).

In this paper, we consider weighted composition operators $M_w C_\varphi$ on the weighted Bergman spaces $\mathfrak{B}^2_\alpha$, with $\alpha > -1$. Note that for such an operator to be bounded from $\mathfrak{B}^2_\alpha$ into itself, it is necessary that $w \in \mathfrak{B}^2_\alpha$ (since $w = (M_w C_\varphi)(\mathbb{1})$).

We show in Section 3 that C_φ can be weighted to become compact on $\mathfrak{B}^2_\alpha$ if and only if the set where φ has an angular derivative has null measure.

In Section 4, we show that the exists a weight w such that $M_w C_\varphi$ is bounded but not compact on $\mathfrak{B}^2_\alpha$ if and only if $\|\varphi\|_\infty = 1$.

In Section 5, we study when $M_w C_\varphi$ can be Hilbert-Schmidt on $\mathfrak{B}^2_\alpha$ for some weight w.

2 Notation and background

The weighted Bergman space $\mathfrak{B}^2_\alpha$, with $\alpha > -1$, is the space of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ on the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{\mathfrak{B}^2_\alpha} = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < \infty \, ,$$

where A is the normalized area measure on $\mathbb{D}$. When $\alpha = 0$, we write simply $\mathfrak{B}^2$ instead of $\mathfrak{B}^2_0$ and call it the Bergman space.

Every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ defines a bounded composition operator $C_\varphi\colon f \mapsto f \circ \varphi$ from $\mathfrak{B}^2_\alpha$ into itself ([16, Proposition 3.4]).

The pull-back measure A_φ of φ is defined as:

$$A_\varphi(B) = A[\varphi^{-1}(B)] \quad \text{for all Borel sets } B \subseteq \mathbb{D} \, .$$

Let μ be a finite Borel measure on $\mathbb{D}$. For $\beta > 1$, the measure μ is said a *β-Carleson measure* if:

$$\sup_{|\xi|=1} \mu[S(\xi, h)] = \mathrm{O}\,(h^\beta) \, , \tag{2.1}$$

where

$$S(\xi, h) = \{z \in \mathbb{D}\, ; \; |z - \xi| < h\}$$

is the Carleson box of size h centered at $\xi \in \mathbb{T} = \partial\mathbb{D}$. The measure μ is said a *vanishing β-Carleson measure* if:

$$\sup_{|\xi|=1} \mu[S(\xi, h)] = \mathrm{o}\,(h^\beta) \quad \text{as } h \to 0 \, . \tag{2.2}$$

Recall the following result (see [7] and [16, Theorem 4.3]).

Theorem 2.1. *Let μ be a finite Borel measure on $\mathbb{D}$. Then:*

(a) $\mathfrak{B}^2_\alpha \subseteq L^2(\mu)$ *if and only if μ is an $(\alpha + 2)$-Carleson measure.*

Moreover, when this happens, the canonical inclusion $J_\mu\colon \mathfrak{B}^2_\alpha \to L^2(\mu)$ is bounded.

(b) *The canonical inclusion $J_\mu\colon \mathfrak{B}^2_\alpha \to L^2(\mu)$ is compact if and only if μ is a vanishing $(\alpha + 2)$-Carleson measure.*

Corollary 2.2. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map and $w \in \mathfrak{B}^2_\alpha$. Set, for every Borel set B in $\mathbb{D}$:*

$$\mu_{w,\varphi}(B) = \int_{\varphi^{-1}(B)} |w(z)|^2 (1 - |z|^2)^\alpha \, dA(z) \, . \tag{2.3}$$

Then:

(a) *The weighted composition operator* $M_w C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$*, defined as:*

$$(M_w C_\varphi) f = w\,(f \circ \varphi)\,, \tag{2.4}$$

is bounded if and only if $\mu_{w,\varphi}$ *is an* $(\alpha+2)$*-Carleson measure.*

(b) *The weighted composition operator* $M_w C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ *is compact if and only if* $\mu_{w,\varphi}$ *is a vanishing* $(\alpha+2)$*-Carleson measure.*

Proof. Observe that, for all $f \in \mathfrak{B}^2_\alpha$, we have

$$\|(M_w C_\varphi) f\|^2_{\mathfrak{B}^2_\alpha} = \int_{\mathbb{D}} |f[\varphi(z)]|^2\, |w(z)|^2 (1-|z|^2)^\alpha\, dA(z) = \|f\|^2_{L^2(\mu_{w,\varphi})}\,. \qquad \square$$

3 Compactification

Recall the following definitions (see [20, Section 4.1]).

Definition 3.1. *A holomorphic self-map* $\varphi \colon \mathbb{D} \to \mathbb{D}$ *has an* angular limit (*or a* non-tangential limit) l *at* $\xi \in \mathbb{T}$ *if* $\varphi(z)$ *converges to* l *whenever* z *tends to* ξ *inside any angular sector in* $\mathbb{D}$ *whose vertex is* ξ*. Then* l *is called the* angular limit *of* φ *at* ξ *and is denoted:*

$$l = \angle \lim_{z \to \xi} \varphi(z)\,.$$

Definition 3.2. *A holomorphic self-map* $\varphi \colon \mathbb{D} \to \mathbb{D}$ *has an* angular derivative *at* $\xi \in \mathbb{T}$ *if it has an angular limit* ζ *at* ξ, with $|\zeta| = 1$ *and:*

$$\angle \lim_{z \to \xi} \frac{\varphi(z) - \zeta}{z - \xi}$$

exists and is finite. This limit is called the angular derivative of φ *at* ξ *and is denoted by* $\varphi'(\xi)$*.*

Let us also recall that the Julia-Carathéodory theorem (see [20, Section 4.2]), says that φ has an angular derivative at $\xi \in \mathbb{T}$ if and only if:

$$\delta := \liminf_{z \to \xi} \frac{1 - |\varphi(z)|}{1 - |z|} < +\infty\,, \tag{3.1}$$

or, equivalently:

$$\limsup_{z \to \xi} \frac{1 - |z|}{1 - |\varphi(z)|} > 0\,, \tag{3.2}$$

and, when this happens, we have $\delta > 0$ and $\varphi'(\xi) = \xi\,\bar{\zeta}\,\delta$, so $|\varphi'(\xi)| = \delta$.

We define

$$\mathcal{AD}(\varphi) = \{\xi \in \mathbb{T}\,;\ \varphi \text{ has an angular derivative at } \xi\}\,, \tag{3.3}$$

and we call it the *angular derivative set of* φ.

B. MacCluer and J. Shapiro proved ([16, Theorem 3.5]) that, for $\alpha > -1$, the composition operator $C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ is compact if and only if:

$$\mathcal{AD}(\varphi) = \emptyset \,. \tag{3.4}$$

Asking for a compactification, we have the following result.

Theorem 3.3. *Let $\varphi \colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map. Then the following assertions are equivalent, for the weighted Bergman space $\mathfrak{B}^2_\alpha$, with $\alpha > -1$:*

1) *there exists a holomorphic function w, with $w \not\equiv 0$, such that the weighted composition operator $M_w C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ is compact;*
2) *there exists a weight $w \in H^\infty$, with $w \not\equiv 0$, such that the weighted composition operator $M_w C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ is compact;*
3) *the angular derivative set of φ has null measure:*

$$m[\mathcal{AD}(\varphi)] = 0 \,, \tag{3.5}$$

where m is the normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$;

4) $\displaystyle\lim_{z\to\xi} \frac{1-|z|}{1-|\varphi(z)|} = 0$ *for almost all $\xi \in \mathbb{T}$.*

For example, if $\varphi(z) = \frac{1+z}{2}$, then C_φ is not compact on $\mathfrak{B}^2_\alpha$, but it is compactifiable by a weight in H^∞.

When the equivalent conditions of Theorem 3.3 are satisfied, we say that composition operator C_φ is *compactifiable*.

The proof will be based on the following result of Moorhouse ([17, Corollary 1]; see also [2, Proposition 1]).

Proposition 3.4 (Moorhouse)**.** *Let $\alpha > -1$. Let φ and w be analytic functions on $\mathbb{D}$. Then:*

1) *If the weighted composition operator $M_w C_\varphi$ is compact on $\mathfrak{B}^2_\alpha$, we have:*

$$\lim_{|z|\to 1} |w(z)|^2 \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2} = 0 \,. \tag{3.6}$$

2) *When w is bounded, $M_w C_\varphi$ is compact on $\mathfrak{B}^2_\alpha$ if and only if*

$$\lim_{|z|\to 1} |w(z)|^2 \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0 \,. \tag{3.7}$$

For 1), we compute:

$$\|(M_w C_\varphi)^*(k_z)\|^2_{(\mathfrak{B}^2_\alpha)^*} = |w(z)|^2 \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2} ,$$

where k_z is the normalized reproducing kernel of $\mathfrak{B}^2_\alpha$, and, using that k_z weakly converges to 0 as $|z| \to 1$, we obtain:

$$\lim_{|z|\to 1} |w(z)|^2 \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2} = 0\,.$$

To obtain the necessary condition in 2), we use the following, easily checked, fact which shows that (3.7) is equivalent to (3.6), since w is bounded in 2).

Lemma 3.5. *Let $f, g\colon \mathbb{D} \to [0,\infty)$ be two bounded functions. Then the following assertions are equivalent:*

a) $\lim_{|z|\to 1} f(z)\, g(z) = 0$*;*
b) $\lim_{|z|\to 1} \min[f(z), g(z)] = 0$*;*
c) $\lim_{|z|\to 1} [f(z)]^a [g(z)]^b = 0$*, for all* $a, b > 0$.

The sufficient condition in 2) is proved by [17, Lemma 1].

Proof of Theorem 3.3. The implication 2) $\Rightarrow$ 1) needs no comment.
3) $\Rightarrow$ 2) Assume that $m[\mathcal{AD}(\varphi)] = 0$. A theorem of Privalov (see [21, Vol. I, bottom of page 276]), asserts the existence a function $w \not\equiv 0$ in H^∞ such that

$$\lim_{z\to\xi} w(z) = 0 \quad \text{for all } \xi \in \mathcal{AD}(\varphi)\,. \tag{3.8}$$

The Schwarz-Pick lemma (see [1, Corollary 2.40]) tells that

$$\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq 2\,\frac{1-|z|}{1-|\varphi(z)|} \leq 2\,\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\,.$$

Hence for $\xi \in \mathcal{AD}(\varphi)$, we have

$$\lim_{z\to\xi} |w(z)|^2\,\frac{1-|z|^2}{1-|\varphi(z)|^2} = 0\,.$$

For $\xi \notin \mathcal{AD}(\varphi)$, thanks to the Julia-Carathéodory theorem and (3.2), we also have

$$\lim_{z\to\xi} |w(z)|^2\,\frac{1-|z|^2}{1-|\varphi(z)|^2} = 0\,.$$

Hence

$$\lim_{z\to\xi} |w(z)|^2\,\frac{1-|z|^2}{1-|\varphi(z)|^2} = 0 \quad \text{for all } \xi \in \mathbb{T}\,. \tag{3.9}$$

By a compactness argument, we obtain that

$$\lim_{|z|\to 1} |w(z)|^2\,\frac{1-|z|^2}{1-|\varphi(z)|^2} = 0\,; \tag{3.10}$$

in fact, if (3.10) failed, there would be a sequence (z_n) such that $|z_n| \underset{n\to\infty}{\longrightarrow} 1$ and for which

$$\limsup_{n\to\infty} |w(z_n)|^2 \, \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2} > 0\,;$$

by compactness a subsequence converges to some $\xi \in \partial\mathbb{D}$, and that would contradict (3.9).

Since w is bounded, it follows from Proposition 3.4, that $M_w C_\varphi$ is compact on $\mathfrak{B}^2_\alpha$.

1) $\Rightarrow$ 3) Assume that $M_w C_\varphi$ is compact with w analytic and $w \not\equiv 0$. By Proposition 3.4, 1), we have:

$$\lim_{|z|\to 1} |w(z)|^2 \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2} = 0\,. \tag{3.11}$$

In particular, for every $\xi \in \mathbb{T}$:

$$\lim_{z\to \xi} |w(z)|^2 \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2} = 0\,. \tag{3.12}$$

Now, for every $\xi \in \mathcal{AD}(\varphi)$, we have, if ζ is the angular limit of φ at ξ:

$$\lim_{z\to\xi} \left|\frac{\varphi(z)-\zeta}{z-\xi}\right| = |\varphi'(\xi)| < \infty\,.$$

If z belongs to an angular sector S_ξ of vertex ξ, there is a positive constant C, depending only on this sector, such that $|z-\xi| \leq C(1-|z|)$; hence

$$\liminf_{z\to\xi, z\in S_\xi} \frac{1-|z|}{1-|\varphi(z)|} \geq \liminf_{z\to\xi, z\in S_\xi} C\,\frac{|z-\xi|}{|\varphi(z)-\zeta|} = \frac{C}{|\varphi'(\xi)|} > 0\,.$$

Then, it follows, with (3.12), that $\lim_{z\to\xi, z\in S_\xi} w(z) = 0$. Since the angular sector S_ξ is arbitrary, we get that $\angle \lim_{z\to\xi} w(z) = 0$.

By another theorem of Privalov (see [21, Chapter XIV, Theorem (1.1) and Theorem (1.9)], [6, Chapter VI, Theorem 2.3], or [5, Chapter II, Exercise 10], where it is called “local Fatou theorem”) , it follows, since $w \not\equiv 0$, that $m[\mathcal{AD}(\varphi)] = 0$.

3) $\Longleftrightarrow$ 4) follows from the Julia-Caratheodory theorem, as stated in (3.2). $\square$

Remark 1. The implication 2) $\Rightarrow$ 3) can be proved using the classical F. and M. Riesz theorem (see [3, Theorem 2.2]) instead of Privalov's theorem.

Remark 2. Condition (3.11) is necessary for the compactness of $M_w C_\varphi$; however, it is not sufficient in general without this assumption that $w \in H^\infty$. An example is given in [2, Section 5, Corollary 4] for which $M_w C_\varphi$, with $w = \varphi'$, is not even bounded on $\mathfrak{B}^2$.

4 Decompactification

4.1 The main result

In the sequel, as usual, $\alpha > -1$.

Definition 4.1. *We say that the composition operator* $C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ *is* decompactifiable *if there exists a weight* $w \in \mathfrak{B}^2$ *such that the weighted composition operator* $M_w C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ *is bounded but not compact.*

Our main result is the following.

Theorem 4.2. *Let* $\varphi \colon \mathbb{D} \to \mathbb{D}$ *be an analytic self-map. Then the composition operator* $C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ *is decompactifiable if and only if* $\|\varphi\|_\infty = 1$.

It is a consequence of this other theorem, whose proof is postponed.

Theorem 4.3. *Let* $\gamma > 1$ *and* ν *be a vanishing* γ*-Carleson measure on* $\mathbb{D}$. *Assume that* ν *satisfies the following property:*

$$\forall t > 0\,, \quad \exists \zeta \in \partial\mathbb{D} \quad \text{such that } \nu[S(\zeta, t)] > 0\,. \tag{4.1}$$

Then there exists a holomorphic function $u \colon \mathbb{D} \to \mathbb{C}$ *such that*

(*i*) $u \in L^2(\nu)$;

(*ii*) $\displaystyle \sup_{|\xi|=1,\, 0<h\leq 1} \frac{1}{h^\gamma} \int_{S(\xi,h)} |u|^2 \, d\nu < \infty$;

(*iii*) *there exist* $\delta > 0$ *and two sequences* (ζ_n) *in* $\partial\mathbb{D}$ *and* (t_n) *in* $(0,1)$ *with* $t_n \underset{n\to\infty}{\longrightarrow} 0^+$ *such that*

$$\frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |u|^2 \, d\nu \geq \delta\,, \quad \text{for all } n \geq 1\,. \tag{4.2}$$

Proof of Theorem 4.2. It is plain that if $\|\varphi\|_\infty < 1$, then $M_w C_\varphi$ is compact for every weight $w \in \mathfrak{B}^2_\alpha$. In fact, if $\mu_{w,\varphi}$ is the measure defined in (2.3), then $\mu_{w,\varphi}[S(\xi, h)] = 0$ for $0 < h < 1 - \|\varphi\|_\infty$; hence Corollary 2.2 gives the result.

Conversely, assume that $\|\varphi\|_\infty = 1$.

Note that if $C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ is not compact, it suffices to take $w = \mathbb{1}$; so we assume that C_φ is compact. Then $\nu := (A_\alpha)_\varphi = \varphi(dA_\alpha)$ is a vanishing $(\alpha + 2)$-Carleson measure.

Since $\|\varphi\|_\infty = 1$, condition (4.1) is satisfied. Set $\gamma = \alpha + 2$ and u be the holomorphic function given by Theorem 4.3 and set $w = u \circ \varphi$. We have

$$\int_{\mathbb{D}} |w|^2 \, dA_\alpha = \int_{\mathbb{D}} |u \circ \varphi|^2 \, dA_\alpha = \int_{\mathbb{D}} |u|^2 \, d\nu < \infty\,;$$

so $w \in \mathfrak{B}^2_\alpha$.

Now, for every $\xi \in \partial\mathbb{D}$ and $h \in [0,1)$, we have, with $\mu = \varphi(|w|^2 dA_\alpha)$:

$$\mu[S(\xi,h)] = \int_{\varphi^{-1}[S(\xi,h)]} |w|^2 \, dA_\alpha = \int_{\mathbb{D}} (\mathbb{I}_{S(\xi,h)} \circ \varphi) \, |u \circ \varphi|^2 \, dA_\alpha$$
$$= \int_{\mathbb{D}} \mathbb{I}_{S(\xi,h)} |u|^2 \, d\nu \, .$$

Hence the properties (*ii*) and (*iii*) of Theorem 4.3 show that μ is a non-vanishing $(\alpha+2)$-Carleson measure, and therefore that $M_w C_\varphi \colon \mathfrak{B}^2_\alpha \to \mathfrak{B}^2_\alpha$ is bounded but not compact. □

4.2 Proof of Theorem 4.3

To prove Theorem 4.3, we need several auxiliary results.

Lemma 4.4. *For every $\omega \in \partial\mathbb{D}$ and $r \in (0,1)$, there exists a bounded analytic function $F \in H^\infty$ such that, for all $z \in \mathbb{D}$:*

a) $\mathfrak{Re}\, F(z) > 0$;

b) $1/2 \leq |F(z)| \leq 2$;

c) $|F(z)| < 1$ *when* $|z-\omega| > r$;

d) $|F(z)| > 1$ *when* $|z-\omega| < r$.

Proof. By composing with a rotation, we can, and do, assume that $\omega = 1$.

Let $C = 1-r$ and let A and B be the points of the intersection of the unit circle $\mathbb{T} = \partial\mathbb{D}$ with the circle of center 1 and radius r, with $\mathfrak{Im}\, A > 0$ and $\mathfrak{Im}\, B < 0$. Consider the Möbius transformation T sending A to 0, C to 1, and B to ∞. The images by T of $\partial\mathbb{D}$ and $\partial D(1,r)$ are straight lines passing through 0. In fact the image of $\partial D(1,r)$ is the extended real line $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Moreover $T[D(1,r)]$ is the open upper half-plane.

Define $g(z) = \sqrt{T(z)}$, where $\sqrt{}$ is the principal branch of the square root.

Then, for $z \in \mathbb{D}$:

$$\begin{cases} \arg\,[g(z)] \in (0, \pi/2) & \text{if } z \in D(1,r)\,, \\ \arg\,[g(z)] \in (-\pi/2, 0) & \text{if } z \in \mathbb{D} \setminus \overline{D(1,r)}\,. \end{cases}$$

Let now U be the Möbius transformation sending 0 to $i/2$, ∞ to $-i/2$, and 1 to 0. We have

- $|U[g(z)]| < 1/2$ for all $z \in \mathbb{D}$;
- $\mathfrak{Re}\, U[g(z)] > 0$ for all $z \in \mathbb{D} \cap D(1,r) = S(1,r)$;
- $\mathfrak{Re}\, U[g(z)] < 0$ for all $z \in \mathbb{D} \setminus \overline{D(1,r)}$.

Finally, the function F defined as $F(z) = \exp U[g(z)]$ suits. □

Lemma 4.5. *Let $\gamma \geq 1$, ν be a γ-Carleson measure on $\mathbb{D}$ and $F \in H^\infty$ such that $1/2 \leq |F(z)| \leq 2$ for all $z \in \mathbb{D}$. For given $\beta \in (0,1]$, we define the function $\Phi\colon \mathbb{R}_+^* \to \mathbb{R}_+$ as:*

$$\Phi(\delta) = \sup_{|\xi|=1;\, 0<h\leq\beta} \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta}\, d\nu\,, \quad \textit{for all } \delta > 0\,.$$

Then Φ is continuous.

Proof. First, we have $\Phi(\delta) < +\infty$ for all $\delta > 0$ because ν is a γ-Carlseon measure; indeed, for all $\xi \in \partial\mathbb{D}$ and all $h \in (0,1]$:

$$\frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta}\, d\nu \leq 4^\delta\, \frac{\nu[S(\xi,h)]}{h^\gamma} \leq C\, 4^\delta < +\infty\,.$$

Now, observe that, since $1/2 \leq |F(z)| \leq 2$, we have, for all $h \in (0,1]$, all $\xi \in \partial\mathbb{D}$, and all $t \in \mathbb{R}$:

$$\frac{1}{4^{|t|}}\, \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta}\, d\nu \leq \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta+2t}\, d\nu \leq 4^{|t|}\, \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta}\, d\nu\,.$$

Taking the supremum, we get

$$4^{-|t|}\, \Phi(\delta) \leq \Phi(\delta + t) \leq 4^{|t|}\, \Phi(\delta)\,,$$

and that proves the continuity of Φ, since $\Phi(\delta) < +\infty$. ☐

Proposition 4.6. *Let ν be a finite γ-Carleson measure on $\mathbb{D}$ with property (4.1). Then, for every $\beta \in (0,1]$ and every $\varepsilon \in (0,1)$, there exists a function $v \in H^\infty$ satisfying:*

(a) *$|v(z)| < \varepsilon$ for all $z \in \mathbb{D}$ such that $|z| < 1 - \beta$;*

(b) *$\dfrac{1}{h^\gamma} \displaystyle\int_{S(\xi,h)} |v|^2\, d\nu \leq 1$ for all $h \in (0,1]$ and all $\xi \in \partial\mathbb{D}$;*

(c) *$\dfrac{1}{h^\gamma} \displaystyle\int_{S(\xi,h)} |v|^2\, d\nu \leq \varepsilon^2$ for all $h \in (\beta,1]$ and all $\xi \in \partial\mathbb{D}$;*

(d) *there exists $t \in (0,\beta]$ and $\zeta \in \partial\mathbb{D}$ such that*

$$\frac{1}{t^\gamma} \int_{S(\zeta,t)} |v|^2\, d\nu \geq \left(\frac{3}{4}\right)^2 .$$

Proof. Since ν is a γ-Carleson measure, there exists a positive constant C (and we can and do assume that $C \geq 1$) such that:

$$\nu[S(\xi,h)] \leq C\, h^\gamma\,, \qquad \forall h \in (0,1]\,,\ \forall \xi \in \partial\mathbb{D}\,. \tag{4.3}$$

Take $r = \beta\, (\varepsilon^2/2)^{1/\gamma}$.

By (4.1), there exists $\omega \in \partial\mathbb{D}$ such that

$$\nu[S(\omega, r)] > 0\,.$$

Let F be the function given by Lemma 4.4.

We define:

$$\Phi(\delta) = \sup_{0<h\leq\beta,\, |\xi|=1} \frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^{2\delta}\, d\nu\,.$$

Thanks to (4.3), we have, for all $\xi \in \partial\mathbb{D}$ and all $h \in (0,\beta]$:

$$\frac{1}{h^\gamma} \int_{S(\xi,h)} |F|^2\, d\nu \leq 4\, \frac{\nu[S(\xi,h)]}{h^\gamma} \leq 4\,C\,,$$

and we get $\Phi(1) \leq 4\,C$.

On the other hand, for all $\delta > 0$:

$$\Phi(\delta) \geq \frac{1}{r^\gamma} \int_{S(\omega,r)} |F|^{2\delta}\, d\nu\,.$$

Since $|F(z)| > 1$ for $z \in S(\omega, r)$ and $\nu[S(\omega,r)] > 0$, we get

$$\lim_{\delta\to+\infty} \int_{S(\omega,r)} |F|^{2\delta}\, d\nu = +\infty\,,$$

and consequently $\lim_{\delta\to+\infty} \Phi(\delta) = +\infty$. Since $\Phi(1) \leq 4\,C < (2C/\varepsilon)^2$ and, thanks to Lemma 4.5, Φ is continuous, there exists $\delta_0 > 1$ such that $\Phi(\delta_0) = (2C/\varepsilon)^2$.

Define

$$v = (\varepsilon/2C)\, F^{\delta_0}\,.$$

Observe that $r < \beta$; so $|z| < 1-\beta$ implies $|z| < 1-r$; hence $z \notin \overline{S(\omega,r)}$ and $|F(z)| < 1$. That means that $|v(z)| < \varepsilon/(2C) < \varepsilon$, and we have proved (a).

By definition of δ_0, (b) is satisfied for all $h \in (0,\beta]$. It will be satisfied as well for $h \in (\beta, 1]$ once we have proved (c).

Let us prove (c). Take $\beta < h \leq 1$. Since $|F(z)| \leq 1$ for $z \in S(\xi,h) \setminus S(\omega,r)$, we have:

$$\begin{aligned}
\int_{S(\xi,h)} |v|^2\, d\nu &\leq \int_{S(\xi,h)\setminus S(\omega,r)} \Big(\frac{\varepsilon}{2C}\Big)^2 d\nu + \Big(\frac{\varepsilon}{2C}\Big)^2 \int_{S(\omega,r)} |F|^{2\delta_0}\, d\nu \\
&\leq \Big(\frac{\varepsilon}{2C}\Big)^2 \nu[S(\xi,h)] + \Big(\frac{\varepsilon}{2C}\Big)^2 r^\gamma \Phi(\delta_0) \\
&\leq \Big(\frac{\varepsilon}{2C}\Big)^2 C\, h^\gamma + r^\gamma = \Big(\frac{\varepsilon}{2C}\Big)^2 C\, h^\gamma + \beta^\gamma \frac{\varepsilon^2}{2} \\
&\leq \Big(\frac{\varepsilon}{2C}\Big)^2 C\, h^\gamma + h^\gamma \frac{\varepsilon^2}{2} \leq \Big(\frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{2}\Big) h^\gamma \leq \varepsilon^2 h^\gamma\,.
\end{aligned}$$

Finally, by definition of δ_0, there exist $t \in (0,\beta]$ and $\zeta \in \partial\mathbb{D}$ such that

$$\frac{1}{t^\gamma} \int_{S(\zeta,t)} |v|^2\, d\nu = \Big(\frac{\varepsilon}{2C}\Big)^2 \frac{1}{t^\gamma} \int_{S(\zeta,t)} |F|^{2\delta_0}\, d\nu \geq \Big(\frac{3}{4}\Big)^2,$$

and (d) if proved. □

Proof of Theorem 4.3. Consider a sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers such that $\sum_{n=1}^\infty \varepsilon_n < 1/4$.

Using Proposition 4.6, we are going to construct by induction four sequences $(v_n)_n$ in H^∞, $(\beta_n)_n$, with $\beta_1 = 1$, and $(t_n)_n$ in $(0,1]$, and $(\zeta_n)_n$ in $\partial\mathbb{D}$ such that, for all $n \geq 1$:

(S 1) $\beta_n \geq t_n > \beta_{n+1} \geq t_{n+1}$;

(S 2) $|v_n(z)| < \varepsilon_n$ for $|z| < 1 - \beta_n$;

(S 3) for all $\xi \in \partial\mathbb{D}$ and all $h \in (0, \beta_{n+1}] \cup [\beta_n, 1]$:

$$\frac{1}{h^\gamma} \int_{S(\xi,h)} |v_n|^2 \, d\nu \leq \varepsilon_n^2 \, ;$$

(S 4) $\displaystyle \frac{1}{h^\gamma} \int_{S(\xi,h)} |v_n|^2 \, d\nu \leq 1$ for all $h \in (\beta_{n+1}, \beta_n]$ and all $\xi \in \partial\mathbb{D}$;

(S 5) $\displaystyle \frac{1}{t_n^\gamma} \int_{S(\zeta_n,t_n)} |v_n|^2 \, d\nu \geq \left(\frac{3}{4}\right)^2,$

and

(S 6) $\displaystyle \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} t_n = 0.$

Take $\beta_1 = 1$. With $\beta = \beta_1$ and $\varepsilon = \varepsilon_1$, let $v_1 = v$ be the function given by Proposition 4.6 and $\zeta_1 = \zeta$ and $t_1 = t \leq \beta_1$ the numbers given by part (d) of that proposition. By Proposition 4.6 (b) and (d) respectively, conditions (S 4) and (S 5) are satisfied for $n = 1$. Condition (S 2) is void for $n = 1$. For condition (S 3), note that since ν is a vanishing γ-Carleson measure, there exists $\beta_2 > 0$ such that

$$\frac{\nu[S(\xi,h)]}{h^\gamma} \leq \varepsilon_1^2 (1 + \|v_1\|_\infty^2)^{-1}$$

for all $h \in (0, \beta_2]$ and all $\xi \in \partial\mathbb{D}$. This implies, for these h's and ξ's:

$$\frac{1}{h^\gamma} \int_{S(\xi,h)} |v_1|^2 \, d\nu \leq \frac{\|v_1\|_\infty^2 \, \nu[S(\xi,h)]}{h^\gamma} \leq \varepsilon_1^2 \, .$$

It follows, with (c) of Proposition 4.6, that (S 3) is satisfied for $n = 1$.

We can of course ask that $\beta_2 \leq 1/2$.

Now, assume that $v_1, \ldots, v_{n+1}$, $\beta_1, \ldots, \beta_{n+1}$, $t_1, \ldots, t_{n+1}$ and $\zeta_1, \ldots, \zeta_{n+1}$ satisfying (S 1), (S 2), (S 3), (S 4) and (S 5) have been constructed.

As above, since ν is a vanishing γ-Carleson measure, there exists a positive number $\beta_{n+2} \leq \min(\beta_{n+1}, 1/(n+2))$ such that

$$\frac{\nu[S(\xi,h)]}{h^\gamma} \leq \varepsilon_n^2 (1 + \|v_{n+1}\|_\infty^2)^{-1}$$

for all $h \in (0, \beta_{n+2}]$ and all $\xi \in \partial\mathbb{D}$. Using Proposition 4.6 with $\beta = \beta_{n+2}$ and $\varepsilon = \varepsilon_{n+1}$, we get $v_{n+2} = v \in H^\infty$, $\zeta_{n+2} = \zeta \in \partial\mathbb{D}$ and $t_{n+2} = t \in (0, \beta_{n+2}]$ and the induction step follows.

We now set:

$$u(z)=\sum_{n=1}^{\infty} v_n(z)\,,\quad z\in\mathbb{D}\,.$$

Thanks to (S 2), this series converges uniformly on compact subsets of $\mathbb{D}$, so u is analytic in $\mathbb{D}$.

Take $\xi\in\partial\mathbb{D}$ and $h\in(0,1]$. There exists a unique $n\geq 1$ such that $h\in(\beta_{n+1},\beta_n]$. By the triangle inequality:

$$\left(\frac{1}{h^\gamma}\int_{S(\xi,h)}|u|^2\,d\nu\right)^{1/2}\leq\sum_{\substack{k=1\\k\neq n}}^{\infty}\left(\frac{1}{h^\gamma}\int_{S(\xi,h)}|v_k|^2\,d\nu\right)^{1/2}+\left(\frac{1}{h^\gamma}\int_{S(\xi,h)}|v_n|^2\,d\nu\right)^{1/2}.$$

Now, since:

$$h\in(0,\beta_{k+1}]\cup(\beta_k,1]\quad\text{for every } k\neq n\,,\tag{4.4}$$

we get, by (S 3) and (S 4):

$$\left(\frac{1}{h^\gamma}\int_{S(\xi,h)}|u|^2\,d\nu\right)^{1/2}\leq\left(\sum_{k\neq n}\varepsilon_k\right)+1\leq\frac{1}{4}+1=\frac{5}{4}\cdot$$

Consequently, u satisfies (ii) of Theorem 4.3. Then condition (ii) implies (i) because ν is a finite measure, u is bounded on $(1/2)\,\mathbb{D}$, and $\mathbb{D}\setminus(1/2)\,\mathbb{D}$ can be covered by a finite number of boxes $S(\xi,1)$, with $\xi\in\partial\mathbb{D}$.

To obtain (iii), we use (4.4) again, with $h=t_n$, to get:

$$\begin{aligned}\left(\frac{1}{t_n^\gamma}\int_{S(\zeta_n,t_n)}|u|^2\,d\nu\right)^{1/2}&\geq\left(\frac{1}{t_n^\gamma}\int_{S(\zeta_n,t_n)}|v_n|^2\,d\nu\right)^{1/2}\\&\qquad-\sum_{k\neq n}\left(\frac{1}{t_n^\gamma}\int_{S(\zeta_n,t_n)}|v_k|^2\,d\nu\right)^{1/2}\\&\geq\frac{3}{4}-\sum_{k\neq n}\varepsilon_k\geq\frac{3}{4}-\frac{1}{4}=\frac{1}{2}\,,\end{aligned}$$

and we have (iii). □

5 Hilbert-Schmidt regularization

We remarked in Section 3 that if $\varphi(z)=\frac{1+z}{2}$, then C_φ is compactifiable on $\mathfrak{B}^2_\alpha$ by a weight in H^∞. Actually, since $|\varphi(\mathrm{e}^{it})|=\cos(t/2)$, we have $\int_{-\pi}^{\pi}\log\frac{1}{1-|\varphi(\mathrm{e}^{it})|}\,dm(t)<\infty$, and [12, Theorem 4.1] tells that the composition operator C_φ can be weighted to have a Hilbert-Schmidt operator on H^2; a fortiori, this weighted composition operator is Hilbert-Schmidt on $\mathfrak{B}^2_\alpha$ (see [13, Theorem 3.12]). We can be more specific on an example, but unfortunately this example shows no difference between the Hardy and Bergman spaces.

Proposition 5.1. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be defined by $\varphi(z) = \frac{1+z}{2}$, and let $w(z) = (1-z)^\beta$ with $\beta > -1/2$, so that $w \in H^2$. Then the weighted composition operators $M_w C_\varphi\colon H^2 \to H^2$ and $M_w C_\varphi\colon \mathfrak{B}^2 \to \mathfrak{B}^2$ are Hilbert-Schmidt if and only if $\beta > 1/2$.*

Proof. The first item was proved in [9, Proposition 2.4]. For the second item, we have to determine those β such that

$$\int_{\mathbb{D}} \frac{|w(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z) < \infty\,.$$

Since $|\varphi(z)|$ approaches 1 only when z approaches 1, we can as well consider

$$I := \int_{\Delta} \frac{|w(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z)\,,$$

where $\Delta = \mathbb{D} \cap D(1,1)$. Passing in polar coordinates centered at 1, we write, for $z \in \Delta$: $z = 1 - r\,\mathrm{e}^{i\theta}$ with $|\theta| < \pi/2$ and $r < 2\cos\theta$. Then, $\left|\frac{1+z}{2}\right|^2 = 1 + \frac{r^2}{4} - r\cos\theta$ and

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \frac{r^{2\beta+1}}{r^2(\cos\theta - r/4)^2}\, dr\, \frac{d\theta}{\pi} = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r^{2\beta-1}}{(\cos\theta - r/4)^2}\, dr\, d\theta\,.$$

Making the change of variable $r = 2t\cos\theta$, $0 \leq t \leq 1$ in the inner integral and observing that $1 \geq 1 - t/2 \geq 1/2$, we see that

$$I \approx \int_0^{\pi/2}\int_0^1 \frac{t^{2\beta-1}(\cos\theta)^{2\beta}}{\cos^2\theta}\, dt\, d\theta = \left(\int_0^1 t^{2\beta-1}\, dt\right)\left(\int_0^{\pi/2} (\sin\theta)^{2\beta-2}\, d\theta\right).$$

So, clearly, $I < \infty$ if and only if $\beta > 1/2$. □

Fortunately, examples showing the difference between Hardy and Bergman spaces exist.

Theorem 5.2. *There exists a Blaschke product B which can be Hilbert-Schmidt regularized, and more, on $\mathfrak{B}^2$, but not on H^2.*

Proof. Any Blaschke product B is an inner function, i.e. $|B^*| = 1$ m-almost everywhere on the unit circle, implying, by [12, Theorem 3.1], that $M_w C_B$ is compact on H^2 for no weight $w \in H^2$, with $w \neq 0$.

On the other hand, as a consequence of [10, Theorem 3.1], we proved ([11, Theorem 4.4]; see also [14, Theorem 13]) that there exist Blaschke products B (which we called *slow Blaschke products*) such that C_B is compact on the Bergman-Orlicz space $\mathfrak{B}^{\Psi_2}$, and hence belong to every Schatten class S_p of $\mathfrak{B}^2$. □

Moreover, we can give the following quantitative precision to Theorem 5.2.

Theorem 5.3. *For any sequence (ε_n) of positive numbers with limit zero, there is a Blaschke product B such that*

$$a_n(C_B\colon \mathfrak{B}^2 \to \mathfrak{B}^2) \lesssim \mathrm{e}^{-n\varepsilon_n}\,.$$

Proof. We can assume that ε_n decreases and that $n\varepsilon_n \uparrow \infty$ with $n\varepsilon_n \geq \sqrt{n}$.

For a given symbol φ, we set

$$\chi(h) = A(\{z\,;\ |\varphi(z)| \geq 1-h\})\,.$$

We use [15, Theorem 5.1] which implies that

$$a_n(C_\varphi) \lesssim \inf_{0<h<1} \left[\sqrt{n}\,\mathrm{e}^{-nh} + \sqrt{\frac{\chi(h)}{h^2}}\,\right]. \tag{5.1}$$

Let $\delta\colon (0,1) \to (0,1)$ be a non-increasing and piecewise linear map, decreasing to 0 so slowly at the origin that

$$\delta(1-|z|) \leq 4\,\varepsilon_n \quad \Longrightarrow \quad 1-|z| \leq \varepsilon_n^2 \exp(-2\,n\,\varepsilon_n)\,.$$

By [10, Theorem 3.1] again, there exists a Blaschke product B such that $|B(z)| \leq \exp\big(-\delta(1-|z|)\big)$. Take $\varphi = B$ in (5.1) and observe that, for $h = 2\,\varepsilon_n \leq 1/2$, we have

$$|B(z)| \geq 1-h \quad \Longrightarrow \quad \exp\big(-\delta(1-|z|)\big) \geq 1-h \geq \exp(-2h)\,.$$

Hence

$$\delta(1-|z|) \leq 4\,\varepsilon_n \quad \text{and} \quad 1-|z| \leq \varepsilon_n^2 \exp(-2\,n\,\varepsilon_n)$$

and

$$\chi(h) \leq 2\,\varepsilon_n^2 \exp(-2\,n\,\varepsilon_n)\,.$$

Inserting this in (5.1), we get the result. □

In order to find a necessary and sufficient condition for a symbol can be weighted in a Hilbert-Schmidt operator, we make some observations.

As recalled, a weighted composition operator which is Hilbert-Schmidt on H^2 is also Hilbert-Schmidt on $\mathfrak{B}^2$. We know that C_φ is Hilbert-Schmidt on H^2 if and only if

$$\int_{\mathbb{T}} \frac{1}{1-|\varphi|^2}\,dm < \infty\,.$$

Equivalently:

$$\sum_{n=0}^{\infty} \|\varphi^n\|_{H^2}^2 < \infty\,. \tag{5.2}$$

On the other hand, C_φ can be weighted to become a Hilbert-Schmidt operator on H^2 if and only if

$$\int_{\mathbb{T}} \log \frac{1}{1-|\varphi|}\, dm < \infty\,,$$

([12, Theorem 4.1]), which is equivalent to

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\, \|C_\varphi(e_n)\|_{H^2}^2 < \infty\,. \tag{5.3}$$

Now, writing $e_n(z) = z^n$, and since $\big((n+1)^{(\alpha+1)/2}\, e_n\big)_n$ is an orthonormal basis of $\mathfrak{B}_\alpha^2$, C_φ is Hilbert-Schmidt on $\mathfrak{B}_\alpha^2$ if and only if

$$\sum_{n=0}^{\infty} (n+1)^{\alpha+1} \|\varphi^n\|_{\mathfrak{B}_\alpha^2}^2 = \sum_{n=0}^{\infty} \|C_\varphi\big((n+1)^{(\alpha+1)/2}\, e_n\big)\|_{\mathfrak{B}_\alpha^2}^2 < \infty\,.$$

By comparison with (5.3), we might think that C_φ can be weighted to become a Hilbert-Schmidt operator on $\mathfrak{B}_\alpha^2$ if and only if

$$\sum_{n=0}^{\infty} (n+1)^{\alpha} \|\varphi^n\|_{\mathfrak{B}_\alpha^2}^2 < \infty\,. \tag{5.4}$$

Since

$$\sum_{n=0}^{\infty} (n+1)^{\alpha} \|\varphi^n\|_{\mathfrak{B}_\alpha^2}^2 = \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+1}}\, dA(z)\,, \tag{5.5}$$

this guesswork takes the following form: is tit true that there exists a weight w such that $M_w C_\varphi\colon \mathfrak{B}_\alpha^2 \to \mathfrak{B}_\alpha^2$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+1}}\, dA(z) < \infty \quad ? \tag{5.6}$$

We do not know if (5.6) implies the existence of a weight $w \not\equiv 0$ for which $M_w C_\varphi$ is Hilbert-Schmidt, but, in any case, it implies that C_φ is compactifiable.

Proposition 5.4. *If, for $\alpha > -1$, we have $\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|)^{\alpha+1}}\, dA(z) < \infty$, then*

$$\lim_{z\to\xi} \frac{1-|z|}{1-|\varphi(z)|} = 0 \qquad \text{for almost all } \xi \in \mathbb{T}\,.$$

Recall that, by Theorem 3.3, this last condition means that C_φ is compactifiable on $\mathfrak{B}_\alpha^2$.

Proof. We set:

$$g(z) = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+1}.$$

Let $r_n = 1 - 2^{-n}$ and:

$$\Gamma_n = \{z \in \mathbb{D}\,;\ r_n \leq |z| < r_{n+1}\}\,.$$

Now, $1/(1-|\varphi|^2)^{\alpha+1}$ is subharmonic (and even logarithmically-subharmonic), because we can write $1/(1-|\varphi|)^{\alpha+1} = \sum_{k=0}^{\infty} c_k(\alpha)|\varphi|^{2k}$ with $c_k(\alpha) \geq 0$. Hence, we have, since $A(\Gamma_n) \approx 1 - r_n^2$:

$$\begin{aligned}\int_{\mathbb{T}} g(r_n \mathrm{e}^{i\theta})\, d\theta &= \int_{\mathbb{T}} \frac{(1-r_n^2)^{\alpha+1}}{(1-|\varphi(r_n\mathrm{e}^{i\theta})|^2)^{\alpha+1}}\, d\theta \\ &\lesssim \frac{1}{1-r_n^2}\int_{\Gamma_n} \frac{(1-r_n^2)^{\alpha+1}}{(1-|\varphi(z)|^2)^{\alpha+1}}\, dA(z) \\ &\lesssim \int_{\Gamma_n} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha+1}}\, dA(z)\end{aligned}$$

(we used that $1-r_n^2 \leq 2(1-r_n) = 2\times 2^{-n} = 4(1-r_{n+1}) \leq 4(1-|z|) \leq 4(1-|z|^2)$ for $z \in \Gamma_n$, so $(1-r_n^2)^\alpha \lesssim (1-|z|^2)^\alpha$ when $\alpha \geq 0$, and, when $-1 < \alpha < 0$, we used that $1-|z|^2 \leq 1-r_n^2$ for $z \in \Gamma_n$). The sets Γ_n being disjoint, we get that:

$$\begin{aligned}\int_{\mathbb{T}} \bigg(\sum_{n=0}^{\infty} g(r_n\mathrm{e}^{i\theta})\bigg)\, d\theta &= \sum_{n=0}^{\infty}\int_{\mathbb{T}} g(r_n\mathrm{e}^{i\theta})\, d\theta \\ &\lesssim \sum_{n=0}^{\infty}\int_{\Gamma_n} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha+1}}\, dA(z) \\ &= \int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^{\alpha+1}}\, dA(z) < \infty\,,\end{aligned}$$

meaning that the function $\sum_{n=0}^{\infty} g(r_n\,\cdot)$ is integrable on $\mathbb{T}$. It follows that $g(r_n\,\cdot) \underset{n\to\infty}{\longrightarrow} 0$ almost everywhere. Since the existence of a radial limit implies that of an angular limit, we obtain that $\angle\lim_{z\to\xi} \frac{1-|z|}{1-|\varphi(z)|} = 0$ for almost all $\xi \in \mathbb{T}$. By the Julia-Caratheodory theorem, it follows that $\lim_{z\to\xi} \frac{1-|z|}{1-|\varphi(z)|} = 0$ for almost all $\xi \in \mathbb{T}$. □

An a priori different condition than (5.6) appears in the following theorem.

Theorem 5.5. *Let $d\lambda_\alpha(r) = 2\,(\alpha+1)\,(1-r^2)^\alpha\, r\, dr$, be the marginal probability measure on $[0,1)$ of dA_α, and*

$$G(\theta) = \int_0^1 \frac{d\lambda_\alpha(r)}{(1-|\varphi(r\mathrm{e}^{i\theta})|^2)^{\alpha+2}}\,. \tag{5.7}$$

Then:

1) *If $\log G \in L^1(0,2\pi)$, then there exists $w \in H^\infty$, $w \not\equiv 0$, such that $M_w C_\varphi$ is Hilbert-Schmidt on $\mathfrak{B}^2_\alpha$.*

2) *Conversely, if there exists such a weight w, then $\log G \in L^{1,\infty}(0,2\pi)$.*

Note that $G \geq 1$, so $\log G \geq 0$.

Recall that $L^{1,\infty}(\mu)$ is the space of (classes of) measurable functions f such that $\sup_{a>0} a\, m(\{|f| > a\}) < \infty$, and that $L^1(\mu) \subseteq L^{1,\infty}(\mu)$, by Markov's inequality.

5.1 Proof of 1) of Theorem 5.5

For convenience, we set

$$U(z) = \frac{1}{(1 - |\varphi(z)|^2)^{\alpha+2}} \cdot \tag{5.8}$$

We will use two lemmas. For that, we denote ρ the pseudo-hyperbolic metric on $\mathbb{D}$. Recall that

$$\rho(u, v) = \left| \frac{u - v}{1 - \bar{u}v} \right|, \quad u, v \in \mathbb{D}\,.$$

Lemma 5.6. *There is a positive constant $C = C(\alpha)$ such that, for $u, v \in \mathbb{D}$:*

$$\rho(u, v) \leq \frac{1}{2} \quad \Longrightarrow \quad \frac{1}{C} \leq \frac{U(u)}{U(v)} \leq C\,. \tag{5.9}$$

Proof. Since

$$\left(\frac{1}{2} \times \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|}\right)^{\alpha+2} \leq \frac{U(u)}{U(v)} \leq \left(2 \times \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|}\right)^{\alpha+2},$$

it suffices to show that there is a positive constant such that

$$\frac{1}{C} \leq \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|} \leq C$$

when $\rho(u, v) \leq 1/2$. Moreover, by the Schwarz-Pick inequality, we have:

$$\rho\big(|\varphi(u)|, |\varphi(v)|\big) \leq \rho\big(\varphi(u), \varphi(v)\big) \leq \rho(u, v)\,,$$

it suffices to majorize $q := \frac{1-a}{1-b}$ when $\rho(a, b) \leq 1/2$ and $0 \leq a, b < 1$ (the minoration will come by exchanging a and b).

If $a \geq b$, then $q \leq 1$

If $a < b$, we remark that $\rho(a, b) \leq 1/2$ writes $T_a(b) := \frac{a-b}{1-ab} \geq -1/2$. Since T_a is decreasing on $[-1, 1]$, we get $b \leq T_a(-1/2)$, i.e. $b \leq \frac{1+2a}{2+a}$, and $1 - b \geq \frac{1-a}{2+a}$. Therefore $q \leq 2 + a \leq 3$. □

Let, for $n \geq 0$:

$$r_n = \exp(-2^{-n})$$

and

$$\Gamma_n = \{z \in \mathbb{D}\,;\ r_n \leq |z| < r_{n+1}\}\,.$$

Lemma 5.7. *For $r_n \leq u, v \leq r_{n+1}$, we have $\rho(u\,\mathrm{e}^{i\theta}, v\,\mathrm{e}^{i\theta}) \leq 1/2$, for every $\theta \in \mathbb{R}$.*

Proof. It is a simple computation:

$$\rho(u\,\mathrm{e}^{i\theta}, v\,\mathrm{e}^{i\theta}) = \frac{|u-v|}{1-uv} \leq \frac{r_{n+1}-r_n}{1-r_{n+1}^2} = \frac{r_{n+1}-r_{n+1}^2}{1-r_{n+1}^2} = \frac{r_{n+1}}{1+r_{n+1}} \leq \frac{1}{2} \cdot \qquad \square$$

Now, we can finish the proof of 1) of Theorem 5.5.

We have to show that there exists a non-null function $w_0 \in H^\infty$ such that

$$\int_{\mathbb{D}} |w_0|^2 U\, dA_\alpha < \infty\,, \tag{5.10}$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha\, dA(z)$.

For every $w \in H^\infty$, we have:

$$\int_{\mathbb{D}} |w|^2 U\, dA_\alpha = \int_{D(0,\mathrm{e}^{-1})} |w|^2 U\, dA_\alpha + \sum_{n=0}^{\infty} \int_{\Gamma_n} |w|^2 U\, dA_\alpha$$

For every $n \geq 0$:

$$\int_{\Gamma_n} |w|^2 U\, dA_\alpha = 2\,(\alpha+1) \int_{r_n}^{r_{n+1}} \left(\frac{1}{2\pi} \int_0^{2\pi} |w(r\mathrm{e}^{i\theta})|^2\, U(r\mathrm{e}^{i\theta})\, d\theta \right) (1-r^2)^\alpha\, r\, dr$$

As said in the proof of Proposition 5.4, U is logarithmically-subharmonic; hence the function $|w|^2\, U$ is also logarithmically-subharmonic; in particular, it is subharmonic; so we have (see [3, Theorem 1.6, page 9]), for $r_n \leq r \leq r_{n+1}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(r\mathrm{e}^{i\theta})|^2\, U(r\mathrm{e}^{i\theta})\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |w(r_{n+1}\mathrm{e}^{i\theta})|^2\, U(r_{n+1}\mathrm{e}^{i\theta})\, d\theta\,.$$

By Lemma 5.6 and Lemma 5.7, we have $U(r_{n+1}\mathrm{e}^{i\theta}) \leq C\, U(r_n\, \mathrm{e}^{i\theta})$. But $r_n = r_{n+1}^2$, so $U(r_{n+1}\mathrm{e}^{i\theta}) \leq C\, U(r_{n+1}^2\, \mathrm{e}^{i\theta})$, and hence

$$\frac{1}{2\pi} \int_0^{2\pi} |w(r\mathrm{e}^{i\theta})|^2\, U(r\mathrm{e}^{i\theta})\, d\theta \leq C\, \frac{1}{2\pi} \int_0^{2\pi} |w(r_{n+1}\mathrm{e}^{i\theta})|^2\, U(r_{n+1}^2\mathrm{e}^{i\theta})\, d\theta\,.$$

By the subharmonicity of $|w|^2\, U$ again, we obtain:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(r\mathrm{e}^{i\theta})|^2\, U(r\mathrm{e}^{i\theta})\, d\theta \leq C\, \frac{1}{2\pi} \int_0^{2\pi} |w(\mathrm{e}^{i\theta})|^2\, U(r_{n+1}\mathrm{e}^{i\theta})\, d\theta\,.$$

Using Lemma 5.6 and Lemma 5.7 again, we have, for every $r_n \leq r < r_{n+1}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(\mathrm{e}^{i\theta})|^2\, U(r_{n+1}\mathrm{e}^{i\theta})\, d\theta \leq C\, \frac{1}{2\pi} \int_0^{2\pi} |w(\mathrm{e}^{i\theta})|^2\, U(r\,\mathrm{e}^{i\theta})\, d\theta\,.$$

Therefore:

$$\int_{\Gamma_n} |w|^2 U\, dA_\alpha \leq C^{\,2} \int_{r_n}^{r_{n+1}} \left(\frac{1}{2\pi} \int_0^{2\pi} |w(\mathrm{e}^{i\theta})|^2\, U(r\,\mathrm{e}^{i\theta})\, d\theta \right) d\lambda_\alpha(r)\,.$$

Using the Fubini theorem, we finally obtain:

$$\int_{\mathbb{D}} |w|^2 U\, dA_\alpha \leq \int_{D(0,\mathrm{e}^{-1})} |w|^2 U\, dA_\alpha + C^{\,2} \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathrm{e}^{-1}}^{1} U(r\,\mathrm{e}^{i\theta})\, d\lambda_\alpha(r) \right) |w(\mathrm{e}^{i\theta})|^2\, d\theta\,.$$

Since

$$G(\theta) = \int_0^1 U(r\,\mathrm{e}^{i\theta})\, d\lambda_\alpha(r)\,,$$

we have:

$$\int_{\mathbb{D}} |w|^2 U\, dA_\alpha \leq \int_{D(0,\mathrm{e}^{-1})} |w|^2 U\, dA_\alpha + C^{\,2} \frac{1}{2\pi} \int_0^{2\pi} G(\theta)\, |w(\mathrm{e}^{i\theta})|^2\, d\theta\,. \tag{5.11}$$

We now use Szegö's theorem (see [5, Theorem 3.1, Chapter IV, page 139], or [18, Section 8.3]):

$$\inf_{w\in H^\infty, w(0)=1} \frac{1}{2\pi} \int_0^{2\pi} |w(\mathrm{e}^{i\theta})|^2 G(\theta)\, d\theta = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log G(\theta)\, d\theta \right).$$

Remarking that the hypothesis of the theorem writes:

$$\int_0^{2\pi} \log G(\theta)\, d\theta < \infty\,,$$

that shows that there exists $w_0 \in H^\infty$ with $w_0(0) = 1$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |w_0(\mathrm{e}^{i\theta})|^2 G(\theta)\, d\theta < \infty\,.$$

With (5.11), that shows that w_0 satisfies (5.10), and that ends the proof of 1) of Theorem 5.5.

Note that the proof shows that we can actually get a polynomial for w_0.

5.2 Proof of 2) of Theorem 5.5

We may, and do, assume that $\|w\|_\infty = 1$.

By hypothesis, we have

$$\int_{\mathbb{D}} |w(z)|^2 \frac{(1-|z|^2)^\alpha}{\big(1-|\varphi(z)|^2\big)^{\alpha+2}}\, dA(z) < \infty\,.$$

Setting, with U defined in (5.8):

$$\psi(\theta) = \int_{1/2}^{1} |w(r\,\mathrm{e}^{i\theta})|^2\, U(r\,\mathrm{e}^{i\theta})\, d\lambda_\alpha(r)\,, \tag{5.12}$$

we hence have $\psi \in L^1(0, 2\pi)$.

Let

$$\widetilde{G}(\theta) = \int_{1/2}^{1} U(r\,\mathrm{e}^{i\theta})\, d\lambda_\alpha(r)$$

and

$$J(\theta) = \inf\{|w(r\,\mathrm{e}^{i\theta})|^2\,;\ 1/2 \leq r < 1\}\,.$$

We have $\psi(\theta) \geq \widetilde{G}(\theta)\, J(\theta)$, so

$$\log \widetilde{G} \leq \log \psi + \log(1/J) \leq \log^+ \psi + \log(1/J) \leq \psi + \log(1/J)\,.$$

Since $U \geq 1$, we have $\widetilde{G}(\theta) \geq C_\alpha$, with $C_\alpha = (3/4)^{\alpha+1} > 0$; hence $\log \widetilde{G}(\theta) \geq \log C_\alpha > -\infty$. Therefore, to get $\log G \in L^{1,\infty}(0, 2\pi)$ and finish the proof of 2) of Theorem 5.5, it suffices to prove that $\log \widetilde{G} \in L^{1,\infty}(0, 2\pi)$, and for that, to prove that $\log(1/J) \in L^{1,\infty}(0, 2\pi)$. This is the object of the following theorem.

Theorem 5.8. *Let $v \in H^\infty$ such that $\|v\|_\infty = 1$ and set*

$$I_v(\theta) = \inf\{|v(r\,\mathrm{e}^{i\theta})|\,;\ 1/2 \leq r < 1\}\,. \tag{5.13}$$

Then $\log(1/I_v) \in L^{1,\infty}(0, 2\pi)$.

Proof. We can write $v(z) = B(z)\, v_0(z)$, where B is the Blaschke product whose zeros are those of v, and v_0 does not vanish. Since

$$I_v \geq I_B \times I_{v_0}\,,$$

it suffices to prove that $\log(1/I_B) \in L^{1,\infty}(0, 2\pi)$ and $\log(1/I_{v_0}) \in L^{1,\infty}(0, 2\pi)$.

Case of a non vanishing function.

We can write $v_0 = \exp(-h)$, where $h\colon \mathbb{D} \to \{\mathfrak{Re}\, z > 0\}$. We have $h = u + i\widetilde{u}$, where $u = \mathfrak{Re}\, h$ and $\widetilde{u}$ is the conjugate function of u. Since $u > 0$, $u = \mathrm{P}[\mu]$ is the Poisson integral of a positive measure μ, and we have

$$u(r\,\mathrm{e}^{i\theta}) \leq C\, M_\mu(\theta) \quad \forall r \in [0, 1)\,,$$

where M_μ is the Hardy-Littlewood maximal function of μ. Then:

$$|v_0(r\,\mathrm{e}^{i\theta})| \geq \exp\big(-C\, M_\mu(\theta)\big)\,;$$

so $I_{v_0}(\theta) \geq \exp\big(-C\, M_\mu(\theta)\big)$, and $\log\big(1/I_{v_0}(\theta)\big) \leq C\, M_\mu(\theta)$. Since $M_\mu \in L^{1,\infty}(0, 2\pi)$, by Kolmogorov's theorem, we obtain that $\log(1/I_{v_0}) \in L^{1,\infty}(0, 2\pi)$.

Case of a Blaschke product.

This case will follow from the next result. We note $\arg z$ the principal argument of z: $-\pi < \arg z \leq \pi$.

Proposition 5.9. *Let B_0 be a Blaschke product whose zeros a_n have modulus greater or equal to some positive constant c, say $c = 3/4$. Then there exist $f \in L^1(-\pi, \pi)$ and $u = \mathrm{P}[w]$, with $w \in L^1(-\pi, \pi)$, such that*

$$\log\big(1/|B_0(z)|\big) \leq f(\arg z) + u(z)\,, \quad \textit{for all } z \in \mathbb{D}\,. \tag{5.14}$$

For $a \in \mathbb{D}$, we denote $\Delta(a, 1/2)$ the pseudo-hyperbolic disk of center a and radius $1/2$.

We begin by two lemmas.

Lemma 5.10. *For $a \in \mathbb{D}$, we set*

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}\,,$$

as well as $I_a = I_{\varphi_a}$ and $G_a = \log(1/I_a)$. Then, for every $a \in \mathbb{D}$, we have $G_a \in L^1(-\pi, \pi)$.

First, we have $G_a \geq 0$. Then:

$$|\varphi_a(z)| = \frac{|a - z|}{|1 - \bar{a}z|} \geq \frac{|z - a|}{2}\,;$$

so, it suffices to give a lower estimate of $|z - a|$.

We separate two cases.

• First case: $|a| \leq 1/4$. Then we have $|r\,\mathrm{e}^{i\theta} - a| \geq 1/4$ when $1/2 \leq r < 1$; hence $G_a(\theta) \leq \log 8$ for all θ and $G_a \in L^1(-\pi, \pi)$.

• Second case: $|a| > 1/4$. We can assume that $1/4 < a < 1$. If $z = r\,\mathrm{e}^{i\theta}$, then, for $|\theta| \leq \pi/2$:

$$|z - a| \geq \mathrm{dist}\,(a, R_\theta) = a\,|\sin\theta|\,,$$

where R_θ is the ray passing through 0 and $\mathrm{e}^{i\theta}$, so

$$G_a(\theta) \leq \left|\log\left(\frac{a}{2}\,|\sin\theta|\right)\right|$$

and $G_a \in L^1(-\pi, \pi)$. □

Lemma 5.11. *There is a positive constant C such that, for $3/4 \leq a < 1$ and $h = 1 - a$, we have:*

$$|\theta| \leq Ch \quad \textit{when } z = r\,\mathrm{e}^{i\theta} \in \Delta(a, 1/2)\,. \tag{5.15}$$

Proof. The pseudo-hyperbolic disk $\Delta(a,1/2)$ is equal to the Euclidean disk $D(\widetilde{a},R)$, with

$$\widetilde{a}=\frac{3}{4-|a|^2}\,a \quad\text{and}\quad R=2\,\frac{1-|a|^2}{4-|a|^2}$$

(see [5, page 3]). For $0<a<1$ and $h=1-a$, we have $R\leq 4\,h/3$ and $3\,a/4\leq\widetilde{a}\leq a$. Hence $\Delta(a,1/2)$ is contained in the angular sector of vertex 0 and half-angle θ_a such that $\sin\theta_a=R/\widetilde{a}\leq(4\,h/3)/(3a/4)$ (see Figure 1). For $3/4\leq a<1$, that gives $\sin\theta_a\leq(64/27)\,h$. It follows that there is $C>0$ ($C=64\pi/54$ works) such that $|\theta|\leq C\,h$ when $z=r\,\mathrm{e}^{i\theta}\in\Delta(a,1/2)$. □

Proof of Proposition 5.9. We restrict ourselves, for the time, to $3/4\leq a<1$.

Note that, since $3/4\leq a<1$ and $a=1-h$, we have $0<h\leq 1/4$.

• Let $z=r\,\mathrm{e}^{i\theta}\in\Delta(a,1/2)$.

We write

$$|\varphi_a(z)|=\frac{|z-a|}{a\left|z-\dfrac{1}{a}\right|}\geq\frac{|z-a|}{a\left[|z-a|+\left(\dfrac{1}{a}-a\right)\right]}=\frac{1}{a+\dfrac{1-a^2}{|z-a|}}\,;$$

so, if $z\in\Delta(a,1/2)$, and $z=r\,\mathrm{e}^{i\theta}$, we have $|\theta|<\pi/2$; hence, when $\theta\neq 0$:

$$|\varphi_a(z)|\geq\frac{1}{a+\dfrac{1-a^2}{a\,|\sin\theta|}}\,\cdot$$

It follows from Lemma 5.11 that, for another constant C:

$$\frac{1}{I_a(\theta)}\leq a+\frac{1-a^2}{a\,|\sin\theta|}\leq C\,\frac{h}{|\theta|}\,,$$

so

$$G_a(\theta)\leq\log\left(C\,\frac{h}{|\theta|}\right)$$

and

$$\int_0^{Ch}G_a(\theta)\,d\theta\leq\left[\theta\log\left(C\,\frac{h}{\theta}\right)+\theta\right]_0^{Ch}=Ch\,.$$

Setting, for $|\theta|\leq\pi$ (recall that $a=1-h$):

$$f_a(\theta)=\log\left(C\,\frac{h}{|\theta|}\right)\mathbb{1}_{[-Ch,Ch]}(\theta)\,, \tag{5.16}$$

we hence have:

$$\log\left(\frac{1}{|\varphi_a(z)|}\right)\leq f_a(\theta)\quad\text{for } z=r\,\mathrm{e}^{i\theta}\in\Delta(a,1/2) \tag{5.17}$$

(since then $|\theta|\leq Ch$). Moreover, we have

$$\|f_a\|_1\leq 2\,Ch\,. \tag{5.18}$$

• Now, let $z \in \mathbb{D} \setminus \Delta(a, 1/2)$.

Let D_a be the (Euclidean) disk of diameter $[c, 1/c]$, where c is the point of the segment $\partial\Delta(a, 1/2) \cap [\,0, 1)$ such that $0 < c < a$, and

$$A_a = \partial\mathbb{D} \cap D_a \,. \tag{5.19}$$

We write simply $A_a = A$ thereafter.

We set

$$w_a = 2 \log 2 \, \mathbb{1}_A \quad \text{and} \quad u_a = \mathrm{P}[w_a] \,, \tag{5.20}$$

the Poisson integral of w_a.

We have, for some positive constant C:

$$\|w_a\|_1 = 2 \log 2 \, m(A) \leq Ch \,. \tag{5.21}$$

In fact, the diameter of D_a is $\frac{1}{c} - c$ and $\frac{1}{2} = |\varphi_a(c)| = \frac{a-c}{1-ac}$, so

$$c = \frac{2\,a - 1}{2 - a} = \frac{1 - 2h}{1 + h} = 1 - 3h + \mathrm{o}\,(h) \,,$$

and the diameter of D_a is equal to $6h + \mathrm{o}\,(h)$.

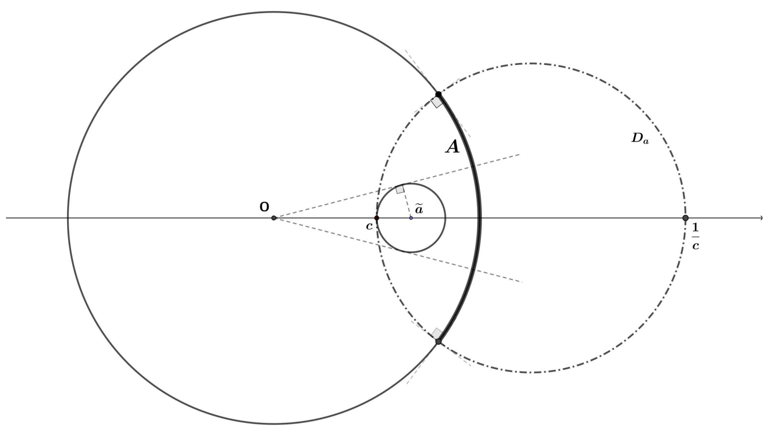

Figure 1: *circles*

Lemma 5.12. *Let $0 < \theta_0 < \pi/2$ and A be the arc of $\partial\mathbb{D}$ with end points $\mathrm{e}^{-i\theta_0}$ and $\mathrm{e}^{i\theta_0}$ and midpoint 1. Then, if D_A is the disk orthogonal to $\partial\mathbb{D}$ passing through $\mathrm{e}^{-i\theta_0}$ and $\mathrm{e}^{i\theta_0}$, we have*

$$\mathrm{P}[\mathbb{1}_A] \geq 1/2 \quad on \quad \mathbb{D} \cap D_A \,.$$

Proof. Let $T \colon \overline{\mathbb{D}} \to \{z \in \mathbb{C} \,;\ \Im\mathrm{m}\, z \geq 0\} \cup \{\infty\}$ be the conformal map mapping A onto $\mathbb{R}_- \cup \{\infty\}$ and $\partial\mathbb{D} \setminus A$ onto $\mathbb{R}_+^*$. The unique bounded solution to the

Dirichlet problem with data 1 on $\mathbb{R}_- \cup \{\infty\}$ and 0 on $\mathbb{R}_+^*$ is $U(\zeta) = \frac{1}{\pi} \arg \zeta$; hence $\mathrm{P}[\mathbb{1}_A](z) = \frac{1}{\pi} \arg(Tz)$.

We have $U(\zeta) \geq 1/2$ if and only if ζ is in the closed left-hand side upper quadrant Q. But $T^{-1}(i\mathbb{R})$ is the arc orthogonal to $\partial\mathbb{D}$ and passing through $\mathrm{e}^{-i\theta_0}$ and $\mathrm{e}^{i\theta_0}$; hence $T^{-1}(i\mathbb{R}) = A$ and, since D_A is orthogonal to $\mathbb{D}$, we have $T\big(\overline{\mathbb{D} \cap D_A}\big) = Q \cup \{\infty\}$. Therefore $\mathrm{P}[\mathbb{1}_A](z) \geq 1/2$ when $z \in \mathbb{D} \cap D_A$. □

By this lemma, we have

$$u_a \geq \log 2 \quad \text{on } \mathbb{D} \cap D_a \,.$$

In particular, since $\Delta(a, 1/2) \subseteq \mathbb{D} \cap D_a$, we have $u_a \geq \log 2$ on $\partial\Delta(a, 1/2)$. Of course u_a is equal to $\mathbb{1}_A$ on $\partial\mathbb{D}$, so is positive on $\partial\mathbb{D}$.

On the other hand, the function $\log(1/|\varphi_a|)$ is harmonic in $\mathbb{D} \setminus \Delta(a, 1/2)$ and is equal to 0 on $\partial\mathbb{D}$ and to $\log 2$ on $\partial\Delta(a, 1/2)$. Therefore, since $\partial\mathbb{D} \cup \partial\Delta(a, 1/2)$ is the boundary of $\mathbb{D} \setminus \Delta(a, 1/2)$, we obtain that

$$u_a(z) \geq \log\big(1/|\varphi_a(z)|\big) \quad \text{for } z \in \mathbb{D} \setminus \Delta(a, 1/2) \,. \tag{5.22}$$

• It follows from (5.17) and (5.22) that

$$\log\big(1/|\varphi_a(z)|\big) \leq f_a(\arg z) + u_a(z) \quad \text{for all } z \in \mathbb{D} \,. \tag{5.23}$$

• We are now able to finish the proof.

We write

$$B_0 = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \, \varphi_{a_n}$$

with $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ and $|a_n| \geq 3/4$. We have, by (5.23):

$$\log\big(1/|B_0(z)|\big) \leq \sum_{n=1}^{\infty} f_{|a_n|}\big(\arg(\bar{a}_n z)\big) + \sum_{n=1}^{\infty} u_{|a_n|}(z \, \mathrm{e}^{-i \arg a_n}) \,,$$

that is

$$\log\big(1/|B_0(z)|\big) \leq f(\arg z) + u(z) \,, \tag{5.24}$$

with

$$f(\theta) = \sum_{n=1}^{\infty} f_{|a_n|}\big(\arg(\bar{a}_n \, \mathrm{e}^{i\theta})\big)$$

and $u = \mathrm{P}[w]$, where

$$w = 2 \log 2 \sum_{n=1}^{\infty} \mathbb{1}_{\mathrm{e}^{i \arg a_n} A_{|a_n|}} \,.$$

We have $f, w \in L^1(-\pi, \pi)$, by invariance of the Lebesgue measure and since, we have $\|f_{|a_n|}\|_1 \leq C\,(1 - |a_n|)$ and $\|w_{|a_n|}\|_1 \leq C\,(1 - |a_n|)$, and $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. That finishes the proof of Proposition 5.9. □

Now, it is easy to end the proof of Theorem 5.8 and hence that of Theorem 5.5.

End of the proof of Theorem 5.8. We only have to write

$$B = \Big(\prod_{|a_n|<3/4} \frac{|a_n|}{a_n}\, \varphi_{a_n} \Big) \times B_0$$

where B_0 is the Blaschke product made with the zeros of B of modulus $\geq 3/4$ (as usual $|a_n|/a_n = 1$ if $a_n = 0$).

Then, with the notation of Lemma 5.10, if

$$G = \sum_{|a_n|<3/4} G_{a_n} \,,$$

we have, by Proposition 5.9, if $|z| \geq 1/2$:

$$\log \big(1/|B(z)|\big) \leq G(\arg z) + f(\arg z) + u(z)\,,$$

with $f \in L^1(-\pi,\pi)$ and $u = \mathrm{P}[w]$ with $w \in L^1(-\pi,\pi)$.

Since the maximal radial function of $u = \mathrm{P}[w]$ is smaller than its Hardy-Littlewood maximal function M_w (actually equivalent: see [19, Theorem 11.20 and Exercise 19]), by a well-known theorem of Hardy and Littlewood, we get:

$$\sup_{1/2\leq r<1} \log \frac{1}{|B(r\,\mathrm{e}^{i\theta})|} \leq G(\theta) + f(\theta) + M_w(\theta)\,. \tag{5.25}$$

Now, $G \in L^1(-\pi,\pi)$, by Lemma 5.10, and $M_w \in L^{1,\infty}(-\pi,\pi)$, by the Kolmogorov theorem; therefore $\log(1/|B|) \in L^{(1,\infty)}(-\pi,\pi)$, and that finishes the proof of Theorem 5.8. □

Remark. The proofs of Theorem 5.5, 2) and Theorem 5.8 show that if the weighted Bergman space $\mathfrak{B}^2_U$ of analytic functions f such that $\int_{\mathbb{D}} |f|^2\, U\, dA < \infty$, contains a function $v \in H^\infty$, with $v(0) = 1$, then $\log(1/I_U) \in L^{1,\infty}(0,2\pi)$.

The result of Theorem 5.8 is essentially sharp, as said by the following result.

Theorem 5.13. *There exists $v \in H^\infty$, $v \not\equiv 0$, such that* $\log(1/I_v) \notin L^1(0,2\pi)$.

Proof. We start with

$$\sigma(\theta) = \begin{cases} 1/[\theta(\log\theta)^2]\,, & 0 < \theta \leq 1/2\,, \\ 0 & \text{elsewhere.} \end{cases}$$

We have $\sigma \in L^1(0,2\pi)$, and we consider $u = \mathrm{P}[\sigma]$. Then u is positive and, since the Poisson kernel is positive and decreasing on $[0,\pi]$, we have, for $0 < \theta \leq 1/2$:

$$\begin{aligned} u(\rho\,\mathrm{e}^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{P}_\rho(\theta - t)\,\sigma(t)\,dt \geq \frac{1}{2\pi}\int_0^{\theta} \mathrm{P}_\rho(\theta - t)\,\sigma(t)\,dt \\ &\geq \frac{1}{2\pi}\int_0^{\theta} \mathrm{P}_\rho(\theta)\,\sigma(t)\,dt = \frac{1}{2\pi}\,\mathrm{P}_\rho(\theta)\,\frac{1}{\log(1/\theta)}\,. \end{aligned}$$

Taking $\rho = 1 - \theta$, we have, as θ goes to 0:

$$\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2} = \frac{1-\rho^2}{(1-\rho)^2+2\rho(1-\cos\theta)} \sim \frac{\theta(2-\theta)}{\theta^2+2(1-\theta)\,\theta^2/2} \sim \frac{1}{\theta}\,.$$

Hence:

$$u\big((1-\theta)\,\mathrm{e}^{i\theta}\big) \geq \frac{C}{\theta}\,\frac{1}{\log(1/\theta)}\,.$$

Therefore

$$\sup_{1/2\leq\rho<1} u(\rho\,\mathrm{e}^{i\theta}) \geq \frac{C}{\theta\log(1/\theta)}\,.$$

Let now $g = u + i\widetilde{u}$ and $v = \exp(-g)$. We have $|v| = \mathrm{e}^{-u}$ and hence

$$I_v(\theta) \leq \exp\left(-\frac{C}{\theta\log(1/\theta)}\right)$$

and

$$\log(1/I_v) \geq \frac{C}{\theta\log(1/\theta)}\,.$$

Therefore $\log(1/I_v) \notin L^1(0,2\pi)$. □

5.3 An example

The fact that, in Theorem 5.5, the function U has the particular form given in (5.8), in particular is logarithmically-subharmonic, is important. In fact, we have the following result.

Theorem 5.14. *There exist a continuous function $U\colon \mathbb{D}\to\mathbb{C}$, with $U\geq 1$ and an analytic function $w\colon \mathbb{D}\to\mathbb{C}$, $w\not\equiv 0$, such that:*

$$\int_{\mathbb{D}} |w|^2\,U\,dA < \infty\,,$$

but

$$\int_0^1 U(r\,\mathrm{e}^{i\theta})\,dr = \infty\,, \quad \textit{for almost all } \theta\,.$$

To prove this, we use the following weak form of a result of Kahane and Katznelson [8].

Theorem 5.15 (Kahane-Katznelson)**.** *Given any positive increasing function $\omega\colon (0,1)\to(0,\infty)$ such that $\omega(r)\underset{r\to 1}{\longrightarrow}\infty$ and any pair of measurable functions $g,h\colon [0,2\pi]\to\overline{\mathbb{R}} = [-\infty,+\infty]$, there exists an analytic function $F\colon \mathbb{D}\to\mathbb{C}$ such that*

1) $\max_{|z|=r}|F(z)| = \mathrm{o}\big(\omega(r)\big)$ *as r goes to 1;*
2) $\lim_{r\to 1}\mathfrak{Re}\,F(r\,\mathrm{e}^{i\theta}) = g(\theta)$ *and* $\lim_{r\to 1}\mathfrak{Im}\,F(r\,\mathrm{e}^{i\theta}) = h(\theta)$, *for almost all $\theta\in[0,2\pi]$.*

Proof of Theorem 5.14. The Kahane-Katznelson theorem shows that there exists a function $w = \exp(-F)$ belonging to $\mathfrak{B}^2$, and even in $\bigcap_{\beta>-1} \mathfrak{B}^2_\beta$, if we want, taking, for instance, $\omega(r) = \sqrt{\log\big(1/(1-r)\big)}$, such that $\lim_{r\to 1} w(r\,\mathrm{e}^{i\theta}) = 0$ for almost every $\theta \in [0, 2\pi]$. We may also assume that $w(0) = 1$.

By the Egorov theorem, we get, for all $n \geq 1$, numbers $\rho_n \in (1 - \frac{1}{n}\,, 1)$ and measurable sets $A_n \subseteq \mathbb{T}$ such that

1) $|w(r\,\mathrm{e}^{i\theta})|^2 \leq 2^{-n}$ for $\mathrm{e}^{i\theta} \in A_n$ and $\rho_n \leq r < 1$;
2) $m(\mathbb{T} \setminus A_n) < 2^{-n}$.

By the regularity of the measure, we can assume that the sets A_n are closed.

Let ρ'_n and ρ''_n such that $\rho_n < \rho'_n < \rho''_n < \frac{1+\rho_n}{2}$ and $\rho''_n - \rho'_n \geq (1-\rho_n)/3$, and

$$E_n = \{r\,\mathrm{e}^{i\theta}\,;\ \mathrm{e}^{i\theta} \in A_n\,,\ \rho'_n \leq r \leq \rho''_n\}\,.$$

By the continuity of w, there is an open neighborhood G_n of E_n, with closure contained in $\mathbb{D}$, such that $\rho_n < |z| < (1+\rho_n)/2$ and $|w(z)|^2 < 2^{-n+1}$ for $z \in G_n$. We have

$$\int_{G_n} \frac{2}{1-\rho_n}\,|w(z)|^2\,dA(z) \leq \frac{1}{\pi}\int_0^{2\pi}\bigg(\int_{\rho_n}^{\frac{1+\rho_n}{2}} \frac{2}{1-\rho_n}\,2^{-n+1}\,dr\bigg)\,d\theta \leq 2^{-n+2}\,.$$

The Urysohn lemma gives a continuous function $U_n\colon \mathbb{D} \to \big[0, \frac{2}{1-\rho_n}\big]$ such that:

a) $U_n(z) = 0$ if $z \notin G_n$;
b) $U_n(z) = 2/(1-\rho_n)$ for $z \in E_n$;
c) $\displaystyle\int_{\mathbb{D}} |w|^2\,U_n\,dA \leq 2^{-n+2}$.

Then

$$U = 1 + \sum_{n=1}^\infty U_n$$

is continuous on $\mathbb{D}$, because the sum is locally finite, since $U_n = 0$ out of G_n. We have, by c), and since $w \in \mathfrak{B}^2$:

$$\int_{\mathbb{D}} |w|^2\,U\,dA = \int_{\mathbb{D}} |w|^2\,dA + \sum_{n=1}^\infty \int_{\mathbb{D}} |w|^2\,U_n\,dA < \infty\,.$$

Moreover, since $\sum_{n=1}^\infty m(\mathbb{T} \setminus A_n) < \infty$, for almost all θ, there exists $N(\theta) \geq 1$ such that $\mathrm{e}^{i\theta} \in A_n$ for all $n \geq N(\theta)$. Hence, for these θ:

$$\begin{aligned}\int_0^1 U(r\,\mathrm{e}^{i\theta})\,dr &\geq \sum_{n=N(\theta)}^\infty \int_{\rho'_n}^{\rho''_n} U_n(r\,\mathrm{e}^{i\theta})\,dr \geq \sum_{n=N(\theta)}^\infty \int_{\rho'_n}^{\rho''_n} \frac{2}{1-\rho_n}\,dr \\ &= \sum_{n=N(\theta)}^\infty (\rho''_n - \rho'_n)\,\frac{2}{1-\rho_n} \geq \sum_{n=N(\theta)}^\infty \frac{2}{3} = \infty\,.\end{aligned}$$

That finishes the proof of Theorem 5.14. □

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