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Boundedness of composition operators on general weighted Hardy spaces of analytic functions

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Abstract. We characterize the (essentially) decreasing sequences of positive numbers $\beta = (\beta_n)$ for which all composition operators on $H^2(\beta)$ are bounded, where $H^2(\beta)$ is the space of analytic functions f in the unit disk such that $\sum_{n=0}^{\infty} |c_n|^2 \beta_n < \infty$ if $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We also give conditions for the boundedness when β is not assumed essentially decreasing.

MSC 2010 primary: 47B33 ; secondary: 30H10

Key-words automorphism of the unit disk; composition operator; Δ_2 -condition; multipliers; weighted Hardy space

1 Introduction

Let $\beta = (\beta_n)_{n \geq 0}$ be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space $H^2(\beta)$ is the set of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty.$$

It is a Hilbert space of analytic functions on \mathbb{D} . When $\beta_n \equiv 1$, we recover the usual Hardy space H^2 .

Recall that a symbol is a (non constant) analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, and the associated composition operator $C_\varphi: H^2(\beta) \rightarrow \mathcal{H}ol(\mathbb{D})$ is defined (formally) as:

$$(1.3) \quad C_\varphi(f) = f \circ \varphi.$$

An important question in the theory is to decide when C_φ is bounded on $H^2(\beta)$, i.e. when $C_\varphi: H^2(\beta) \rightarrow H^2(\beta)$. When $H^2(\beta)$ is the usual Hardy space H^2 (i.e. when $\beta_n \equiv 1$), it is well-known ([18, pp. 13–17]) that all symbols generate bounded composition operators. On the other hand, for the Dirichlet space, corresponding to $\beta_n = n + 1$, not all composition operators are bounded. Note that, by definition of the norm of $H^2(\beta)$, all rotations R_θ , $\theta \in \mathbb{R}$, induce bounded composition operators on $H^2(\beta)$ and send isometrically $H^2(\beta)$ into itself.

Our goal in this paper is characterizing the (non-increasing) sequences β for which all composition operators act boundedly on the space $H^2(\beta)$, i.e. send $H^2(\beta)$ into itself. We will prove the following result, where $T_a(z) = \frac{a+z}{1+\bar{a}z}$ for $a, z \in \mathbb{D}$.

Theorem 1.1. *Let β be an essentially decreasing sequence of positive numbers. The following assertions are equivalent:*

- 1) *all composition operators are bounded on $H^2(\beta)$;*
- 2) *all maps T_a , for $0 < a < 1$, induce bounded composition operators C_{T_a} on $H^2(\beta)$;*
- 3) *for some $a \in (0, 1)$, the map T_a induces a bounded composition operator C_{T_a} on $H^2(\beta)$;*
- 4) *β satisfies the Δ_2 -condition.*

For the notion of essentially decreasing sequence, see Definition 2.1; the Δ_2 -condition is defined in (2.4). Actually, we will obtain Theorem 1.1 by gluing Theorem 3.3 and Theorem 4.1.

Most of the existing works with weighted Hardy spaces concern the case

$$(1.4) \quad \beta_n = \int_0^1 t^n d\sigma(t)$$

where σ is a positive measure on $(0, 1)$. More specifically the following definition is often used. Let $G: (0, 1) \rightarrow \mathbb{R}_+$ be an integrable function and let H_G^2 be the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that:

$$(1.5) \quad \|f\|_{H_G^2}^2 := \int_{\mathbb{D}} |f(z)|^2 G(1 - |z|^2) dA(z) < \infty.$$

Such weighted Bergman type spaces are used, for instance, in [11], [10] and in [14]. We have $H_G^2 = H^2(\beta)$ with:

$$(1.6) \quad \beta_n = 2 \int_0^1 r^{2n+1} G(1 - r^2) dr = \int_0^1 t^n G(1 - t) dt,$$

and the sequence $\beta = (\beta_n)_n$ is non-increasing.

In Shapiro's presentation, the main point is the case $\varphi(0) = 0$ and a subordination principle for subharmonic functions (Littlewood's subordination principle). The case of automorphisms is claimed simple, using an integral representation for the norm and some change of variable. When β is defined as in (1.4), one disposes of integral representations for the norm in $H^2(\beta)$, and, as in the Hardy space case, this integral representation rather easily gives the boundedness of C_{T_a} on H^2 , where

$$(1.7) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}$$

for $a \in \mathbb{D}$. But the above representation (1.4) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence β , namely its *complete monotony*. When integral representations fail, we have to work with bare hands. If the symbol vanishes at the origin, Kacnel'son's theorem gives a positive answer when β is essentially decreasing (see [2] or [13, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [8] (see [5, Theorem 6.3]), which itself uses a refinement by Rogosinski of Littlewood's principle ([5, Theorem 6.2]). So that the main issue remains the boundedness of C_{T_a} .

A polynomial minoration (see Definition 2.3 below) for β is necessary for any C_{T_a} to be bounded on $H^2(\beta)$ (Proposition 2.5) and we showed in [13, end of Section 3] that for $\beta_n = \exp(-\sqrt{n})$, C_{T_a} is never bounded on $H^2(\beta)$. But this polynomial minoration is not sufficient, as we will see in Theorem 1.1 below, which also evidences the basic role of the maps T_a in the question.

However, we construct a weight β which is not essentially decreasing and for which all composition operators with symbol vanishing at 0 are bounded (Theorem 3.9), though no map T_a with $0 < a < 1$ induces a bounded composition operator (Proposition 4.4).

For spaces of Bergman type $A_G^2 := H_G^2$, where $\tilde{G}(r) = G(1-r^2)$, defined as the spaces of analytic functions in \mathbb{D} such that $\int_{\mathbb{D}} |f(z)|^2 \tilde{G}(|z|) dA < \infty$, for a positive non-increasing continuous function \tilde{G} on $[0, 1)$, Kriete and MacCluer studied in [11] some analogous problems. They proved, in particular [11, Theorem 3] that, for:

$$\tilde{G}(r) = \exp\left(-B \frac{1}{(1-r)^\alpha}\right), \quad B > 0, \quad 0 < \alpha \leq 2,$$

and

$$\varphi(z) = z + t(1-z)^\beta, \quad 1 < \beta \leq 3, \quad 0 < t < 2^{1-\beta},$$

then C_φ is bounded on A_G^2 if and only if $\beta \geq \alpha + 1$.

Here

$$\beta_n = \int_0^1 t^n e^{-B/(1-\sqrt{t})^\alpha} dt;$$

and, since $\beta_n \approx \exp(-cn^{\alpha/(\alpha+1)})$, the sequence (β_n) does not satisfy the Δ_2 -condition, accordingly to our Theorem 3.3 below.

We end the paper with some miscellaneous remarks.

2 Definitions, notation, and preliminary results

The open unit disk of \mathbb{C} is denoted \mathbb{D} and we write \mathbb{T} its boundary $\partial\mathbb{D}$. We set $e_n(z) = z^n$, $n \geq 0$.

The weighted Hardy space $H^2(\beta)$ defined in the introduction is a Hilbert space with the canonical orthonormal basis

$$(2.1) \quad e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}} z^n, \quad n \geq 0,$$

and the reproducing kernel K_w given for all $w \in \mathbb{D}$ by

$$(2.2) \quad K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z) \overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \bar{w}^n z^n.$$

Note that (1.1) is necessary for $H^2(\beta)$ to consist of analytic functions in \mathbb{D} . Indeed the fact that $\sum_{n \geq 1} \frac{1}{n\sqrt{\beta_n}} z^n$ belongs to $H^2(\beta)$ and is analytic in \mathbb{D} implies (1.1). Note also that H^2 is continuously embedded in $H^2(\beta)$ if and only if β is bounded above. In particular, this is the case when β is non-increasing. In this paper, we need a slightly more general notion.

Definition 2.1. *A sequence of positive numbers $\beta = (\beta_n)$ is said essentially decreasing if, for some constant $C \geq 1$, we have, for all $m \geq n \geq 0$:*

$$(2.3) \quad \beta_m \leq C \beta_n.$$

Note that saying that β is essentially decreasing means that the shift operator on $H^2(\beta)$ is power bounded.

If β is essentially decreasing, and if we set:

$$\tilde{\beta}_n = \sup_{m \geq n} \beta_m,$$

the sequence $\tilde{\beta} = (\tilde{\beta}_n)$ is non-increasing and we have $\beta_n \leq \tilde{\beta}_n \leq C \beta_n$. In particular, the space $H^2(\beta)$ is isomorphic to $H^2(\tilde{\beta})$ and H^2 is continuously embedded in $H^2(\beta)$.

Definition 2.2. *The sequence of positive numbers $\beta = (\beta_n)$ is said to satisfy the Δ_2 -condition if there is a positive constant $\delta < 1$ such that, for all integers $n \geq 0$:*

$$(2.4) \quad \beta_{2n} \geq \delta \beta_n.$$

This terminology is given by analogy with that used for Orlicz functions.

Definition 2.3. *The sequence of positive numbers $\beta = (\beta_n)$ is said to have a polynomial minoration if there are positive constants δ and α such that, for all integers $n \geq 1$:*

$$(2.5) \quad \beta_n \geq \delta n^{-\alpha}.$$

That means that $H^2(\beta)$ is continuously embedded in the weighted Bergman space \mathfrak{B}_α^2 of the analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\|f\|_{\mathfrak{B}_\alpha^2}^2 := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty$$

since $\mathfrak{B}_\alpha^2 = H^2(\gamma)$ with $\gamma_n \approx n^{-\alpha}$.

The following simple proposition links those notions.

Proposition 2.4. *Let β be an essentially decreasing sequence of positive numbers. Then if β satisfies the Δ_2 -condition, it has a polynomial minoration.*

The converse does not hold.

Proof. Assume $\beta_m \leq C \beta_n$ for $m \geq n$ and $\beta_{2^k} \geq e^{-A} \beta_k$. Let now n be an integer ≥ 2 , and $k \geq 1$ the smallest integer such that $2^k \geq n$, so that $k \leq a \log n$ with a a positive constant. We get:

$$\beta_n \geq C^{-1} \beta_{2^k} \geq C^{-1} e^{-kA} \beta_1 \geq C^{-1} \beta_1 e^{-aA \log n} =: \rho n^{-\alpha},$$

with $\rho = C^{-1} \beta_1$ and $\alpha = aA$.

Let us now see that the converse does not hold. Let $\delta > 0$. We set $\beta_0 = \beta_1 = 1$ and for $n \geq 2$:

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)!$$

The sequence β is non-increasing.

For n and k as above, we have:

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta};$$

hence β has arbitrary polynomial minoration. However we have, for $k \geq 2$:

$$\frac{\beta_{2(k!)}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \xrightarrow[k \rightarrow \infty]{} 0,$$

so β fails to satisfy the Δ_2 -condition. □

For $a \in \mathbb{D}$, we define:

$$(2.6) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}, \quad z \in \mathbb{D}.$$

Recall that T_a is an automorphism of \mathbb{D} and that $T_a(0) = a$ and $T_a(-a) = 0$.

Though we do not need this, we may remark that $(T_a)_{a \in (-1,1)}$ is a group and $(T_a)_{a \in (0,1)}$ is a semigroup. It suffices to see that $T_a \circ T_b = T_{a*b}$, with:

$$(2.7) \quad a * b = \frac{a+b}{1+ab}.$$

Proposition 2.5. *Let $a \in (0, 1)$ and assume that T_a induces a bounded composition operator on $H^2(\beta)$. Then β has a polynomial minoration.*

Proof. Since

$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n},$$

we have $\|K_x\| \leq \|K_y\|$ for $0 \leq x \leq y < 1$.

We define by induction a sequence $(u_n)_{n \geq 0}$ with:

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = T_a(u_n).$$

Since $T_a(1) = 1$ (recall that $a \in (0, 1)$), we have:

$$1 - u_{n+1} = \int_{u_n}^1 T_a'(t) dt = \int_{u_n}^1 \frac{1 - a^2}{(1 + at)^2} dt;$$

hence

$$\frac{1 - a}{1 + a} (1 - u_n) \leq 1 - u_{n+1} \leq (1 - a^2)(1 - u_n).$$

Let $0 < x < 1$. We can find $N \geq 0$ such that $u_N \leq x < u_{N+1}$. Then:

$$1 - x \leq 1 - u_N \leq (1 - a^2)^N.$$

On the other hand, since $C_{T_a}^* K_z = K_{T_a(z)}$ for all $z \in \mathbb{D}$, we have:

$$\|K_x\| \leq \|K_{u_{N+1}}\| \leq \|C_{T_a}\| \|K_{u_N}\| \leq \|C_{T_a}\|^{N+1} \|K_{u_0}\| = \|C_{T_a}\|^{N+1}.$$

Let $s \geq 0$ such that $(1 - a^2)^{-s} = \|C_{T_a}\|$. We obtain:

$$(2.8) \quad \|K_x\| \leq \|C_{T_a}\| \frac{1}{(1 - x)^s}.$$

But

$$\|K_x\|^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k};$$

so we get, for any $k \geq 2$:

$$\frac{x^{2k}}{\beta_k} \leq \|C_{T_a}\|^2 \frac{1}{(1 - x)^{2s}}.$$

Taking $x = 1 - \frac{1}{k}$, we obtain $\beta_k \geq C k^{-2s}$. □

Remarks. 1) For example, when $\beta_n = \exp[-c(\log(n+1))^2]$, with $c > 0$, no T_a induces a bounded composition operator on $H^2(\beta)$, though all symbols φ with $\varphi(0) = 0$ are bounded, since β is decreasing, as we will see in Proposition 3.2.

2) For the Dirichlet space \mathcal{D}^2 , we have $\beta_n = n+1$, but all the maps T_a induce bounded composition operators on \mathcal{D}^2 (see [13, Remark before Theorem 3.12]). In this case β has a polynomial minoration though it is not bounded above.

3) However, even for decreasing sequences, a polynomial minoration for β is not enough for some T_a to induce a bounded composition operator. Indeed, we saw in Proposition 2.4 an example of a decreasing sequence β with polynomial minoration, but not sharing the Δ_2 -condition, and we will see in Theorem 4.1 that the Δ_2 -condition is needed for having some T_a inducing a bounded composition operator.

4) In [7], Eva Gallardo-Gutiérrez and Jonathan Partington give estimates for the norm of C_{T_a} , with $a \in (0, 1)$, when C_{T_a} is bounded on $H^2(\beta)$. More precisely, they proved that if β is bounded above and C_{T_a} is bounded, then

$$\|C_{T_a}\| \geq \left(\frac{1+a}{1-a}\right)^\sigma,$$

where $\sigma = \inf\{s \geq 0; (1-z)^{-s} \notin H^2(\beta)\}$, and

$$\|C_{T_a}\| \leq \left(\frac{1+a}{1-a}\right)^\tau,$$

where $\tau = \frac{1}{2} \sup \Re W(A)$, with A the infinitesimal generator of the continuous semigroup (S_t) defined as $S_t = C_{T_{\tanh t}}$, namely $(Af)(z) = f'(z)(1-z^2)$, and $W(A)$ its numerical range.

For $\beta_n = 1/(n+1)^\nu$ with $0 \leq \nu \leq 1$, the two bounds coincide, so they get $\|C_{T_a}\| = \left(\frac{1+a}{1-a}\right)^{(\nu+1)/2}$.

5) We saw in the proof of Proposition 2.5 that if C_{T_a} is bounded on $H^2(\beta)$ for some $a \in (0, 1)$, then the reproducing kernels K_w have, by (2.8), a *slow growth*:

$$(2.9) \quad \|K_w\| \leq \frac{C}{(1-|w|)^s}$$

for positive constants C and s . Actually, we have the following equivalence.

Proposition 2.6. *The sequence β has a polynomial minoration if and only if the reproducing kernels K_w of $H^2(\beta)$ have a slow growth.*

Proof. The sufficiency is easy and seen at the end of the proof of Proposition 2.5. For the necessity, we only have to see that:

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^\alpha |w|^{2n} \leq \frac{C}{(1-|w|^2)^{\alpha+1}}. \quad \square$$

3 Boundedness of composition operators

We study in this section conditions ensuring that all composition operators on $H^2(\beta)$ are bounded.

3.1 Conditions on the weight

We begin with this simple observation.

Proposition 3.1. *If all composition operators, and even if all composition operators with symbol vanishing at 0, are bounded on $H^2(\beta)$, then the sequence β is bounded above.*

Proof. Let $f \in H^\infty$. Write $f = A\varphi + f(0)$ where A is a constant and φ a symbol vanishing at 0. We have $\varphi = C_\varphi(z) \in H^2(\beta)$, by hypothesis. So that $f \in H^2(\beta)$ and $H^\infty \subseteq H^2(\beta)$. It follows (by the closed graph theorem) that there exists a constant M such that $\|f\|_{H^2(\beta)} \leq M\|f\|_\infty$ for all $f \in H^\infty$. Testing that with $f(z) = z^n$, we get $\beta_n \leq M^2$. \square

For symbols vanishing at 0, we have the following characterization.

Proposition 3.2. *The following assertions are equivalent:*

1) *all symbols φ such that $\varphi(0) = 0$ induce bounded composition operators C_φ on $H^2(\beta)$ and*

$$(3.1) \quad \sup_{\varphi(0)=0} \|C_\varphi\| < \infty;$$

2) *β is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3.1) is equivalent to:

$$\sup_{\varphi(0)=0} \|f \circ \varphi\| < \infty \quad \text{for all } f \in H^2(\beta).$$

Proof. 2) \Rightarrow 1) We may assume that β is non-increasing. Then the Goluzin-Rogosinski theorem ([5, Theorem 6.3]) gives the result; in fact, writing $f(z) = \sum_{n=0}^\infty c_n z^n$ and $(C_\varphi f)(z) = \sum_{n=0}^\infty d_n z^n$, it says that:

$$\|C_\varphi f\|^2 = |d_0|^2 \beta_0 + \sum_{n=1}^\infty |d_n|^2 \beta_n \leq |c_0|^2 \beta_0 + \sum_{n=1}^\infty |c_n|^2 \beta_n = \|f\|^2,$$

leading to C_φ bounded and $\|C_\varphi\| \leq 1$. Alternatively, we can use a result of Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]). This result was also proved by C. Cowen [4, Corollary of Theorem 7].

1) \Rightarrow 2) Set $M = \sup_{\varphi(0)=0} \|C_\varphi\|$. Let $m > n$, and take:

$$\varphi(z) = \varphi_{m,n}(z) = z \left(\frac{1 + z^{m-n}}{2} \right)^{1/n}.$$

Then $\varphi(0) = 0$ and $[\varphi(z)]^n = \frac{z^n + z^m}{2}$; hence

$$\frac{1}{4}(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \leq \|C_\varphi\|^2 \|e_n\|^2 \leq M^2 \beta_n,$$

so β is essentially decreasing. \square

Boundedness of β_n does not suffice. For example, let (β_n) be a sequence such that $\beta_{2k+2}/\beta_{2k+1} \xrightarrow[k \rightarrow \infty]{} \infty$ (for instance $\beta_{2k} = 1$ and $\beta_{2k+1} = 1/(k+1)$); if $\varphi(z) = z^2$, then $\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)}$; since $\|z^{2n+1}\|^2 = \beta_{2n+1}$, C_φ is not bounded on $H^2(\beta)$.

A more interesting example is the following. For $0 < r < 1$, let $\beta_n = \pi n r^{2n}$. This sequence is eventually decreasing, so it is essentially decreasing. The square $\|f\|_{H^2(\beta)}^2$ of the norm $\|f\|_{H^2(\beta)}$ is the area of the part of the Riemann surface on which $r\mathbb{D}$ is mapped by f . E. Reich [17], generalizing Goluzin's result [8] (see [5, Theorem 6.3]), proved that for all symbols φ such that $\varphi(0) = 0$, the composition operator C_φ is bounded on $H^2(\beta)$ and

$$\|C_\varphi\| \leq \sup_{n \geq 1} \sqrt{n} r^{n-1} \leq \frac{1}{\sqrt{2e}} \frac{1}{r \sqrt{\log(1/r)}}.$$

For $0 < r < 1/\sqrt{2}$, Goluzin's theorem asserts that $\|C_\varphi\| \leq 1$.

Note that this sequence β does not satisfy the Δ_2 -condition since $\beta_{2n}/\beta_n = 2r^{2n}$, Theorem 4.1 below states that no composition operator C_{T_a} is bounded.

However that the weight β is essentially decreasing is not necessary for the boundedness of all composition operators C_φ , with symbol φ vanishing at 0, as we will see later (Theorem 3.9).

3.2 Sufficient condition for the boundedness of composition operators – Part I

We now have one of the the main results of this section.

Theorem 3.3. *Let $H^2(\beta)$ be a weighted Hardy space with $\beta = (\beta_n)$ essentially decreasing and satisfying the Δ_2 -condition. Then all composition operators on $H^2(\beta)$ are bounded.*

For the proof, we need a lemma.

Lemma 3.4. *For $0 < a < 1$, we write:*

$$(3.2) \quad (T_a z)^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

Then, there are constants $b > 0$, $0 < C_1 < 1$, $C_2 > 1$ such that

$$|a_{m,n}| \leq \begin{cases} e^{-bn} & \text{if } m \leq C_1 n, \\ e^{-bm} & \text{if } m \geq C_2 n. \end{cases}$$

Proof. First take $0 < r < 1$; let:

$$M(r) = \sup_{|z|=r} |T_a(z)| = \sup_{|z|=r} \left| \frac{z+a}{1+az} \right|.$$

We have $M(r) < 1$, so we can write $M(r) = r^\rho$, for some positive $\rho = \rho(a)$.

The Cauchy inequalities give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = r^{\rho n - m},$$

and we obtain the first inequality by taking $r = e^{-\alpha}$ and adjusting C_1 .

Next, we use that T_a is analytic on $D(0, 1/a)$. Fix $1 < r =: e^\beta < 1/a$, with $\beta > 0$. Let $M(r) = e^\alpha$ with $\alpha > 0$. The Cauchy inequalities again give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = e^{\alpha n - \beta m},$$

and we obtain the second inequality by adjusting C_2 . \square

We will also need the following result of V. È Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]).

Theorem 3.5 (V. È Kacnel'son). *Let H be a separable complex Hilbert space and $(e_i)_{i \geq 0}$ a fixed orthonormal basis of H . Let $M: H \rightarrow H$ be a bounded linear operator. We assume that the matrix of M with respect to this basis is lower-triangular: $\langle M e_j | e_i \rangle = 0$ for $i < j$.*

Let $(\gamma_j)_{j \geq 0}$ be a non-decreasing sequence of positive real numbers and Γ the (possibly unbounded) diagonal operator such that $\Gamma(e_j) = \gamma_j e_j$, $j \geq 0$. Then the operator $\Gamma^{-1} M \Gamma: H \rightarrow H$ is bounded and moreover:

$$\|\Gamma^{-1} M \Gamma\| \leq \|M\|.$$

Proof of Theorem 3.3. We may, and do, assume that β is non-increasing.

Proposition 3.2 gives the result when $\varphi(0) = 0$.

It remains to show that all C_{T_a} , $a \in \mathbb{D}$, are bounded. Indeed, if $a = \varphi(0)$ and $\psi = T_{-a} \circ \varphi$, then $\psi(0) = 0$ and $\varphi = T_a \circ \psi$, so $C_\varphi = C_\psi \circ C_{T_a}$. Moreover, we have only to show that when $a \in [0, 1)$. Indeed, if $a \in \mathbb{D}$ and $a = |a| e^{i\theta}$, we have $T_a = R_\theta \circ T_{|a|} \circ R_{-\theta}$, so $C_{T_a} = C_{R_{-\theta}} \circ C_{T_{|a|}} \circ C_{R_\theta}$.

We consider the matrices

$$A = (a_{m,n})_{m,n \geq 0} \quad \text{and} \quad A_\beta = \left(\sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} \right)_{m,n \geq 0}.$$

Since $C_{T_a} e_n = T_a^n e_n$, the formula (3.2) shows that A is the matrix of C_{T_a} in H^2 with respect to the basis $(e_n)_{n \geq 0}$. On the other hand, A_β is the matrix of C_{T_a} in $H^2(\beta)$ with respect to the basis $(e_n^\beta)_{n \geq 0}$. We note that $A_\beta = B A B^{-1}$, where B is the diagonal matrix with values $\sqrt{\beta_0}, \sqrt{\beta_1}, \dots$ on the diagonal.

Since C_{T_a} is a bounded composition operators on H^2 , the matrix A defines a bounded operator on ℓ_2 . We have to show that A_β also, i.e. $\|A_\beta\| < \infty$.

For that purpose, we split A and A_β into several sub-matrices.

Let N be an integer such that $N \geq 2/C_1$, where C_1 is defined in Lemma 3.4 (actually, the proof of that lemma shows that we can take C_1 such that $1/C_1$ is

an integer, so we could take $N = 2/C_1$). Let $I_0 = [0, N[$ $J_0 = [N, +\infty[$ and for $k = 1, 2, \dots$:

$$I_k = [N^k, N^{k+1}[\quad \text{and} \quad J_k = [N^{k+1}, +\infty[.$$

We define the matrices D_β and R_β , whose entries are respectively:

$$d_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if} & (m, n) \in \bigcup_{k=0}^{\infty} (I_k \times I_k) \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$r_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if} & (m, n) \in \bigcup_{k=0}^{\infty} (I_k \times I_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

We also define the matrix S_β with entries:

$$s_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if} & (m, n) \in \bigcup_{k=0}^{\infty} (J_k \times I_k) \\ 0 & \text{elsewhere.} \end{cases}$$

Matrices D , R , and S are constructed in the same way from A and we set $U = A - (D + R + S)$.

Now, let H_k be the subspace of the sequences $(x_n)_{n \geq 0}$ in ℓ_2 such that $x_n = 0$ for $n \notin I_k$, i.e. $H_k = \text{span}\{e_n; n \in I_k\}$, and let P_k be (the matrix of) the orthogonal projection of ℓ_2 with range H_k . We have:

$$D = \sum_{k=0}^{\infty} P_k A P_k \quad \text{and} \quad R = \sum_{k=0}^{\infty} P_k A P_{k+1},$$

where $D_k = P_k A P_k$ is the matrix with entries $a_{m,n}$ when $(m, n) \in I_k \times I_k$ and 0 elsewhere, and $R_k = P_k A P_{k+1}$ the matrix with entries $a_{m,n}$ when $(m, n) \in I_k \times I_{k+1}$ and 0 elsewhere.

$$\left(\begin{array}{c|c|c|c} \boxed{D_0} & \boxed{R_0} & & \\ \hline & D_1 & \boxed{R_1} & \\ \hline S_0 & S_1 & D_2 & R_2 \\ \hline & & S_2 & \end{array} \right) U$$

Since the subspaces H_k are orthogonal, the matrices D and R induce bounded operators on ℓ_2 , and

$$(3.3) \quad \|D\| \leq \|A\|, \quad \|R\| \leq \|A\|.$$

Now, for $k \geq 1$, let B_k be the diagonal matrix whose entries are $b_{m,m} = \sqrt{\beta_m}$ if $m \in I_k$ and $b_{m,n} = 0$ otherwise.

Then $P_k D_\beta P_k = P_k B_k D B_k^{-1} P_k$, so

$$\|P_k D_\beta P_k\| \leq \|B_k\| \|B_k^{-1}\| \|D\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_k} \frac{1}{\sqrt{\beta_j}} \|A\|.$$

But the weight β satisfies the Δ_2 -condition: $\beta_{2l} \geq \delta_0 \beta_l$, and it follows that for every $l \geq 1$:

$$\beta_{N^{2l}} \geq \delta^2 \beta_l,$$

for some other constant δ , chosen small enough to have $\|P_0 D_\beta P_0\| \leq \delta^{-1} \|A\|$. Since β is non-increasing, we have $\beta_j \geq \delta^2 \beta_{N^k}$ for $N^k \leq j \leq N^{k+1}$. In particular $\max_{j \in I_k} \sqrt{\beta_j} \leq \delta^{-1} \min_{j \in I_k} \sqrt{\beta_j}$ and

$$\|P_k D_\beta P_k\| \leq \delta^{-1} \|A\|.$$

Hence, by orthogonality of the subspaces H_k :

$$(3.4) \quad \|D_\beta\| \leq \delta^{-1} \|D\|.$$

In the same way, we have $P_k R_\beta P_k = P_k B_k D B_{k+1}^{-1} P_k$, so:

$$\|P_k R_\beta P_k\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_{k+1}} \frac{1}{\sqrt{\beta_j}} \|A\| = \max_{(m,n) \in I_k \times I_{k+1}} \sqrt{\frac{\beta_m}{\beta_n}} \|A\|.$$

But, when $(m,n) \in I_k \times I_{k+1}$, we get

$$\beta_n \geq \beta_{N^{k+2}} \geq \delta^2 \beta_{N^k} \geq \delta^2 \beta_m.$$

Hence

$$(3.5) \quad \|R_\beta\| \leq \delta^{-1} \|R\|.$$

Next, consider $U = A - (D + R + S)$; we can compute its Hilbert-Schmidt norm using Lemma 3.4. Note that $u_{m,n} \neq 0$ only (if it happens) when $m \in I_k$ for some $k \geq 0$ and $n \geq \min I_{k+2} \geq N^{k+2} > Nm$, since $m \in I_k$, so only when $m \leq C_1 n$. We have, since then $u_{m,n} = a_{m,n}$:

$$\|U\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m \leq C_1 n} |a_{m,n}|^2 \leq \sum_{n=0}^{\infty} \sum_{m \leq C_1 n} e^{-2bn} \leq \sum_{n=0}^{\infty} C_1 n e^{-2bn} < \infty.$$

Consequently, with (3.3), we have $\|S\| \leq \|A\| + \|D\| + \|R\| + \|U\| < \infty$.

Now, since S is a lower-triangular matrix and β is non-increasing, we can use the result of V. È Kacnel'son (Theorem 3.5), with $\gamma_j = 1/\sqrt{\beta_j}$. We get that S_β defines a bounded operator and $\|S_\beta\| \leq \|S\|$.

Further, Proposition 2.4 says that β has a polynomial minoration:

$$\beta_n \geq c n^{-\sigma}$$

for positive constants c and σ . Then, if $U_\beta = A_\beta - (D_\beta + R_\beta + S_\beta)$, we have:

$$\|U_\beta\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m < \rho n/2} \frac{|a_{m,n}|^2}{\beta_n} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2} e^{-\alpha \rho n} \frac{n^\sigma}{c} < \infty.$$

Putting this together with (3.4) and (3.5), we finally obtain that $A_\beta = S_\beta + D_\beta + R_\beta + U_\beta$ is the matrix of a bounded operator, and that ends the proof of Theorem 3.3. \square

Remark. We could have done here without the theorem of Kacnel'son, using Lemma 3.4 to show that the matrix $\left(a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}}\right)$ is Hilbert-Schmidt as well “far below” the main diagonal. Indeed, we see from this lemma that

$$\sum_{m \geq C_2 n} |a_{m,n}|^2 \frac{\beta_m}{\beta_n} \lesssim \sum_{m \geq C_2 n} n^\sigma |a_{m,n}|^2 \lesssim \sum_{m \geq C_2 n} n^\sigma e^{-2bm} \lesssim \sum_{n \geq 0} n^\sigma e^{-2bC_2 n} < +\infty.$$

See also Theorem 6.2 of the final section. Kacnel'son's theorem will really be needed in the forthcoming Theorem 3.6.

3.3 Sufficient condition for the boundedness of composition operators – Part II

In this section, we give a sufficient condition for the boundedness of composition operators with symbol vanishing at 0, of a different nature than the one given in Proposition 3.2.

Theorem 3.6. *Let $\beta = (\beta_n)_{n=0}^\infty$ be a bounded sequence of positive numbers with a polynomial minoration. Assume that:*

$$(3.6) \quad \text{For every } \delta > 0, \text{ there exists a positive constant } C = C(\delta) \text{ such that} \\ \beta_m \leq C \beta_n \quad \text{whenever } m > (1 + \delta)n.$$

Then, for all symbols $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ vanishing at 0, the composition operator C_φ is bounded on $H^2(\beta)$.

To prove Theorem 3.6, we need several lemmas.

Lemma 3.7. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map such that $\varphi(0) = 0$ and $|\varphi'(0)| < 1$. Then there exists $\rho > 0$ such that*

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1 + \rho)n - m]\right).$$

Proof. It is the same as that of Lemma 3.4. Since $\varphi(0) = 0$, we can write $\varphi(z) = z \varphi_1(z)$. Since $|\varphi'(0)| < 1$, we have $\varphi_1: \mathbb{D} \rightarrow \mathbb{D}$. Let $M(r) = \sup_{|z|=r} |\varphi_1(z)|$. The Cauchy inequalities say that $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n / r^m$. We have $M(r) < 1$, so there exists a positive number $\rho = \rho(r)$ such that $M(r) = r^\rho$. We get:

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^{\rho n}}{r^{m-n}} = r^{(1+\rho)n-m},$$

and the result follows, by taking $r = e^{-1/2}$. \square

The next lemma is a variant of the result of V. È Kacnel'son quoted before.

Lemma 3.8. *Let $A: \ell_2 \rightarrow \ell_2$ be a bounded operator represented by the matrix $(a_{m,n})_{m,n}$, i.e. $a_{m,n} = \langle A e_n, e_m \rangle$, where $(e_n)_{n \geq 1}$ is the canonical basis of ℓ_2 .*

Let (d_n) be a sequence of positive numbers such that, for every m and n :

$$(3.7) \quad d_m < d_n \quad \implies \quad a_{m,n} = 0.$$

Then, D being the (possibly unbounded) diagonal operator with entries d_n , we have:

$$\|D^{-1}AD\| \leq \|A\|.$$

For the convenience of the reader, we reproduce the proof.

Proof. Let \mathbb{C}_0 be the right-half plane $\mathbb{C}_0 = \{z \in \mathbb{C}; \Re z > 0\}$. We set $H_N = \text{span}\{e_n; n \leq N\}$ and

$$A_N = P_N A J_N,$$

where P_N is the orthogonal projection from ℓ_2 onto H_N and J_N the canonical injection from H_N into ℓ_2 . We consider, for $z \in \overline{\mathbb{C}_0}$:

$$A_N(z) = D^{-z} A_N D^z: H_N \rightarrow H_N,$$

where $D^z(e_n) = d_n^z e_n$.

If $(a_{m,n}(z))_{m,n}$ is the matrix of $A_N(z)$ on the basis $\{e_n; n \leq N\}$ of H_N , we clearly have:

$$a_{m,n}(z) = a_{m,n}(d_n/d_m)^z.$$

In particular, we have, thanks to (3.7):

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n,$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M, \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Since $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n \leq N} |a_{m,n}(z)|^2 \leq (N+1)^2 M^2$, we get:

$$\|A_N(z)\| \leq (N+1) M \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Let us consider the function $u: \overline{\mathbb{C}_0} \rightarrow \overline{\mathbb{C}_0}$ defined by:

$$(3.8) \quad u_N(z) = \|A_N(z)\|.$$

This function u_N is continuous on $\overline{\mathbb{C}_0}$, bounded above by $(N+1)M$, and subharmonic in \mathbb{C}_0 . Moreover, thanks to (3.7), the maximum principle gives:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) = \sup_{\partial\mathbb{C}_0} u_N(z).$$

Since $\|D^z\| = \|D^{-z}\| = 1$ for $z \in \partial\mathbb{C}_0$, we have $\|A_N(z)\| \leq \|A_N\|$ for $z \in \partial\mathbb{C}_0$, and we get:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) \leq \|A_N\| \leq \|A\|.$$

In particular $u_N(1) \leq \|A\|$, and, letting N going to infinity, we get $\|D^{-1}AD\| \leq \|A\|$. \square

Proof of Theorem 3.6. First, if $|\varphi'(0)| = 1$, we have $\varphi(z) = \alpha z$ for some α with $|\alpha| = 1$, and the result is trivial.

So, we assume that $|\varphi'(0)| < 1$. Then, by Lemma 3.7, there exists $\rho > 0$ such that, for all m, n :

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1+\rho)n - m]\right).$$

Since $\varphi(0) = 0$, we also know that $\widehat{\varphi^n}(m) = 0$ if $m < n$.

Take $\delta = \rho/2$ and use property (3.6): there exists $C > 0$ such that:

$$\frac{\beta_m}{\beta_n} \leq C \quad \text{when } m \geq (1+\delta)n.$$

Define now a new sequence $\gamma = (\gamma_n)$ as:

$$\gamma_n = \max\left\{\beta_n, \sup_{m > (1+\delta)n} \beta_m\right\}.$$

We have:

- 1) $\beta_n \leq \gamma_n \leq C\beta_n$;
- 2) $\gamma_m \leq \gamma_n$ if $m \geq (1+\delta)n$.

Item 1) implies that $H^2(\gamma) = H^2(\beta)$, and we are reduced to prove that $C_\varphi: H^2(\gamma) \rightarrow H^2(\gamma)$ is bounded.

Let $A = (a_{m,n}) = (\widehat{\varphi^n}(m))$. We have to prove that

$$B = (\gamma_m^{1/2} \gamma_n^{-1/2} a_{m,n})_{m,n}$$

represents a bounded operator on ℓ_2 .

Define the matrix

$$A_1 = (a_{m,n} \mathbb{1}_{\{(m,n); m \leq (1+\delta)n\}})_{m,n}$$

and set $A_2 = A - A_1$. Define analogously B_1 and $B_2 = B - B_1$.

Then A_1 is a Hilbert-Schmidt operator, because (recall that $a_{m,n} = 0$ if $m < n$)

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{(1+\delta)n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \exp(-[(1+\rho)n - m]) \\ &\leq \sum_{n=1}^{\infty} (\delta n + 1) \exp(-\delta n) < \infty. \end{aligned}$$

Now, β is bounded above and has a polynomial minoration, so, for some positive constants C_1 , C_2 , and α , we have:

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{\gamma_m}{\gamma_n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{C_1}{n^{-\alpha}} \exp(-\delta n) \\ &\leq \sum_{n=1}^{\infty} C_2 n^{\alpha+1} \exp(-\delta n) < \infty, \end{aligned}$$

meaning that B_1 is a Hilbert-Schmidt operator.

Since A is bounded, it follows that $A_2 = A - A_1$ is bounded. Remark that, writing $A_2 = (\alpha_{m,n})_{m,n}$, we have, with $d_n = 1/\sqrt{\gamma_n}$:

$$d_m < d_n \implies \gamma_m > \gamma_n \implies m < (1+\delta)n \implies \alpha_{m,n} = 0.$$

Hence we can apply Lemma 3.8 to the matrix A_2 , and it ensues that B_2 is bounded, and therefore that $B = B_1 + B_2$ is bounded as well, as wanted. \square

As a corollary of Theorem 3.6, we can provide the following example.

Theorem 3.9. *There exists a bounded sequence β , with a polynomial minoration, but which is not essentially decreasing, for which every composition operator with symbol vanishing at 0 is bounded on $H^2(\beta)$, with nevertheless $\sup_{\varphi(0)=0} \|C_\varphi\| = \infty$.*

It should be noted that for this weight, the composition operators are not all bounded, as we will see in Proposition 4.4.

Proof. Define $\beta_n = 1$ for $n \leq 3!$, and, for $k \geq 3$:

$$\left\{ \begin{array}{l} \beta_n = \frac{1}{k!} \quad \text{for } k! < n \leq (k+1)! - 2 \text{ and for } n = (k+1)! \\ \beta_n = \frac{1}{(k+1)!} \quad \text{for } n = (k+1)! - 1. \end{array} \right.$$

Note that, for $m > n$, we have $\beta_m > \beta_n$ only if $n = (k+1)! - 1$ and $m = (k+1)! = n + 1$, for some $k \geq 3$.

However, β is not essentially decreasing since, for every $k \geq 3$, we have $\beta_{n+1}/\beta_n = k + 1$ if $n = (k+1)! - 1$.

The sequence β has a polynomial minoration because $\beta_n \geq 1/(2n)$ for all $n \geq 1$. In fact, for $k \geq 3$, we have $\beta_n \geq (k+1)/n \geq 1/n$ if $k! < n \leq (k+1)! - 2$ or if $n = (k+1)!$; and for $n = (k+1)! - 1$, we have $n\beta_n = [(k+1)! - 1]/(k+1)! \geq 1/2$.

Now, it remains to check (3.6) in order to apply Theorem 3.6 and finish the proof of Theorem 3.9. Note first that we have $\beta_m/\beta_n \leq 1$ if $m \geq n + 2$. Next, for given $\delta > 0$, there exists an integer N such that $(1 + \delta)n \geq n + 2$ for every $n \geq N$, so $\beta_m/\beta_n \leq 1$ if $m \geq (1 + \delta)n$ and $n \geq N$. It suffices to take $C = \max_{1 \leq n \leq N} \beta_{n+1}/\beta_n$ to obtain (3.6). The last assertion follows from Proposition 3.2. \square

4 Necessity of the Δ_2 -condition

In this section, we will show that, for essentially decreasing sequences β , the Δ_2 -condition is necessary for having boundedness of composition operators on $H^2(\beta)$. We will indeed show slightly more.

Theorem 4.1. *Let β be such that, for some $a \in (0, 1)$, T_a induces a bounded composition operator on $H^2(\beta)$. Then β satisfies:*

$$(\exists 0 < \delta < 1/3) (\forall n \geq 1) (\exists E_n \subseteq [(1 - 2\delta)n, (1 - \delta)n]) \quad \text{with}$$

$$|E_n| \geq \delta n \quad \text{and} \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m.$$

In particular, if β is essentially decreasing, then β satisfies the Δ_2 -condition.

In order to prove Theorem 4.1, we need several preliminary lemmas. The first one is standard, but we give it for convenience.

Lemma 4.2. *Let $a \in (0, 1)$ and let*

$$P_{-a}(x) = \frac{1 - a^2}{1 + 2a \cos x + a^2}$$

be the Poisson kernel at the point $-a$. Then, for all $x \in [-\pi, \pi]$:

$$(4.1) \quad T_a(e^{ix}) = \exp[i h_a(x)],$$

where

$$(4.2) \quad h_a(x) = \int_0^x P_{-a}(t) dt.$$

Proof. For $t \in [-\pi, \pi]$, write:

$$\psi(t) := \frac{e^{it} + a}{1 + a e^{it}} = \exp(i u(t)),$$

with u a real-valued, \mathcal{C}^1 function on $[-\pi, \pi]$ such that $u(0) = 0$. This is possible since $|\psi(e^{it})| = 1$ and $\psi(0) = 1$. Differentiating both sides with respect to t , we get:

$$i e^{it} \frac{1 - a^2}{(1 + a e^{it})^2} = i u'(t) \frac{e^{it} + a}{1 + a e^{it}}.$$

This implies

$$u'(t) = \frac{1 - a^2}{|1 + a e^{it}|^2} = P_{-a}(t),$$

and the result follows since $u(0) = 0 = h_a(0)$. \square

4.1 Main lemma and proof of Theorem 4.1

To prove Theorem 4.1, we need the following lemma. The proof of this lemma uses Lemma 4.2 and a van der Corput type estimate, inspired from [21, pp. 72–73]. We thank R. Zarouf [22] for interesting recent informations in this respect, related to his joint work with O. Szehr on the Schäffer problem (see [20], in which the authors are primarily concerned with upper bounds).

The first version of our paper was put on arXiv at the end of November 2020. Since then, the paper [1] was put on arXiv on July 2021, where sharp estimates of powers of Blaschke factors are given (see also K. Fouchet's thesis [6]), with different purposes (strongly annular analytic functions). However, in our case, our proof is much simpler.

Recall that we have set:

$$(4.3) \quad [T_a(z)]^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

Lemma 4.3. *Let $a \in (0, 1)$. We set:*

$$(4.4) \quad \tau = \frac{1 + a}{1 - a} > 1$$

and write:

$$(4.5) \quad \tau^{-1} = 1 - 3\mu,$$

with $\mu = \mu_a \in (0, 1/3)$. For every fixed positive integer n , let:

$$(4.6) \quad J_n = [(1 - 2\mu)n, (1 - \mu)n].$$

Then, there exists $\delta = \delta_a > 0$ such that, for every n large enough, there exists a set of indices $E_n \subseteq J_n$ with cardinality $|E_n| \geq \delta n$ and such that:

$$(4.7) \quad m \in E_n \implies |a_{m,n}| \geq \delta n^{-1/2}.$$

Proof of Theorem 4.1. Set $M = \|C_{T_a}\|$. We have:

$$(4.8) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|T_a^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n.$$

so, by Lemma 4.3, since $E_n \subseteq J_n = [(1 - 2\mu)n, (1 - \mu)n]$:

$$M^2 \beta_n \geq \sum_{m \in E_n} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} \sum_{m \in E_n} \beta_m \geq \delta^3 |E_n|^{-1} \sum_{m \in E_n} \beta_m.$$

This proves (changing δ) the first part of Theorem 4.1. Next, assume that β is essentially decreasing. We may, and do, assume that β is non-increasing. We set, for x not an integer, $\beta_x = \beta_k$ with k the least integer greater than x . The above implies, for all integers $n \geq 1$:

$$\beta_n \geq (\delta^3/M^2) |E_n|^{-1} |E_n| \beta_{(1-\mu)n} \geq c \beta_{(1-\mu)n}.$$

Let $r \geq 1$ such that $(1 - \mu)^r \leq 1/2$; we have:

$$\beta_n \geq c^r \beta_{(1-\mu)^r n} \geq c^r \beta_{n/2},$$

so β satisfies the Δ_2 -condition. \square

A consequence of Theorem 4.1 is the following result.

Proposition 4.4. *For the weight β constructed in the proof of Theorem 3.9, no automorphism T_a with $0 < a < 1$ can be bounded.*

Proof. Consider the necessary condition for the boundedness of C_{T_a} in Theorem 4.1:

$$(4.9) \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m.$$

For the weight β constructed in the proof of Theorem 3.9, we are going to see that this condition (4.9) is not satisfied for $n = (k + 1)! - 1 =: n_k$.

Indeed, for this n , the left-hand side of (4.9) is equal to $1/(k + 1)!$ and the right-hand side to $\delta/k!$ since $(1 - 2\delta)n_k > k!$ for k large, so that all β_m are equal to $1/k!$ for $m \in E_{n_k}$. This ends the proof. \square

4.2 Proof of Lemma 4.3

To prove Lemma 4.3, we will use a variant of [21, Lemma 4.6 p. 72] on the stationary phase method. A careful reading of the proof in [21, p. 72] gives the version below, which allows the derivative F' of F to vanish at some point, as occurs in our situation. For sake of completeness, we will give a proof, however postponed.

Proposition 4.5 (Stationary phase). *Let F be real function on the interval $[A, B]$, with continuous derivatives up to the third order and $F'' > 0$ throughout $]A, B[$. Assume that there is a (unique) point c in $]A, B[$ such that $F'(c) = 0$, and that, for some positive numbers λ_2, λ_3 , and η , the following assertions hold:*

- 1) $[c - \eta, c + \eta] \subseteq [A, B]$;
- 2) $F''(x) \geq \lambda_2$ for all $x \in [c - \eta, c + \eta]$;
- 3) $|F'''(x)| \leq \lambda_3$ for all $x \in [A, B]$.

Then:

$$(4.10) \quad \int_A^B e^{iF(x)} dx = \sqrt{2\pi} \frac{e^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\right),$$

where the O involves an absolute constant.

Proof of Lemma 4.3. We turn to the problem of bounding $a_{m,n}$ from below, in the case $m \in J_n$, and only in that case. Since $\inf_{[0,\pi]} P_{-a} = \tau^{-1} < \sup_{[0,\pi]} P_{-a} = \tau$, there exists a unique point $x_m = x_{m,n} \in [0, \pi]$ such that

$$nP_{-a}(x_m) - m = n \frac{(1 - a^2)}{1 + 2a \cos x_m + a^2} - m = 0,$$

or else:

$$(4.11) \quad \cos x_m = \frac{n}{m} \frac{1 - a^2}{2a} - \frac{1 + a^2}{2a}.$$

The point is that if $m \in J_n$, x_m can approach neither 0 nor π , so that $\sin x_m \geq \delta_a > 0$; more precisely, the definition of J_n and (4.11) imply that $\pi/4 \leq x_m \leq \pi/2$.

With h_a the function of Lemma 4.2, the Fourier formulas give, since $a_{m,n}$ is real, or since $h_a(x) - mx$ is odd:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp i[nh_a(x) - mx] dx = 2 \Re I_{m,n},$$

where

$$(4.12) \quad I_{m,n} = \int_0^{\pi} \exp i[nh_a(x) - mx] dx.$$

Write:

$$(4.13) \quad I_{m,n} = \int_0^{\pi} \exp[i F_m(x)] dx,$$

with:

$$(4.14) \quad F_m(x) = nh_a(x) - mx = n \int_0^x P_{-a}(t) dt - mx.$$

We have:

$$(4.15) \quad F'_m(x) = n P_{-a}(x) - m.$$

We will now proceed in two steps, first giving good lower bounds for $|I_{m,n}|$, then showing that the argument of $I_{m,n}$ is often far from $\pi/2 \bmod \pi$. Then, we will be done.

First step. We will prove that:

$$(4.16) \quad I_{m,n} = \sqrt{2\pi} n^{-1/2} \frac{e^{i(F_m(x_m)+\pi/4)}}{\sqrt{|h''_a(x_m)|}} + O(n^{-3/5}),$$

where the O only depends on a .

Note that $3/5 > 1/2$ and $F''_m = n h''_a$.

To get (4.16), we will show that Theorem 4.5 is applicable with:

$$[A, B] = [0, \pi], \quad c = x_m, \quad \lambda_2 = \kappa_0 n, \quad \lambda_3 = C_0 n, \quad \eta = (\lambda_2 \lambda_3)^{-1/5}.$$

The parameter η is chosen in order to make both error terms in Theorem 4.5 equal: $\frac{1}{\eta \lambda_2} = \eta^4 \lambda_3$; so:

$$\eta = \kappa n^{-2/5}$$

and

$$(4.17) \quad \frac{1}{\eta \lambda_2} + \eta^4 \lambda_3 = \tilde{\kappa} n^{-3/5} = O(n^{-3/5})$$

(with $\kappa = (\kappa_0 C_0)^{-1/5}$ and $\tilde{\kappa} = 2/\kappa_0 \kappa$).

The slight technical difficulty encountered here is that $F''_m(x)$ vanishes at 0 and π . But Theorem 4.5 covers this case. We have

$$F''_m(x) = n P'_{-a}(x) = 2a(1-a^2) \frac{\sin x}{(1+2a \cos x + a^2)^2} n,$$

and there are some positive κ_0 and σ such that

$$(4.18) \quad F''_m(x) \geq \kappa_0 n = \lambda_2 \quad \text{for } x \in [\sigma, \pi - \sigma].$$

Now (for n large enough), $[x_m - \eta, x_m + \eta] \subseteq [\sigma, \pi - \sigma]$. Hence the assumptions 1) and 2) of Proposition 4.5 are satisfied.

Finally, since $F_m(x) = n h_a(x) - m x$, and h_a is \mathcal{C}^∞ on \mathbb{R} , we have, for all $x \in [0, \pi]$:

$$|F'''_m(x)| \leq C_0 n = \lambda_3,$$

and assertion 3) of Proposition 4.5 holds.

With (4.17) this ends the proof of (4.16), once we remarked that $n h''_a(x_m) = F''_m(x_m)$.

Note that, since $|h_a''(x_m)| \leq M_a$, we get that $|I_{m,n}| \geq \delta n^{-1/2}$ when $m \in J_n$.

Second step. The mean-value theorem gives, for $m \in J_n$:

$$(4.19) \quad |\sin x_m| \geq \delta \quad \text{and} \quad x_{m+1} - x_m \approx \cos x_m - \cos x_{m+1}.$$

We also have, for $x \in \mathcal{J} = [1 - 2\mu, 1 - \mu]$, with another constant δ :

$$(4.20) \quad \delta \leq P'_{-a}(x) = 2a(1 - a^2) \frac{\sin x}{(1 + 2a \cos x + a^2)^2} \leq \delta^{-1}.$$

We now claim that

$$(4.21) \quad x_{m+1} - x_m \approx n^{-1} \quad \text{for } m \in J_n.$$

Indeed, since $m \in J_n$, we have, using (4.11):

$$\cos x_m - \cos x_{m+1} = \frac{1 - a^2}{2a} \frac{n}{m(m+1)} \approx \frac{n}{m^2} \approx n^{-1}.$$

In view of (4.19), this proves (4.21).

Now, according to (4.16), when $m \in J$, the main term in $I_{m,n}$ is

$$A_{m,n} := n^{-1/2} \frac{\sqrt{2\pi}}{\sqrt{|h_a''(x_m)|}} e^{i(F_m(x_m) + \pi/4)},$$

and its argument θ_m is $F_m(x_m) + \pi/4$. Going from m to $m+1$, the variation $F_{m+1}(x_{m+1}) - F_m(x_m)$ of this argument is

$$\begin{aligned} \theta_{m+1} - \theta_m &= n \int_{x_m}^{x_{m+1}} \left(P_{-a}(t) - \frac{m}{n} \right) dt - x_{m+1} \\ &= n \int_{x_m}^{x_{m+1}} \left(P_{-a}(t) - P_{-a}(x_m) \right) dt - x_{m+1}. \end{aligned}$$

But, due to (4.20), this implies:

$$\begin{aligned} \theta_{m+1} - \theta_m &= -x_{m+1} + O\left(n \int_{x_m}^{x_{m+1}} (t - x_m) dt\right) \\ &= -x_{m+1} + O(n(x_{m+1} - x_m)^2) = -x_{m+1} + O(n^{-1}). \end{aligned}$$

Since $-\sigma \leq -x_{m+1} \leq \pi - \sigma$, the variation of θ_m is thus regular. As a consequence, for a positive proportion E_n of the indices $m \in J_n$, the argument θ_m will belong to a subarc of \mathbb{T} which lies δ -apart from $\pm\pi/2$, implying $\cos \theta_m \geq \delta$, or else:

$$|E_n| \geq \delta n,$$

and, for all $m \in E_n$:

$$\Re A_{m,n} \geq \delta |A_{m,n}| \geq \delta n^{-1/2}.$$

It follows that, for $m \in E_n$:

$$\Re I_{m,n} \geq \delta n^{-1/2} - C n^{-3/5} \geq \tilde{\delta} n^{-1/2}.$$

Since $a_{m,n} = \pi^{-1} \Re I_{m,n}$, that ends the proof of Lemma 4.3. \square

We now pass to the proof of Proposition 4.5. The following lemma can be found in [15, Lemma 1, page 47].

Lemma 4.6. *Let $F: [u, v] \rightarrow \mathbb{R}$, with $u < v$, be a C^2 -function with $F'' > 0$, and F' not vanishing on $[u, v]$. Let*

$$J = \int_u^v e^{iF(x)} dx.$$

Then:

- a) if $F' > 0$ on $[u, v]$, then $|J| \leq \frac{2}{F'(u)}$;
- b) If $F' < 0$ on $[u, v]$, then $|J| \leq \frac{2}{|F'(v)|}$.

Proof of Proposition 4.5. Write now the integral I of Proposition 4.5 on $[A, B]$ as $I = I_1 + I_2 + I_3$ with:

$$I_1 = \int_A^{c-\eta} e^{iF(x)} dx, \quad I_2 = \int_{c-\eta}^{c+\eta} e^{iF(x)} dx, \quad I_3 = \int_{c+\eta}^B e^{iF(x)} dx.$$

Lemma 4.6 with $u = A$ and $v = c - \eta$ implies:

$$(4.22) \quad |I_1| \leq \frac{2}{|F'(c-\eta)|} \leq \frac{2}{\eta\lambda_2},$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta F''(\xi)$$

for some $\xi \in [c-\eta, c]$ so that $F''(\xi) \geq \lambda_2$.

Similarly, Lemma 4.6 with $u = c + \eta$ and $v = B$ implies

$$(4.23) \quad |I_3| \leq \frac{2}{F'(c+\eta)} \leq \frac{2}{\eta\lambda_2}.$$

We can now estimate I_2 . The Taylor formula shows that

$$F(x) = F(c) + \frac{(x-c)^2}{2} F''(c) + R,$$

with

$$|R| \leq \frac{|x-c|^3}{6} \lambda_3.$$

Hence

$$I_2 = e^{iF(c)} \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx + S$$

with

$$|S| \leq \lambda_3 \int_0^\eta \frac{x^3}{3} dx = \frac{\eta^4}{12} \lambda_3.$$

Finally, set

$$K = \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx.$$

We make the change of variable $x = \sqrt{\frac{2}{F''(c)}} \sqrt{t}$. Recall that $\int_0^\infty \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\pi} e^{i\pi/4}$ is the classical Fresnel integral, and that an integration by parts gives, for $m > 0$:

$$\left| \int_m^\infty \frac{e^{it}}{\sqrt{t}} dt \right| \leq \frac{2}{\sqrt{m}}.$$

Therefore, with $m = \frac{\eta^2}{2} F''(c)$:

$$K = \sqrt{\frac{2}{F''(c)}} \int_0^m \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\frac{2\pi}{F''(c)}} e^{i\pi/4} + R_m,$$

with

$$|R_m| \leq C \sqrt{\frac{1}{F''(c)}} \frac{1}{\sqrt{m}} \leq \frac{C}{\eta \lambda_2}.$$

All in all, we proved that

$$(4.24) \quad I_2 = \sqrt{\frac{2\pi}{F''(c)}} \exp[i(F(c) + \pi/4)] + O\left(\frac{1}{\eta \lambda_2} + \eta^4 \lambda_3\right).$$

and the same estimate holds for I , thanks to (4.22) and (4.23).

We have hence proved Proposition 4.5. □

5 Some results on multipliers

The set $\mathcal{M}(H^2(\beta))$ of multipliers of $H^2(\beta)$ is by definition the vector space of functions h analytic on \mathbb{D} and such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. When $h \in \mathcal{M}(H^2(\beta))$, the operator M_h of multiplication by h is bounded on $H^2(\beta)$ by the closed graph theorem. The space $\mathcal{M}(H^2(\beta))$ equipped with the operator norm is a Banach space. We note the obvious property:

$$(5.1) \quad \mathcal{M}(H^2(\beta)) \hookrightarrow H^\infty \quad \text{contractively.}$$

Indeed, if $h \in \mathcal{M}(H^2(\beta))$, we easily get for all $w \in \mathbb{D}$:

$$M_h^*(K_w) = \overline{h(w)} K_w;$$

so that taking norms and simplifying, we are left with $|h(w)| \leq \|M_h\|$, showing that $h \in H^\infty$ with $\|h\|_\infty \leq \|M_h\|$.

Proposition 5.1. *We have $\mathcal{M}(H^2(\beta)) = H^\infty$ isomorphically if and only if β is essentially decreasing.*

Proof. The sufficient condition is proved in [13, beginning of the proof of Proposition 3.16]. For the necessity, we have $\|M_h\| \approx \|h\|_\infty$ for every $h \in H^\infty$. Now, for $m > n$ (recall that $e_n(z) = z^n$):

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z);$$

so, since $\|M_{e_{m-n}}\| \leq C \|e_{m-n}\|_\infty = C$ for some positive constant C :

$$\beta_m = \|e_m\|^2 \leq C^2 \|e_n\|^2 = C^2 \beta_n. \quad \square$$

In [13, Section 3.6], we gave the following notion of *admissible* Hilbert space of analytic functions.

Definition 5.2. *A Hilbert space H of analytic functions on \mathbb{D} , containing the constants, and with reproducing kernels K_a , $a \in \mathbb{D}$, is said admissible if:*

- (i) H^2 is continuously embedded in H ;
- (ii) $\mathcal{M}(H) = H^\infty$;
- (iii) the automorphisms of \mathbb{D} induce bounded composition operators on H ;
- (iv) $\frac{\|K_a\|_H}{\|K_b\|_H} \leq h\left(\frac{1-|b|}{1-|a|}\right)$ for $a, b \in \mathbb{D}$, where $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-decreasing function.

We proved in that paper that every weighted Hilbert space $H^2(\beta)$ with β non-increasing is admissible, under the additional hypothesis that the automorphisms of \mathbb{D} induce bounded composition operators. In view of Theorem 3.3, we get the following result.

Proposition 5.3. *Let β be essentially decreasing that satisfies the Δ_2 -condition. Then $H^2(\beta)$ is admissible.*

Let us give a different proof.

Proof. Because β is essentially decreasing, item (i) holds, as well as item (ii), by Proposition 5.1. Item (iii) is Theorem 3.3. It remains to show (iv). We may assume that β is non-increasing.

Let $0 < s < r < 1$.

Without loss of generality, we may assume that $r, s \geq 1/2$. It is enough to prove:

$$(5.2) \quad \|K_r\|^2 \leq C \|K_{r^2}\|^2$$

for some constant $C > 1$. Indeed, iteration of (5.2) gives:

$$\|K_r\|^2 \leq C^k \|K_{r^{2^k}}\|^2$$

and if k is the smallest integer such that $r^{2^k} \leq s$, we have $2^{k-1} \log r > \log s$ and $2^k \leq D \frac{1-s}{1-r}$ where D is a numerical constant. Writing $C = 2^\alpha$ with $\alpha > 1$, we obtain:

$$\left(\frac{\|K_r\|}{\|K_s\|} \right)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \left(\frac{1-s}{1-r} \right)^\alpha.$$

To prove (5.2), we pick some $M > 1$ such that $\beta_{2n} \geq M^{-1}\beta_n$ for all $n \geq 1$ and write $t = r^2$. We have:

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\beta_{2n-1}},$$

implying, since $\beta_{2n-1} \geq \beta_{2n} \geq M^{-1}\beta_n$ and $t^{2n-1} \leq 4t^{2n}$:

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M \|K_t\|^2. \quad \square$$

The notion of admissible Hilbert space H is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{w \in H; w(f \circ \varphi) \in H \text{ for all } f \in H\}.$$

As a corollary of [13, Theorem 3.18] we get:

Corollary 5.4. *Let β be essentially decreasing and satisfying the Δ_2 -condition. Then:*

- 1) $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}(H^2(\beta), \varphi)$;
- 2) $\mathcal{M}(H^2(\beta), \varphi) = H^2(\beta)$ if and only if $\|\varphi\|_\infty < 1$;
- 3) $\mathcal{M}(H^2(\beta), \varphi) = H^\infty$ if and only if φ is a finite Blaschke product.

6 Miscellaneous remarks

Some of the results of this paper can slightly be improved.

6.1 Conditions on the weight β

First, we say that a sequence (β_n) of positive numbers is *slowly oscillating* if there is a function $\rho: (0, \infty) \rightarrow (0, \infty)$ that is bounded on each compact subset of $(0, \infty)$ for which:

$$\frac{\beta_m}{\beta_n} \leq \rho\left(\frac{m}{n}\right).$$

This clearly amounts to say that, for some positive constants $C_1 < C_2$, we have:

$$C_1 \leq \frac{\beta_m}{\beta_n} \leq C_2 \quad \text{when } n/2 \leq m \leq 2n.$$

Every essentially decreasing sequence with the Δ_2 -condition is slowly oscillating.

Proposition 6.1. *The following holds:*

- 1) *every slowly oscillating sequence has a polynomial minoration;*
- 2) *there are bounded sequences which are slowly oscillating, but not essentially decreasing.*

Proof. 1) is clear, because if $2^j \leq n < 2^{j+1}$, then

$$\beta_n \geq C^{-1} \beta_{2^j} \geq C^{-j-1} \beta_1 \geq C^{-1} \beta_1 n^{-\alpha},$$

with $\alpha = \log C / \log 2$.

2) We define β_n as follows. Let (a_k) be an increasing sequence of positive square integers such that $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \infty$, for example $a_k = 4^{k^2}$, and let $b_k = \sqrt{a_k a_{k+1}}$; with our choice, this is an integer and we clearly have $a_k < b_k < a_{k+1}$. We set:

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k \\ (a_k/b_k^2)n = (1/a_{k+1})n & \text{for } b_k \leq n < a_{k+1}. \end{cases}$$

This sequence (β_n) is slowly oscillating by construction. Indeed, it suffices to check that for $a_k \leq n/2 < b_k \leq n < a_{k+1}$, the quotient β_m/β_n remains lower and upper bounded when $n/2 \leq m \leq n$. But for $n/2 \leq m < b_k$, we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn},$$

which is $\leq 2b_k^2/n^2 \leq 2$ and $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = 1/4$; and for $b_k \leq m$, we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in [1/2, 1].$$

However, though (β_n) is bounded, since $\beta_n \leq 1$ for $a_k \leq n < b_k$ and, for $b_k \leq n < a_{k+1}$,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}} (a_{k+1} - 1) \leq 1,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}} (a_{k+1} - 1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \xrightarrow{k \rightarrow \infty} \infty. \quad \square$$

By a slight modification (change the value of the constants in the definition of β_n), we could obtain a sequence which is slowly oscillating, tends to zero, yet again not essentially decreasing.

Now, Theorem 3.3 admits the following variant.

Theorem 6.2. *Let (β_n) be a sequence of positive numbers which is bounded above and slowly oscillating. Then all symbols that extend analytically in a neighborhood of \mathbb{D} induce a bounded composition operator on $H^2(\beta)$.*

It is the case, for example, for finite Blaschke products. The proof follows that of Theorem 3.3, with the help of the following lemma.

Lemma 6.3. *Let (β_n) be a sequence of positive numbers which is bounded above and slowly oscillating. Let $A = (a_{m,n})_{m,n}$ be the matrix of a bounded operator on ℓ_2 . Assume that, for constants $C_1 < 1$, $C_2 > 1$, and c, b , we have:*

- 1) $|a_{m,n}| \leq c e^{-bn}$ for $m \leq C_1 n$;
- 2) $|a_{m,n}| \leq c e^{-bm}$ for $m \geq C_2 n$.

Then the matrix $\tilde{A} = \left(a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}} \right)$ also defines a bounded operator on ℓ_2 .

Sketch of proof. The matrix \tilde{A} is Hilbert-Schmidt far from the diagonal since, with $\lambda \geq \beta_n \geq \delta n^{-\alpha}$, we have:

$$\sum_{m < C_1 n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m < C_1 n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^{\alpha+1} e^{-bn} < \infty,$$

and

$$\sum_{m > C_2 n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m > C_2 n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^\alpha \left(\sum_{m > C_2 n} e^{-bm} \right).$$

Since β_m / β_n remains bounded from above and below around the diagonal, the matrix \tilde{A} behaves like A near the diagonal. \square

Remark. The proof shows that, instead of 1) and 2), it is enough to have:

$$\sum_{m < C_1 n} n^{\alpha+1} |a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m > C_2 n} m^\alpha |a_{m,n}|^2 < \infty.$$

Moreover the proof also shows that when β is slowly oscillating, if we set $E = \{(m, n); C_1 n \leq m \leq C_2 n\}$, then the matrix $(\sqrt{\beta_m / \beta_n} \mathbb{1}_E(m, n))$ is a Schur multiplier over *all* the bounded matrices, while Kacnel'son's theorem (Theorem 3.5) says that, if $\gamma = (\gamma_n)$ is non-increasing, the matrix (γ_m / γ_n) is a Schur multiplier of all bounded *lower-triangular* matrices.

Proof of Theorem 6.2. We first prove that the assumptions of Lemma 6.3 are satisfied with $a_{m,n} = \widehat{\varphi}^n(m)$. This works nearly as in Lemma 3.4. For every symbol φ , let $M(r) = \sup_{|z|=r} |\varphi(z)|$ and write $M(1/e) = e^{-\delta}$, with $\delta > 0$. Let A be the matrix of C_φ , with respect to the canonical basis of H^2 . The Cauchy inequalities give, if $m \leq (\delta/2)n$:

$$|a_{m,n}| \leq [M(1/e)]^n e^m \leq e^{m-\delta n} \leq e^{-\delta/2n}$$

When φ is analytic in a neighborhood of $\overline{D(0, R)}$ with $R > 1$, the Cauchy inequalities give, writing $M(R) = e^\rho$ and $R = e^\delta$, for $m \geq C_2 n$, and a suitable constant C_2 (take $C_2 = (2\rho)/\delta$ for instance):

$$|a_{m,n}| \leq \frac{[M(R)]^n}{R^m} \leq e^{n\rho - \delta m} \leq e^{-(\delta/2)m}.$$

We now conclude with Lemma 6.3. \square

6.2 Singular inner functions

Let $a > 0$ and let I_a be the singular inner function defined by

$$(6.1) \quad I_a(z) = \exp\left(-a \frac{1+z}{1-z}\right) = \sum_{m=0}^{\infty} c_m(a) z^m.$$

If n is a positive integer, we have $I_n(z) = [I_1(z)]^n$ and we write

$$I_n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m, \quad \text{with } a_{m,n} = c_m(n).$$

We rely on the following lemma, familiar to experts in orthogonal polynomials and special functions, but maybe not so much as regards the uniformity, essential for our present purposes (see [16] or [12]).

Lemma 6.4. *It holds*

$$(6.2) \quad c_m(n) = c n^{1/4} m^{-3/4} \cos(2\sqrt{2nm} + \pi/4) + R_m(n) =: M_m(n) + R_m(n),$$

where $c = \pi^{-1/2} 2^{1/4}$ and where $|R_m(n)| \leq K \sqrt{n} m^{-5/4}$, with K some numerical constant.

We use the following [16, p. 253] and [19, p. 198], where $L_m^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree m with parameter α .

Theorem 6.5. *With the notation of (6.1), we have*

$$c_m(a) = e^{-a} L_m^{(-1)}(2a).$$

Moreover, we have, uniformly for $0 < \varepsilon \leq x \leq M < \infty$:

$$L_m^{(\alpha)}(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} m^{\alpha/2-1/4} \cos(2\sqrt{mx} - \alpha\pi/2 - \pi/4) + R_m(x),$$

where $|R_m(x)| \leq K_\alpha m^{\alpha/2-3/4}$, and $K_\alpha > 0$ only depending on α .

Now, using Theorem 6.5 with $\alpha = -1$ and $a = n$, we get Lemma 6.4. Actually, to get Lemma 6.4, we need some uniformity with respect to a in Theorem 6.5, which is not given by the above statement of that theorem; but provided we change $m^{-5/4}$ into $\sqrt{n} m^{-5/4}$ in R_m , a careful examination of Fejér's proof of Theorem 6.5 shows that this uniformity holds. Alternatively, write $c_m(a)$ as a Fourier coefficient

$$(6.3) \quad c_m(n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp[i(n \cot x + 2mx)] dx$$

and use the previous van der Corput estimates.

Once this lemma is at our disposal, we can prove again the following theorem.

Theorem 6.6. *Assume that β is a non-increasing sequence and that the composition operator C_{I_1} maps $H^2(\beta)$ to itself. Then β satisfies the Δ_2 -condition.*

Proof. Our assumption implies, with $M = \|C_{I_1}\|^2$:

$$(6.4) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \leq M \beta_n.$$

We use Lemma 6.4 with $C^{-2}n \leq m < n$, where $C > 1$ satisfies $C\sqrt{2} < \pi/2$, which is possible since $2\sqrt{2} < \pi$. The term $R_m(n)$ is dominated by $KC^{-1/2}m^{-3/4}$. Let $\theta_m = 2\sqrt{2}\sqrt{nm} + \pi/4$ be the argument of the number appearing in (6.2). We have

$$\theta_{m+1} - \theta_m = \frac{2\sqrt{2}\sqrt{n}}{\sqrt{m} + \sqrt{m+1}};$$

hence

$$\sqrt{2} \leq \theta_{m+1} - \theta_m \leq C\sqrt{2} < \frac{\pi}{2}.$$

The argument θ_m then varies regularly, and there is a positive constant δ and a subset E of integers in the interval $[C^{-2}n, n]$ such that $|E| \geq \delta n$ and

$$M_m(n) \geq 2\delta n^{1/4}m^{-3/4} \geq 2\delta n^{-1/2}$$

for all $m \in E$. Therefore, for n large enough, we have, for all $m \in E$:

$$|a_{m,n}| \geq 2\delta n^{-1/2} - K\sqrt{n}m^{-5/4} \geq 2\delta n^{-1/2} - KC^{5/2}n^{-3/4} \geq \delta n^{-1/2}.$$

With this information, (6.4) gives:

$$M\beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} |E| \beta_{\lfloor C^{-2}n \rfloor} \geq \delta^3 \beta_{\lfloor C^{-2}n \rfloor},$$

where $\lfloor \cdot \rfloor$ stands for the integer part, and this proves the theorem. \square

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