# Boundedness of composition operators on general weighted Hardy spaces of analytic functions

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# Boundedness of composition operators on general weighted Hardy spaces of analytic functions

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**Abstract.** We characterize the (essentially) decreasing sequences of positive numbers $\beta = (\beta_n)$ for which all composition operators on $H^2(\beta)$ are bounded, where $H^2(\beta)$ is the space of analytic functions $f$ in the unit disk such that $\sum_{n=0}^{\infty} |c_n|^2 \beta_n < \infty$ if $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We also give conditions for the boundedness when $\beta$ is not assumed essentially decreasing.

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## 1 Introduction

Let $\beta = (\beta_n)_{n\geq 0}$ be a sequence of positive numbers such that

$$\liminf_{n\to\infty} \beta_n^{1/n} \geq 1 \,. \tag{1.1}$$

The associated weighted Hardy space $H^2(\beta)$ is the set of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty \,. \tag{1.2}$$

It is a Hilbert space of analytic functions on $\mathbb{D}$. When $\beta_n \equiv 1$, we recover the usual Hardy space $H^2$.

Recall that a symbol is a (non constant) analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, and the associated composition operator $C_\varphi\colon H^2(\beta) \to \mathcal{H}ol(\mathbb{D})$ is defined (formally) as:

$$C_\varphi(f) = f \circ \varphi \,. \tag{1.3}$$

An important question in the theory is to decide when $C_\varphi$ is bounded on $H^2(\beta)$, i.e. when $C_\varphi\colon H^2(\beta) \to H^2(\beta)$. When $H^2(\beta)$ is the usual Hardy space $H^2$ (i.e. when $\beta_n \equiv 1$), it is well-known ([18, pp. 13–17]) that all symbols generate bounded composition operators. On the other hand, for the Dirichlet space, corresponding to $\beta_n = n+1$, not all composition operators are bounded. Note that, by definition of the norm of $H^2(\beta)$, all rotations $R_\theta$, $\theta \in \mathbb{R}$, induce bounded composition operators on $H^2(\beta)$ and send isometrically $H^2(\beta)$ into itself.

Our goal in this paper is characterizing the (non-increasing) sequences $\beta$ for which all composition operators act boundedly on the space $H^2(\beta)$, i.e. send $H^2(\beta)$ into itself. We will prove the following result, where $T_a(z) = \frac{a+z}{1+\bar{a}\,z}$ for $a, z \in \mathbb{D}$.

**Theorem 1.1.** *Let $\beta$ be an essentially decreasing sequence of positive numbers. The following assertions are equivalent:*

1) *all composition operators are bounded on $H^2(\beta)$;*
2) *all maps $T_a$, for $0 < a < 1$, induce bounded composition operators $C_{T_a}$ on $H^2(\beta)$;*
3) *for some $a \in (0,1)$, the map $T_a$ induces a bounded composition operator $C_{T_a}$ on $H^2(\beta)$;*
4) *$\beta$ satisfies the $\Delta_2$-condition.*

For the notion of essentially decreasing sequence, see Definition 2.1; the $\Delta_2$-condition is defined in (2.4). Actually, we will obtain Theorem 1.1 by gluing Theorem 3.3 and Theorem 4.1.

Most of the existing works with weighted Hardy spaces concern the case

$$\beta_n = \int_0^1 t^n \, d\sigma(t) \tag{1.4}$$

where $\sigma$ is a positive measure on $(0,1)$. More specifically the following definition is often used. Let $G\colon (0,1) \to \mathbb{R}_+$ be an integrable function and let $H^2_G$ be the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|^2_{H^2_G} := \int_{\mathbb{D}} |f(z)|^2 \, G(1 - |z|^2) \, dA(z) < \infty \, . \tag{1.5}$$

Such weighted Bergman type spaces are used, for instance, in [11], [10] and in [14]. We have $H^2_G = H^2(\beta)$ with:

$$\beta_n = 2 \int_0^1 r^{2n+1} G(1 - r^2) \, dr = \int_0^1 t^n \, G(1-t) \, dt \, , \tag{1.6}$$

and the sequence $\beta = (\beta_n)_n$ is non-increasing.

In Shapiro's presentation, the main point is the case $\varphi(0) = 0$ and a subordination principle for subharmonic functions (Littlewood's subordination principle). The case of automorphisms is claimed simple, using an integral representation for the norm and some change of variable. When $\beta$ is defined as in (1.4), one disposes of integral representations for the norm in $H^2(\beta)$, and, as in the Hardy space case, this integral representation rather easily gives the boundedness of $C_{T_a}$ on $H^2$, where

$$T_a(z) = \frac{a + z}{1 + \bar{a}\, z} \tag{1.7}$$

for $a \in \mathbb{D}$. But the above representation (1.4) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence $\beta$, namely its *complete monotony*. When integral representations fail, we have to work with bare hands. If the symbol vanishes at the origin, Kacnel'son's theorem gives a positive answer when $\beta$ is essentially decreasing (see [2] or [13, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [8] (see [5, Theorem 6.3]), which itself uses a refinement by Rogosinski of Littlewood's principle ([5, Theorem 6.2]). So that the main issue remains the boundedness of $C_{T_a}$.

A polynomial minoration (see Definition 2.3 below) for $\beta$ is necessary for any $C_{T_a}$ to be bounded on $H^2(\beta)$ (Proposition 2.5) and we showed in [13, end of Section 3] that for $\beta_n = \exp(-\sqrt{n})$, $C_{T_a}$ is never bounded on $H^2(\beta)$. But this polynomial minoration is not sufficient, as we will see in Theorem 1.1 below, which also evidences the basic role of the maps $T_a$ in the question.

However, we construct a weight $\beta$ which is not essentially decreasing and for which all composition operators with symbol vanishing at 0 are bounded (Theorem 3.9), though no map $T_a$ with $0 < a < 1$ induces a bounded composition operator (Proposition 4.4).

For spaces of Bergman type $A^2_{\widetilde{G}} := H^2_G$, where $\widetilde{G}(r) = G(1 - r^2)$, defined as the spaces of analytic functions in $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^2\, \widetilde{G}(|z|)\, dA < \infty$, for a positive non-increasing continuous function $\widetilde{G}$ on $[0, 1)$, Kriete and MacCluer studied in [11] some analogous problems. They proved, in particular [11, Theorem 3] that, for:

$$\widetilde{G}(r) = \exp\left( - B\, \frac{1}{(1 - r)^{\alpha}} \right), \qquad B > 0\,,\ 0 < \alpha \leq 2\,,$$

and

$$\varphi(z) = z + t(1 - z)^{\beta}\,, \qquad 1 < \beta \leq 3\,,\ 0 < t < 2^{1-\beta}\,,$$

then $C_\varphi$ is bounded on $A^2_{\widetilde{G}}$ if and only if $\beta \geq \alpha + 1$.

Here

$$\beta_n = \int_0^1 t^n \mathrm{e}^{-B/(1-\sqrt{t})^{\alpha}}\, dt\,;$$

and, since $\beta_n \approx \exp(-c\, n^{\alpha/(\alpha+1)})$, the sequence $(\beta_n)$ does not satisfy the $\Delta_2$-condition, accordingly to our Theorem 3.3 below.

We end the paper with some miscellaneous remarks.

## 2 Definitions, notation, and preliminary results

The open unit disk of $\mathbb{C}$ is denoted $\mathbb{D}$ and we write $\mathbb{T}$ its boundary $\partial\mathbb{D}$. We set $e_n(z) = z^n$, $n \geq 0$.

The weighted Hardy space $H^2(\beta)$ defined in the introduction is a Hilbert space with the canonical orthonormal basis

$$e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}}\, z^n\,, \quad n \geq 0\,, \tag{2.1}$$

and the reproducing kernel $K_w$ given for all $w \in \mathbb{D}$ by

$$K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z)\, \overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n}\, \overline{w}^n\, z^n\,. \tag{2.2}$$

Note that (1.1) is necessary for $H^2(\beta)$ to consist of analytic functions in $\mathbb{D}$. Indeed the fact that $\sum_{n\geq 1} \frac{1}{n\sqrt{\beta_n}}\, z^n$ belongs to $H^2(\beta)$ and is analytic in $\mathbb{D}$ implies (1.1). Note also that $H^2$ is continuously embedded in $H^2(\beta)$ if and only if $\beta$ is bounded above. In particular, this is the case when $\beta$ is non-increasing. In this paper, we need a slightly more general notion.

**Definition 2.1.** *A sequence of positive numbers $\beta = (\beta_n)$ is said* essentially decreasing *if, for some constant $C \geq 1$, we have, for all $m \geq n \geq 0$:*

$$\beta_m \leq C\, \beta_n\,. \tag{2.3}$$

Note that saying that $\beta$ is essentially decreasing means that the shift operator on $H^2(\beta)$ is power bounded.

If $\beta$ is essentially decreasing, and if we set:

$$\widetilde{\beta}_n = \sup_{m\geq n} \beta_m\,,$$

the sequence $\widetilde{\beta} = (\widetilde{\beta}_n)$ is non-increasing and we have $\beta_n \leq \widetilde{\beta}_n \leq C\, \beta_n$. In particular, the space $H^2(\beta)$ is isomorphic to $H^2(\widetilde{\beta})$ and $H^2$ is continuously embedded in $H^2(\beta)$.

**Definition 2.2.** *The sequence of positive numbers $\beta = (\beta_n)$ is said to satisfy the* $\Delta_2$-condition *if there is a positive constant $\delta < 1$ such that, for all integers $n \geq 0$:*

$$\beta_{2n} \geq \delta\, \beta_n\,. \tag{2.4}$$

This terminology is given by analogy with that used for Orlicz functions.

**Definition 2.3.** *The sequence of positive numbers $\beta = (\beta_n)$ is said to have a* polynomial minoration *if there are positive constants $\delta$ and $\alpha$ such that, for all integers $n \geq 1$:*

$$\beta_n \geq \delta\, n^{-\alpha}. \tag{2.5}$$

That means that $H^2(\beta)$ is continuously embedded in the weighted Bergman space $\mathfrak{B}^2_\alpha$ of the analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{\mathfrak{B}^2_\alpha} := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < \infty$$

since $\mathfrak{B}^2_\alpha = H^2(\gamma)$ with $\gamma_n \approx n^{-\alpha}$.

The following simple proposition links those notions.

**Proposition 2.4.** *Let $\beta$ be an essentially decreasing sequence of positive numbers. Then if $\beta$ satisfies the $\Delta_2$-condition, it has a polynomial minoration.*

*The converse does not hold.*

*Proof.* Assume $\beta_m \leq C\,\beta_n$ for $m \geq n$ and $\beta_{2p} \geq \mathrm{e}^{-A}\,\beta_p$. Let now $n$ be an integer $\geq 2$, and $k \geq 1$ the smallest integer such that $2^k \geq n$, so that $k \leq a \log n$ with $a$ a positive constant. We get:

$$\beta_n \geq C^{-1} \beta_{2^k} \geq C^{-1} \mathrm{e}^{-kA} \beta_1 \geq C^{-1} \beta_1 \mathrm{e}^{-aA \log n} =: \rho\, n^{-\alpha}\,,$$

with $\rho = C^{-1}\beta_1$ and $\alpha = aA$.

Let us now see that the converse does not hold. Let $\delta > 0$. We set $\beta_0 = \beta_1 = 1$ and for $n \geq 2$:

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)!\,.$$

The sequence $\beta$ is non-increasing.

For $n$ and $k$ as above, we have:

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta}\,;$$

hence $\beta$ has arbitrary polynomial minoration. However we have, for $k \geq 2$:

$$\frac{\beta_{2(k!)}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \underset{k\to\infty}{\longrightarrow} 0\,,$$

so $\beta$ fails to satisfy the $\Delta_2$-condition. □

For $a \in \mathbb{D}$, we define:

$$T_a(z) = \frac{a + z}{1 + \bar{a}\, z}\,, \quad z \in \mathbb{D}\,. \tag{2.6}$$

Recall that $T_a$ is an automorphism of $\mathbb{D}$ and that $T_a(0) = a$ and $T_a(-a) = 0$.

Though we do not need this, we may remark that $(T_a)_{a\in(-1,1)}$ is a group and $(T_a)_{a\in(0,1)}$ is a semigroup. It suffices to see that $T_a \circ T_b = T_{a*b}$, with:

$$a * b = \frac{a + b}{1 + ab} \,\cdot \tag{2.7}$$

**Proposition 2.5.** *Let $a \in (0,1)$ and assume that $T_a$ induces a bounded composition operator on $H^2(\beta)$. Then $\beta$ has a polynomial minoration.*

*Proof.* Since
$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n}\,,$$
we have $\|K_x\| \leq \|K_y\|$ for $0 \leq x \leq y < 1$.

We define by induction a sequence $(u_n)_{n\geq 0}$ with:
$$u_0 = 0 \qquad \text{and} \qquad u_{n+1} = T_a(u_n)\,.$$
Since $T_a(1) = 1$ (recall that $a \in (0,1)$), we have:
$$1 - u_{n+1} = \int_{u_n}^{1} T_a'(t)\,dt = \int_{u_n}^{1} \frac{1-a^2}{(1+at)^2}\,dt\,;$$
hence
$$\frac{1-a}{1+a}\,(1-u_n) \leq 1 - u_{n+1} \leq (1-a^2)(1-u_n)\,.$$
Let $0 < x < 1$. We can find $N \geq 0$ such that $u_N \leq x < u_{N+1}$. Then:
$$1 - x \leq 1 - u_N \leq (1-a^2)^N\,.$$

On the other hand, since $C_{T_a}^{*} K_z = K_{T_a(z)}$ for all $z \in \mathbb{D}$, we have:
$$\|K_x\| \leq \|K_{u_{N+1}}\| \leq \|C_{T_a}\|\,\|K_{u_N}\| \leq \|C_{T_a}\|^{N+1}\|K_{u_0}\| = \|C_{T_a}\|^{N+1}\,.$$
Let $s \geq 0$ such that $(1-a^2)^{-s} = \|C_{T_a}\|$. We obtain:
$$\|K_x\| \leq \|C_{T_a}\|\,\frac{1}{(1-x)^s}\,. \tag{2.8}$$
But
$$\|K_x\|^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k}\,;$$
so we get, for any $k \geq 2$:
$$\frac{x^{2k}}{\beta_k} \leq \|C_{T_a}\|^2 \frac{1}{(1-x)^{2s}}\,.$$
Taking $x = 1 - \frac{1}{k}$, we obtain $\beta_k \geq C\,k^{-2s}$. □

**Remarks.** 1) For example, when $\beta_n = \exp\big[-c\,\big(\log(n+1)\big)^2\big]$, with $c > 0$, no $T_a$ induces a bounded composition operator on $H^2(\beta)$, though all symbols $\varphi$ with $\varphi(0) = 0$ are bounded, since $\beta$ is decreasing, as we will see in Proposition 3.2.

2) For the Dirichlet space $\mathcal{D}^2$, we have $\beta_n = n+1$, but all the maps $T_a$ induce bounded composition operators on $\mathcal{D}^2$ (see [13, Remark before Theorem 3.12]). In this case $\beta$ has a polynomial minoration though it is not bounded above.

3) However, even for decreasing sequences, a polynomial minoration for $\beta$ is not enough for some $T_a$ to induce a bounded composition operator. Indeed, we saw in Proposition 2.4 an example of a decreasing sequence $\beta$ with polynomial minoration, but not sharing the $\Delta_2$-condition, and we will see in Theorem 4.1 that the $\Delta_2$-condition is needed for having some $T_a$ inducing a bounded composition operator.

4) In [7], Eva Gallardo-Gutiérrez and Jonathan Partington give estimates for the norm of $C_{T_a}$, with $a \in (0,1)$, when $C_{T_a}$ is bounded on $H^2(\beta)$. More precisely, they proved that if $\beta$ is bounded above and $C_{T_a}$ is bounded, then

$$\|C_{T_a}\| \geq \left(\frac{1+a}{1-a}\right)^{\sigma},$$

where $\sigma = \inf\{s \geq 0\,;\ (1-z)^{-s} \notin H^2(\beta)\}$, and

$$\|C_{T_a}\| \leq \left(\frac{1+a}{1-a}\right)^{\tau},$$

where $\tau = \frac{1}{2} \sup \mathfrak{Re}\, W(A)$, with $A$ the infinitesimal generator of the continuous semigroup $(S_t)$ defined as $S_t = C_{T_{\tanh t}}$, namely $(Af)(z) = f'(z)(1-z^2)$, and $W(A)$ its numerical range.

For $\beta_n = 1/(n+1)^{\nu}$ with $0 \leq \nu \leq 1$, the two bounds coincide, so they get $\|C_{T_a}\| = \big(\frac{1+a}{1-a}\big)^{(\nu+1)/2}$.

5) We saw in the proof of Proposition 2.5 that if $C_{T_a}$ is bounded on $H^2(\beta)$ for some $a \in (0,1)$, then the reproducing kernels $K_w$ have, by (2.8), a *slow growth*:

$$\|K_w\| \leq \frac{C}{(1-|w|)^s} \tag{2.9}$$

for positive constants $C$ and $s$. Actually, we have the following equivalence.

**Proposition 2.6.** *The sequence $\beta$ has a polynomial minoration if and only if the reproducing kernels $K_w$ of $H^2(\beta)$ have a slow growth.*

*Proof.* The sufficiency is easy and seen at the end of the proof of Proposition 2.5. For the necessity, we only have to see that:

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^{\alpha} |w|^{2n} \leq \frac{C}{(1-|w|^2)^{\alpha+1}} \cdot \qquad \square$$

# 3 Boundedness of composition operators

We study in this section conditions ensuring that all composition operators on $H^2(\beta)$ are bounded.

## 3.1 Conditions on the weight

We begin with this simple observation.

**Proposition 3.1.** *If all composition operators, and even if all composition operators with symbol vanishing at* $0$*, are bounded on* $H^2(\beta)$*, then the sequence* $\beta$ *is bounded above.*

*Proof.* Let $f \in H^\infty$. Write $f = A\,\varphi + f(0)$ where $A$ is a constant and $\varphi$ a symbol vanishing at 0. We have $\varphi = C_\varphi(z) \in H^2(\beta)$, by hypothesis. So that $f \in H^2(\beta)$ and $H^\infty \subseteq H^2(\beta)$. It follows (by the closed graph theorem) that there exists a constant $M$ such that $\|f\|_{H^2(\beta)} \leq M\,\|f\|_\infty$ for all $f \in H^\infty$. Testing that with $f(z) = z^n$, we get $\beta_n \leq M^2$. □

For symbols vanishing at 0, we have the following characterization.

**Proposition 3.2.** *The following assertions are equivalent:*

1) *all symbols* $\varphi$ *such that* $\varphi(0) = 0$ *induce bounded composition operators* $C_\varphi$ *on* $H^2(\beta)$ *and*

$$\sup_{\varphi(0)=0} \|C_\varphi\| < \infty\,; \tag{3.1}$$

2) $\beta$ *is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3.1) is equivalent to:

$$\sup_{\varphi(0)=0} \|f \circ \varphi\| < \infty \quad \text{for all } f \in H^2(\beta)\,.$$

*Proof.* 2) ⇒ 1) We may assume that $\beta$ is non-increasing. Then the Goluzin-Rogosinski theorem ([5, Theorem 6.3]) gives the result; in fact, writing $f(z) = \sum_{n=0}^\infty c_n z^n$ and $(C_\varphi f)(z) = \sum_{n=0}^\infty d_n z^n$, it says that:

$$\|C_\varphi f\|^2 = |d_0|^2\beta_0 + \sum_{n=1}^\infty |d_n|^2\beta_n \leq |c_0|^2\beta_0 + \sum_{n=1}^\infty |c_n|^2\beta_n = \|f\|^2\,,$$

leading to $C_\varphi$ bounded and $\|C_\varphi\| \leq 1$. Alternatively, we can use a result of Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]). This result was also proved by C. Cowen [4, Corollary of Theorem 7].

1) ⇒ 2) Set $M = \sup_{\varphi(0)=0} \|C_\varphi\|$. Let $m > n$, and take:

$$\varphi(z) = \varphi_{m,n}(z) = z\left(\frac{1 + z^{m-n}}{2}\right)^{1/n}.$$

Then $\varphi(0) = 0$ and $[\varphi(z)]^n = \frac{z^n + z^m}{2}$; hence

$$\frac{1}{4}\,(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \leq \|C_\varphi\|^2\|e_n\|^2 \leq M^2\,\beta_n\,,$$

so $\beta$ is essentially decreasing. □

Boundedness of $\beta_n$ does not suffice. For example, let $(\beta_n)$ be a sequence such that $\beta_{2k+2}/\beta_{2k+1} \underset{k\to\infty}{\longrightarrow} \infty$ (for instance $\beta_{2k} = 1$ and $\beta_{2k+1} = 1/(k+1)$); if $\varphi(z) = z^2$, then $\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)}$; since $\|z^{2n+1}\|^2 = \beta_{2n+1}$, $C_\varphi$ is not bounded on $H^2(\beta)$.

A more interesting example is the following. For $0 < r < 1$, let $\beta_n = \pi\, n\, r^{2n}$. This sequence is eventually decreasing, so it is essentially decreasing. The square $\|f\|^2_{H^2(\beta)}$ of the norm $\|f\|_{H^2(\beta)}$ is the area of the part of the Riemann surface on which $r\mathbb{D}$ is mapped by $f$. E. Reich [17], generalizing Goluzin's result [8] (see [5, Theorem 6.3]), proved that for all symbols $\varphi$ such that $\varphi(0) = 0$, the composition operator $C_\varphi$ is bounded on $H^2(\beta)$ and

$$\|C_\varphi\| \leq \sup_{n\geq 1} \sqrt{n}\, r^{n-1} \leq \frac{1}{\sqrt{2\,\mathrm{e}}}\, \frac{1}{r\sqrt{\log(1/r)}} \,\cdot$$

For $0 < r < 1/\sqrt{2}$, Goluzin's theorem asserts that $\|C_\varphi\| \leq 1$.

Note that this sequence $\beta$ does not satisfy the $\Delta_2$-condition since $\beta_{2n}/\beta_n = 2\, r^{2n}$, Theorem 4.1 below states that no composition operator $C_{T_a}$ is bounded.

However that the weight $\beta$ is essentially decreasing is not necessary for the boundedness of all composition operators $C_\varphi$, with symbol $\varphi$ vanishing at 0, as we will see later (Theorem 3.9).

## 3.2 Sufficient condition for the boundedness of composition operators – Part I

We now have one of the the main results of this section.

**Theorem 3.3.** *Let $H^2(\beta)$ be a weighted Hardy space with $\beta = (\beta_n)$ essentially decreasing and satisfying the $\Delta_2$-condition. Then all composition operators on $H^2(\beta)$ are bounded.*

For the proof, we need a lemma.

**Lemma 3.4.** *For $0 < a < 1$, we write:*

$$(T_a z)^n = \sum_{m=0}^{\infty} a_{m,n} z^m \,. \tag{3.2}$$

*Then, there are constants $b > 0,\ 0 < C_1 < 1,\ C_2 > 1$ such that*

$$|a_{m,n}| \leq \begin{cases} \mathrm{e}^{-bn} & \text{if} \quad m \leq C_1 n\,, \\ \mathrm{e}^{-bm} & \text{if} \quad m \geq C_2 n\,. \end{cases}$$

*Proof.* First take $0 < r < 1$; let:

$$M(r) = \sup_{|z|=r} |T_a(z)| = \sup_{|z|=r} \left| \frac{z+a}{1+az} \right| \cdot$$

We have $M(r) < 1$, so we can write $M(r) = r^\rho$, for some positive $\rho = \rho(a)$.

The Cauchy inequalities give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = r^{\rho n - m} \, ,$$

and we obtain the first inequality by taking $r = \mathrm{e}^{-\alpha}$ and adjusting $C_1$.

Next, we use that $T_a$ is analytic on $D(0, 1/a)$. Fix $1 < r =: \mathrm{e}^\beta < 1/a$, with $\beta > 0$. Let $M(r) = \mathrm{e}^\alpha$ with $\alpha > 0$. The Cauchy inequalities again give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = \mathrm{e}^{\alpha n - \beta m} \, ,$$

and we obtain the second inequality by adjusting $C_2$. □

We will also need the following result of V. È Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]).

**Theorem 3.5** (V. È Kacnel'son)**.** *Let $H$ be a separable complex Hilbert space and $(e_i)_{i\geq 0}$ a fixed orthonormal basis of $H$. Let $M\colon H \to H$ be a bounded linear operator. We assume that the matrix of $M$ with respect to this basis is lower-triangular: $\langle M e_j \mid e_i \rangle = 0$ for $i < j$.*

*Let $(\gamma_j)_{j\geq 0}$ be a non-decreasing sequence of positive real numbers and $\Gamma$ the (possibly unbounded) diagonal operator such that $\Gamma(e_j) = \gamma_j e_j$, $j \geq 0$. Then the operator $\Gamma^{-1} M \, \Gamma\colon H \to H$ is bounded and moreover:*

$$\| \Gamma^{-1} M \, \Gamma \| \leq \| M \| \, .$$

*Proof of Theorem 3.3.* We may, and do, assume that $\beta$ is non-increasing.

Proposition 3.2 gives the result when $\varphi(0) = 0$.

It remains to show that all $C_{T_a}$, $a \in \mathbb{D}$, are bounded. Indeed, if $a = \varphi(0)$ and $\psi = T_{-a} \circ \varphi$, then $\psi(0) = 0$ and $\varphi = T_a \circ \psi$, so $C_\varphi = C_\psi \circ C_{T_a}$. Moreover, we have only to show that when $a \in [0, 1)$. Indeed, if $a \in \mathbb{D}$ and $a = |a| \, \mathrm{e}^{i\theta}$, we have $T_a = R_\theta \circ T_{|a|} \circ R_{-\theta}$, so $C_{T_a} = C_{R_{-\theta}} \circ C_{T_{|a|}} \circ C_{R_\theta}$.

We consider the matrices

$$A = (a_{m,n})_{m,n\geq 0} \quad \text{and} \quad A_\beta = \left( \sqrt{\frac{\beta_m}{\beta_n}} \, a_{m,n} \right)_{m,n\geq 0} .$$

Since $C_{T_a} e_n = {T_a}^n$, the formula (3.2) shows that $A$ is the matrix of $C_{T_a}$ in $H^2$ with respect to the basis $(e_n)_{n\geq 0}$. On the other hand, $A_\beta$ is the matrix of $C_{T_a}$ in $H^2(\beta)$ with respect to the basis $(e_n^\beta)_{n\geq 0}$. We note that $A_\beta = BAB^{-1}$, where $B$ is the diagonal matrix with values $\sqrt{\beta_0}, \sqrt{\beta_1}, \ldots$ on the diagonal.

Since $C_{T_a}$ is a bounded composition operators on $H^2$, the matrix $A$ defines a bounded operator on $\ell_2$. We have to show that $A_\beta$ also, i.e. $\| A_\beta \| < \infty$.

For that purpose, we split $A$ and $A_\beta$ into several sub-matrices.

Let $N$ be an integer such that $N \geq 2/C_1$, where $C_1$ is defined in Lemma 3.4 (actually, the proof of that lemma shows that we can take $C_1$ such that $1/C_1$ is an integer, so we could take $N = 2/C_1$). Let $I_0 = [0, N[ \;\; J_0 = [N, +\infty[$ and for $k = 1, 2, \ldots$:

$$I_k = [N^k, N^{k+1}[ \quad \text{and} \quad J_k = [N^{k+1}, +\infty[\,.$$

We define the matrices $D_\beta$ and $R_\beta$, whose entries are respectively:

$$d_{m,n} = \begin{cases} \sqrt{\dfrac{\beta_m}{\beta_n}}\, a_{m,n} & \text{if} \qquad\qquad (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_k) \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$r_{m,n} = \begin{cases} \sqrt{\dfrac{\beta_m}{\beta_n}}\, a_{m,n} & \text{if} \qquad\qquad (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

We also define the matrix $S_\beta$ with entries:

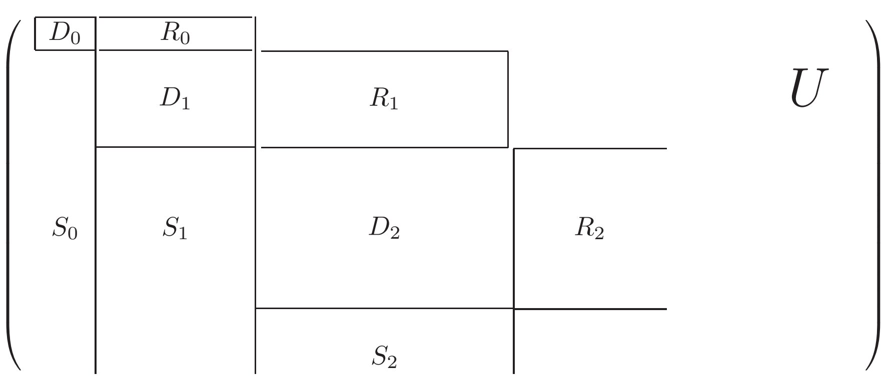
$$s_{m,n} = \begin{cases} \sqrt{\dfrac{\beta_m}{\beta_n}}\, a_{m,n} & \text{if} \qquad\qquad (m,n) \in \bigcup_{k=0}^{\infty} (J_k \times I_k) \\ 0 & \text{elsewhere.} \end{cases}$$

Matrices $D$, $R$, and $S$ are constructed in the same way from $A$ and we set $U = A - (D + R + S)$.

Now, let $H_k$ be the subspace of the sequences $(x_n)_{n\geq 0}$ in $\ell_2$ such that $x_n = 0$ for $n \notin I_k$, i.e. $H_k = \operatorname{span}\{e_n\,;\; n \in I_k\}$, and let $P_k$ be (the matrix of) the orthogonal projection of $\ell_2$ with range $H_k$. We have:

$$D = \sum_{k=0}^{\infty} P_k A P_k \quad \text{and} \quad R = \sum_{k=0}^{\infty} P_k A P_{k+1}\,,$$

where $D_k = P_k A P_k$ is the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times I_k$ and 0 elsewhere, and $R_k = P_k A P_{k+1}$ the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times I_{k+1}$ and 0 elsewhere.



Since the subspaces $H_k$ are orthogonal, the matrices $D$ and $R$ induce bounded operators on $\ell_2$, and

$$\|D\| \leq \|A\|\,, \quad \|R\| \leq \|A\|\,. \tag{3.3}$$

Now, for $k \geq 1$, let $B_k$ be the diagonal matrix whose entries are $b_{m,m} = \sqrt{\beta_m}$ if $m \in I_k$ and $b_{m,n} = 0$ otherwise.

Then $P_k D_\beta P_k = P_k B_k D B_k^{-1} P_k$, so

$$\|P_k D_\beta P_k\| \leq \|B_k\|\, \|B_k^{-1}\|\, \|D\| \leq \max_{j\in I_k} \sqrt{\beta_j}\, \max_{j \in I_k} \frac{1}{\sqrt{\beta_j}}\, \|A\|\,.$$

But the weight $\beta$ satisfies the $\Delta_2$-condition: $\beta_{2l} \geq \delta_0\, \beta_l$, and it follows that for every $l \geq 1$:

$$\beta_{N^2 l} \geq \delta^2\, \beta_l\,,$$

for some other constant $\delta$, chosen small enough to have $\|P_0 D_\beta P_0\| \leq \delta^{-1} \|A\|$. Since $\beta$ is non-increasing, we have $\beta_j \geq \delta^2\, \beta_{N^k}$ for $N^k \leq j \leq N^{k+1}$. In particular $\max_{j\in I_k} \sqrt{\beta_j} \leq \delta^{-1} \min_{j \in I_k} \sqrt{\beta_j}$ and

$$\|P_k D_\beta P_k\| \leq \delta^{-1} \|A\|\,.$$

Hence, by orthogonality of the subspaces $H_k$:

$$\|D_\beta\| \leq \delta^{-1} \|D\|\,. \tag{3.4}$$

In the same way, we have $P_k R_\beta P_k = P_k B_k D B_{k+1}^{-1} P_k$, so:

$$\|P_k R_\beta P_k\| \leq \max_{j \in I_k} \sqrt{\beta_j}\, \max_{j\in I_{k+1}} \frac{1}{\sqrt{\beta_j}}\, \|A\| \,=\, \max_{(m,n) \in I_k \times I_{k+1}} \sqrt{\frac{\beta_m}{\beta_n}}\, \|A\|\,.$$

But, when $(m,n) \in I_k \times I_{k+1}$, we get

$$\beta_n \geq \beta_{N^{k+2}} \geq \delta^2 \beta_{N^k} \geq \delta^2 \beta_m\,.$$

Hence

$$\|R_\beta\| \leq \delta^{-1} \|R\|\,. \tag{3.5}$$

Next, consider $U = A - (D + R + S)$; we can compute its Hilbert-Schmidt norm using Lemma 3.4. Note that $u_{m,n} \neq 0$ only (if it happens) when $m \in I_k$ for some $k \geq 0$ and $n \geq \min I_{k+2} \geq N^{k+2} > Nm$, since $m \in I_k$, so only when $m \leq C_1\, n$. We have, since then $u_{m,n} = a_{m,n}$:

$$\|U\|_{HS}^2 \leq \sum_{n=0}^\infty \sum_{m \leq C_1\, n} |a_{m,n}|^2 \leq \sum_{n=0}^\infty \sum_{m \leq C_1 n} \mathrm{e}^{-2bn} \leq \sum_{n=0}^\infty C_1 n\, \mathrm{e}^{-2bn} < \infty\,.$$

Consequently, with (3.3), we have $\|S\| \leq \|A\| + \|D\| + \|R\| + \|U\| < \infty$.

Now, since $S$ is a lower-triangular matrix and $\beta$ is non-increasing, we can use the result of V. È Kacnel'son (Theorem 3.5), with $\gamma_j = 1/\sqrt{\beta_j}$. We get that $S_\beta$ defines a bounded operator and $\|S_\beta\| \leq \|S\|$.

Further, Proposition 2.4 says that $\beta$ has a polynomial minoration:

$$\beta_n \geq c\, n^{-\sigma}$$

for positive constants $c$ and $\sigma$. Then, if $U_\beta = A_\beta - (D_\beta + R_\beta + S_\beta)$, we have:

$$\|U_\beta\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m<\rho n/2} \frac{|a_{m,n}|^2}{\beta_n} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2}\, \mathrm{e}^{-\alpha \rho n}\, \frac{n^\sigma}{c} < \infty\,.$$

Putting this together with (3.4) and (3.5), we finally obtain that $A_\beta = S_\beta + D_\beta + R_\beta + U_\beta$ is the matrix of a bounded operator, and that ends the proof of Theorem 3.3. □

**Remark.** We could have done here without the theorem of Kacnel'son, using Lemma 3.4 to show that the matrix $\Big(a_{m,n}\sqrt{\frac{\beta_m}{\beta_n}}\Big)$ is Hilbert-Schmidt as well "far below" the main diagonal. Indeed, we see from this lemma that

$$\sum_{m\geq C_2 n} |a_{m,n}|^2 \frac{\beta_m}{\beta_n} \lesssim \sum_{m\geq C_2 n} n^\sigma |a_{m,n}|^2 \lesssim \sum_{m\geq C_2 n} n^\sigma \mathrm{e}^{-2bm} \lesssim \sum_{n\geq 0} n^\sigma \mathrm{e}^{-2bC_2 n} < +\infty\,.$$

See also Theorem 6.2 of the final section. Kacnel'son's theorem will really be needed in the forthcoming Theorem 3.6.

## 3.3 Sufficient condition for the boundedness of composition operators – Part II

In this section, we give a sufficient condition for the boundedness of composition operators with symbol vanishing at 0, of a different nature than the one given in Proposition 3.2.

**Theorem 3.6.** *Let $\beta = (\beta_n)_{n=0}^\infty$ be a bounded sequence of positive numbers with a polynomial minoration. Assume that:*

(3.6) *For every $\delta > 0$, there exists a positive constant $C = C(\delta)$ such that*

$$\beta_m \leq C\, \beta_n \quad \text{whenever } m > (1+\delta)\, n\,.$$

*Then, for all symbols $\varphi\colon \mathbb{D} \to \mathbb{D}$ vanishing at $0$, the composition operator $C_\varphi$ is bounded on $H^2(\beta)$.*

To prove Theorem 3.6, we need several lemmas.

**Lemma 3.7.** *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map such that $\varphi(0) = 0$ and $|\varphi'(0)| < 1$. Then there exists $\rho > 0$ such that*

$$|\widehat{\varphi^n}(m)| \leq \exp\Big(-\frac{1}{2}\,[(1+\rho)\,n - m]\Big)\,.$$

*Proof.* It is the same as that of Lemma 3.4. Since $\varphi(0) = 0$, we can write $\varphi(z) = z\,\varphi_1(z)$. Since $|\varphi'(0)| < 1$, we have $\varphi_1 \colon \mathbb{D} \to \mathbb{D}$. Let $M(r) = \sup_{|z|=r} |\varphi_1(z)|$. The Cauchy inequalities say that $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n / r^m$. We have $M(r) < 1$, so there exists a positive number $\rho = \rho(r)$ such that $M(r) = r^\rho$. We get:

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^{\rho n}}{r^{m-n}} = r^{(1+\rho)\, n - m}\,,$$

and the result follows, by taking $r = \mathrm{e}^{-1/2}$. □

The next lemma is a variant of the result of V. È Kacnel'son quoted before.

**Lemma 3.8.** *Let $A \colon \ell_2 \to \ell_2$ be a bounded operator represented by the matrix $\big(a_{m,n}\big)_{m,n}$, i.e. $a_{m,n} = \langle A\, e_n, e_m\rangle$, where $(e_n)_{n\geq 1}$ is the canonical basis of $\ell_2$.*

*Let $(d_n)$ be a sequence of positive numbers such that, for every $m$ and $n$:*

$$d_m < d_n \quad \Longrightarrow \quad a_{m,n} = 0\,. \tag{3.7}$$

*Then, $D$ being the (possibly unbounded) diagonal operator with entries $d_n$, we have:*

$$\|D^{-1} A D\| \leq \|A\|\,.$$

For the convenience of the reader, we reproduce the proof.

*Proof.* Let $\mathbb{C}_0$ be the right-half plane $\mathbb{C}_0 = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}$. We set $H_N = \mathrm{span}\,\{e_n\,;\ n \leq N\}$ and

$$A_N = P_N A J_N\,,$$

where $P_N$ is the orthogonal projection from $\ell_2$ onto $H_N$ and $J_N$ the canonical injection from $H_N$ into $\ell_2$. We consider, for $z \in \overline{\mathbb{C}_0}$:

$$A_N(z) = D^{-z} A_N D^z \colon H_N \to H_N\,,$$

where $D^z(e_n) = d_n^z e_n$.

If $\big(a_{m,n}(z)\big)_{m,n}$ is the matrix of $A_N(z)$ on the basis $\{e_n\,;\ n \leq N\}$ of $H_N$, we clearly have:

$$a_{m,n}(z) = a_{m,n} (d_n/d_m)^z\,.$$

In particular, we have, thanks to (3.7):

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n\,,$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M\,, \qquad \text{for all } z \in \overline{\mathbb{C}_0}\,.$$

Since $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n\leq N} |a_{m,n}(z)|^2 \leq (N+1)^2 M^2$, we get:

$$\|A_N(z)\| \leq (N+1)\, M \qquad \text{for all } z \in \overline{\mathbb{C}_0}\,.$$

Let us consider the function $u\colon \overline{\mathbb{C}_0} \to \overline{\mathbb{C}_0}$ defined by:

$$u_N(z) = \|A_N(z)\| \,. \tag{3.8}$$

This function $u_N$ is continuous on $\overline{\mathbb{C}_0}$, bounded above by $(N+1)M$, and subharmonic in $\mathbb{C}_0$. Moreover, thanks to (3.7), the maximum principle gives:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) = \sup_{\partial\mathbb{C}_0} u_N(z) \,.$$

Since $\|D^z\| = \|D^{-z}\| = 1$ for $z \in \partial\mathbb{C}_0$, we have $\|A_N(z)\| \leq \|A_N\|$ for $z \in \partial\mathbb{C}_0$, and we get:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) \leq \|A_N\| \leq \|A\| \,.$$

In particular $u_N(1) \leq \|A\|$, and, letting $N$ going to infinity, we get $\|D^{-1}AD\| \leq \|A\|$. □

*Proof of Theorem 3.6.* First, if $|\varphi'(0)| = 1$, we have $\varphi(z) = \alpha\, z$ for some $\alpha$ with $|\alpha| = 1$, and the result is trivial.

So, we assume that $|\varphi'(0)| < 1$. Then, by Lemma 3.7, there exists $\rho > 0$ such that, for all $m$, $n$:

$$|\widehat{\varphi^n}(m)| \leq \exp\Big( - \frac{1}{2}\,[(1+\rho)\,n - m]\Big) \,.$$

Since $\varphi(0) = 0$, we also know that $\widehat{\varphi^n}(m) = 0$ if $m < n$.

Take $\delta = \rho/2$ and use property (3.6): there exists $C > 0$ such that:

$$\frac{\beta_m}{\beta_n} \leq C \quad \text{when } m \geq (1+\delta)\,n \,.$$

Define now a new sequence $\gamma = (\gamma_n)$ as:

$$\gamma_n = \max\bigg\{\beta_n,\ \sup_{m > (1+\delta)\,n} \beta_m\bigg\} \,.$$

We have:

1) $\beta_n \leq \gamma_n \leq C\,\beta_n$;

2) $\gamma_m \leq \gamma_n \quad$ if $m \geq (1+\delta)\,n$.

Item 1) implies that $H^2(\gamma) = H^2(\beta)$, and we are reduced to prove that $C_\varphi\colon H^2(\gamma) \to H^2(\gamma)$ is bounded.

Let $A = \big(a_{m,n}\big) = \big(\widehat{\varphi^n}(m)\big)$. We have to prove that

$$B = \big(\gamma_m^{1/2}\gamma_n^{-1/2} a_{m,n}\big)_{m,n}$$

represents a bounded operator on $\ell_2$.

Define the matrix

$$A_1 = \big(a_{m,n} \mathbb{1}_{\{(m,n)\,;\; m \leq (1+\delta)\, n\}}\big)_{m,n}$$

and set $A_2 = A - A_1$. Define analogously $B_1$ and $B_2 = B - B_1$.

Then $A_1$ is a Hilbert-Schmidt operator, because (recall that $a_{m,n} = 0$ if $m < n$)

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{m=1}^{(1+\delta)\, n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)\, n} \exp\big(-[(1+\rho)\, n - m]\big) \\ &\leq \sum_{n=1}^{\infty} (\delta\, n + 1)\, \exp(-\delta\, n) < \infty\,.\end{aligned}$$

Now, $\beta$ is bounded above and has a polynomial minoration, so, for some positive constants $C_1$, $C_2$, and $\alpha$, we have:

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)\, n} \frac{\gamma_m}{\gamma_n}\, |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)\, n} \frac{C_1}{n^{-\alpha}}\, \exp(-\delta\, n) \\ &\leq \sum_{n=1}^{\infty} C_2\, n^{\alpha+1} \exp(-\delta\, n) < \infty\,,\end{aligned}$$

meaning that $B_1$ is a Hilbert-Schmidt operator.

Since $A$ is bounded, it follows that $A_2 = A - A_1$ is bounded. Remark that, writing $A_2 = \big(\alpha_{m,n}\big)_{m,n}$, we have, with $d_n = 1/\sqrt{\gamma_n}$:

$$d_m < d_n \quad \Longrightarrow \quad \gamma_m > \gamma_n \quad \Longrightarrow \quad m < (1+\delta)\, n \quad \Longrightarrow \quad \alpha_{m,n} = 0\,.$$

Hence we can apply Lemma 3.8 to the matrix $A_2$, and it ensues that $B_2$ is bounded, and therefore that $B = B_1 + B_2$ is bounded as well, as wanted. □

As a corollary of Theorem 3.6, we can provide the following example.

**Theorem 3.9.** *There exists a bounded sequence $\beta$, with a polynomial minoration, but which is* not essentially decreasing*, for which every composition operator with symbol vanishing at* 0 *is bounded on $H^2(\beta)$, with nevertheless* $\sup_{\varphi(0)=0} \|C_\varphi\| = \infty$.

It should be noted that for this weight, the composition operators are not all bounded, as we will see in Proposition 4.4.

*Proof.* Define $\beta_n = 1$ for $n \leq 3!$, and, for $k \geq 3$:

$$\begin{cases} \beta_n = \dfrac{1}{k!} & \text{for } k! < n \leq (k+1)! - 2 \text{ and for } n = (k+1)! \\[2ex] \beta_n = \dfrac{1}{(k+1)!} & \text{for } n = (k+1)! - 1\,. \end{cases}$$

Note that, for $m > n$, we have $\beta_m > \beta_n$ only if $n = (k+1)! - 1$ and $m = (k+1)! = n+1$, for some $k \geq 3$.

However, $\beta$ is not essentially decreasing since, for every $k \geq 3$, we have $\beta_{n+1}/\beta_n = k+1$ if $n = (k+1)! - 1$.

The sequence $\beta$ has a polynomial minoration because $\beta_n \geq 1/(2\,n)$ for all $n \geq 1$. In fact, for $k \geq 3$, we have $\beta_n \geq (k{+}1)/n \geq 1/n$ if $k! < n \leq (k{+}1)!-2$ or if $n = (k{+}1)!$; and for $n = n = (k{+}1)!-1$, we have $n\,\beta_n = [(k{+}1)!-1]/(k{+}1)! \geq 1/2$.

Now, it remains to check (3.6) in order to apply Theorem 3.6 and finish the proof of Theorem 3.9. Note first that we have $\beta_m/\beta_n \leq 1$ if $m \geq n+2$. Next, for given $\delta > 0$, there exists an integer $N$ such that $(1+\delta)\,n \geq n+2$ for every $n \geq N$, so $\beta_m/\beta_n \leq 1$ if $m \geq (1+\delta)\,n$ and $n \geq N$. It suffices to take $C = \max_{1 \leq n \leq N} \beta_{n+1}/\beta_n$ to obtain (3.6). The last assertion follows from Proposition 3.2. □

# 4 Necessity of the $\Delta_2$-condition

In this section, we will show that, for essentially decreasing sequences $\beta$, the $\Delta_2$-condition is necessary for having boundedness of composition operators on $H^2(\beta)$. We will indeed show slightly more.

**Theorem 4.1.** *Let $\beta$ be such that, for some $a \in (0,1)$, $T_a$ induces a bounded composition operator on $H^2(\beta)$. Then $\beta$ satisfies:*

$$(\exists\, 0 < \delta < 1/3)\ (\forall n \geq 1)\ (\exists\, E_n \subseteq [(1-2\delta)n, (1-\delta)n]) \quad \textit{with}$$
$$|E_n| \geq \delta n \quad \textit{and} \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m\,.$$

*In particular, if $\beta$ is essentially decreasing, then $\beta$ satisfies the $\Delta_2$-condition.*

In order to prove Theorem 4.1, we need several preliminary lemmas. The first one is standard, but we give it for convenience.

**Lemma 4.2.** *Let $a \in (0,1)$ and let*

$$P_{-a}(x) = \frac{1-a^2}{1 + 2\,a \cos x + a^2}$$

*be the Poisson kernel at the point $-a$. Then, for all $x \in [-\pi, \pi]$:*

$$T_a(\mathrm{e}^{ix}) = \exp\big[i\,h_a(x)\big]\,, \tag{4.1}$$

*where*

$$h_a(x) = \int_0^x P_{-a}(t)\,dt\,. \tag{4.2}$$

*Proof.* For $t \in [-\pi, \pi]$, write:

$$\psi(t) := \frac{\mathrm{e}^{it} + a}{1 + a\,\mathrm{e}^{it}} = \exp(i\,u(t))\,,$$

with $u$ a real-valued, $\mathcal{C}^1$ function on $[-\pi, \pi]$ such that $u(0) = 0$. This is possible since $|\psi(\mathrm{e}^{it})| = 1$ and $\psi(0) = 1$. Differentiating both sides with respect to $t$, we get:

$$i\,\mathrm{e}^{it}\,\frac{1 - a^2}{(1 + a\,\mathrm{e}^{it})^2} = i\,u'(t)\,\frac{\mathrm{e}^{it} + a}{1 + a\,\mathrm{e}^{it}}\,.$$

This implies

$$u'(t) = \frac{1 - a^2}{|1 + a\,\mathrm{e}^{it}|^2} = P_{-a}(t)\,,$$

and the result follows since $u(0) = 0 = h_a(0)$. □

## 4.1 Main lemma and proof of Theorem 4.1

To prove Theorem 4.1, we need the following lemma. The proof of this lemma uses Lemma 4.2 and a van der Corput type estimate, inspired from [21, pp. 72–73]. We thank R. Zarouf [22] for interesting recent informations in this respect, related to his joint work with O. Szehr on the Schäffer problem (see [20], in which the authors are primarily concerned with upper bounds).

The first version of our paper was put on arXiv at the end of November 2020. Since then, the paper [1] was put on arXiv on July 2021, where sharp estimates of powers of Blaschke factors are given (see also K. Fouchet's thesis [6]), with different purposes (strongly annular analytic functions). However, in our case, our proof is much simpler.

Recall that we have set:

$$[T_a(z)]^n = \sum_{m=0}^{\infty} a_{m,n} z^m\,. \tag{4.3}$$

**Lemma 4.3.** *Let $a \in (0, 1)$. We set:*

$$\tau = \frac{1 + a}{1 - a} > 1 \tag{4.4}$$

*and write:*

$$\tau^{-1} = 1 - 3\mu\,, \tag{4.5}$$

*with $\mu = \mu_a \in (0, 1/3)$. For every fixed positive integer $n$, let:*

$$J_n = [(1 - 2\mu)\,n, (1 - \mu)\,n]\,. \tag{4.6}$$

*Then, there exists $\delta = \delta_a > 0$ such that, for every $n$ large enough, there exists a set of indices $E_n \subseteq J_n$ with cardinality $|E_n| \geq \delta n$ and such that:*

$$m \in E_n \quad \Longrightarrow \quad |a_{m,n}| \geq \delta\, n^{-1/2}\,. \tag{4.7}$$

*Proof of Theorem 4.1.* Set $M = \|C_{T_a}\|$. We have:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|{T_a}^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n \,. \tag{4.8}$$

so, by Lemma 4.3, since $E_n \subseteq J_n = [(1 - 2\mu)n, (1 - \mu)n]$:

$$M^2 \beta_n \geq \sum_{m \in E_n} |a_{m,n}|^2 \beta_m \geq \delta^2 \, n^{-1} \sum_{m \in E_n} \beta_m \geq \delta^3 \, |E_n|^{-1} \sum_{m \in E_n} \beta_m \,.$$

This proves (changing $\delta$) the first part of Theorem 4.1. Next, assume that $\beta$ is essentially decreasing. We may, and do, assume that $\beta$ is non-increasing. We set, for $x$ not an integer, $\beta_x = \beta_k$ with $k$ the least integer greater than $x$. The above implies, for all integers $n \geq 1$:

$$\beta_n \geq (\delta^3 / M^2) \, |E_n|^{-1} |E_n| \, \beta_{(1-\mu)n} \geq c \, \beta_{(1-\mu)n} \,.$$

Let $r \geq 1$ such that $(1 - \mu)^r \leq 1/2$; we have:

$$\beta_n \geq c^r \, \beta_{(1-\mu)^r n} \geq c^r \, \beta_{n/2} \,,$$

so $\beta$ satisfies the $\Delta_2$-condition. □

A consequence of Theorem 4.1 is the following result.

**Proposition 4.4.** *For the weight $\beta$ constructed in the proof of Theorem 3.9, no automorphism $T_a$ with $0 < a < 1$ can be bounded.*

*Proof.* Consider the necessary condition for the boundedness of $C_{T_a}$ in Theorem 4.1:

$$\beta_n \geq \delta \, \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m \,. \tag{4.9}$$

For the weight $\beta$ constructed in the proof of Theorem 3.9, we are going to see that this condition (4.9) is not satisfied for $n = (k + 1)! - 1 =: n_k$.

Indeed, for this $n$, the left-hand side of (4.9) is equal to $1/(k + 1)!$ and the right-hand side to $\delta/k!$ since $(1-2\delta)n_k > k!$ for $k$ large, so that all $\beta_m$ are equal to $1/k!$ for $m \in E_{n_k}$. This ends the proof. □

## 4.2 Proof of Lemma 4.3

To prove Lemma 4.3, we will use a variant of [21, Lemma 4.6 p. 72] on the stationary phase method. A careful reading of the proof in [21, p. 72] gives the version below, which allows the derivative $F'$ of $F$ to vanish at some point, as occurs in our situation. For sake of completeness, we will give a proof, however postponed.

**Proposition 4.5** (Stationary phase)**.** *Let $F$ be real function on the interval $[A,B]$, with continuous derivatives up to the third order and $F''>0$ throughout $]A,B[$. Assume that there is a (unique) point $c$ in $]A,B[$ such that $F'(c)=0$, and that, for some positive numbers $\lambda_2$, $\lambda_3$, and $\eta$, the following assertions hold:*

1) $[c-\eta, c+\eta] \subseteq [A,B]$*;*
2) $F''(x) \geq \lambda_2$ *for all* $x \in [c-\eta, c+\eta]$*;*
3) $|F'''(x)| \leq \lambda_3$ *for all* $x \in [A,B]$*.*

*Then:*

$$\int_A^B \mathrm{e}^{iF(x)}\,dx = \sqrt{2\pi}\,\frac{\mathrm{e}^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\right), \tag{4.10}$$

*where the $O$ involves an absolute constant.*

*Proof of Lemma 4.3.* We turn to the problem of bounding $a_{m,n}$ from below, *in the case $m \in J_n$, and only in that case.* Since $\inf_{[0,\pi]} P_{-a} = \tau^{-1} < \sup_{[0,\pi]} P_{-a} = \tau$, there exists a unique point $x_m = x_{m,n} \in [0,\pi]$ such that

$$nP_{-a}(x_m) - m = n\,\frac{(1-a^2)}{1+2\,a\cos x_m + a^2} - m = 0\,,$$

or else:

$$\cos x_m = \frac{n}{m}\,\frac{1-a^2}{2\,a} - \frac{1+a^2}{2\,a}\,. \tag{4.11}$$

The point is that if $m \in J_n$, $x_m$ can approach neither 0 nor $\pi$, so that $\sin x_m \geq \delta_a > 0$; more precisely, the definition of $J_n$ and (4.11) imply that $\pi/4 \leq x_m \leq \pi/2$.

With $h_a$ the function of Lemma 4.2, the Fourier formulas give, since $a_{m,n}$ is real, or since $h_a(x) - mx$ is odd:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp i[nh_a(x) - mx]\,dx = 2\,\Re\mathrm{e}\, I_{m,n}\,,$$

where

$$I_{m,n} = \int_0^{\pi} \exp i[nh_a(x) - mx]\,dx\,. \tag{4.12}$$

Write:

$$I_{m,n} = \int_0^{\pi} \exp[i\,F_m(x)]\,dx\,, \tag{4.13}$$

with:

$$F_m(x) = nh_a(x) - mx = n\int_0^x P_{-a}(t)\,dt - mx\,. \tag{4.14}$$

We have:

$$F'_m(x) = n\,P_{-a}(x) - m\,. \tag{4.15}$$

We will now proceed in two steps, first giving good lower bounds for $|I_{m,n}|$, then showing that the argument of $I_{m,n}$ is often far from $\pi/2$ mod. $\pi$. Then, we will be done.

**First step.** We will prove that:

$$I_{m,n} = \sqrt{2\pi}\,n^{-1/2}\ \frac{\mathrm{e}^{i\,(F_m(x_m)+\pi/4)}}{\sqrt{|h''_a(x_m)|}} + O\,(n^{-3/5})\,, \tag{4.16}$$

where the $O$ only depends on $a$.

Note that $3/5 > 1/2$ and $F''_m = n\,h''_a$.

To get (4.16), we will show that Theorem 4.5 is applicable with:

$$[A,B] = [0,\pi]\,,\quad c = x_m\,,\quad \lambda_2 = \kappa_0\,n\,,\quad \lambda_3 = C_0 n\,,\quad \eta = (\lambda_2\lambda_3)^{-1/5}\,.$$

The parameter $\eta$ is chosen in order to make both error terms in Theorem 4.5 equal: $\frac{1}{\eta\lambda_2} = \eta^4\lambda_3$; so:

$$\eta = \kappa\,n^{-2/5}$$

and

$$\frac{1}{\eta\lambda_2} + \eta^4\lambda_3 = \widetilde{\kappa}\,n^{-3/5} = O\,(n^{-3/5}) \tag{4.17}$$

(with $\kappa = (\kappa_0 C_0)^{-1/5}$ and $\widetilde{\kappa} = 2/\kappa_0\kappa$).

The slight technical difficulty encountered here is that $F''_m(x)$ vanishes at 0 and $\pi$. But Theorem 4.5 covers this case. We have

$$F''_m(x) = nP'_{-a}(x) = 2\,a(1-a^2)\,\frac{\sin x}{(1+2a\cos x + a^2)^2}\,n\,,$$

and there are some positive $\kappa_0$ and $\sigma$ such that

$$F''_m(x) \geq \kappa_0\,n = \lambda_2 \quad \text{for } x \in [\sigma, \pi-\sigma]\,. \tag{4.18}$$

Now (for $n$ large enough), $[x_m-\eta, x_m+\eta] \subseteq [\sigma, \pi-\sigma]$. Hence the assumptions 1) and 2) of Proposition 4.5 are satisfied.

Finally, since $F_m(x) = nh_a(x) - mx$, and $h_a$ is $\mathcal{C}^\infty$ on $\mathbb{R}$, we have, for all $x \in [0,\pi]$:

$$|F'''_m(x)| \leq C_0\,n = \lambda_3\,,$$

and assertion 3) of Proposition 4.5 holds.

With (4.17) this ends the proof of (4.16), once we remarked that $nh''_a(x_m) = F''_m(x_m)$.

Note that, since $|h_a''(x_m)| \le M_a$, we get that $|I_{m,n}| \ge \delta\, n^{-1/2}$ when $m \in J_n$.

**Second step.** The mean-value theorem gives, for $m \in J_n$:

$$|\sin x_m| \ge \delta \quad \text{and} \quad x_{m+1} - x_m \approx \cos x_m - \cos x_{m+1}\,. \tag{4.19}$$

We also have, for $x \in \mathcal{J} = [1-2\mu, 1-\mu]$, with another constant $\delta$:

$$\delta \le P'_{-a}(x) = 2\,a(1-a^2)\,\frac{\sin x}{(1+2a\cos x + a^2)^2} \le \delta^{-1}\,. \tag{4.20}$$

We now claim that

$$x_{m+1} - x_m \approx n^{-1} \quad \text{for } m \in J_n\,. \tag{4.21}$$

Indeed, since $m \in J_n$, we have, using (4.11):

$$\cos x_m - \cos x_{m+1} = \frac{1-a^2}{2a}\,\frac{n}{m(m+1)} \approx \frac{n}{m^2} \approx n^{-1}\,.$$

In view of (4.19), this proves (4.21).

Now, according to (4.16), when $m \in J$, the main term in $I_{m,n}$ is

$$A_{m,n} := n^{-1/2}\,\frac{\sqrt{2\pi}}{\sqrt{|h_a''(x_m)|}}\,\mathrm{e}^{i(F_m(x_m)+\pi/4)}\,,$$

and its argument $\theta_m$ is $F_m(x_m) + \pi/4$. Going from $m$ to $m+1$, the variation $F_{m+1}(x_{m+1}) - F_m(x_m)$ of this argument is

$$\begin{aligned}\theta_{m+1} - \theta_m &= n\int_{x_m}^{x_{m+1}} \left(P_{-a}(t) - \frac{m}{n}\right) dt - x_{m+1}\\ &= n\int_{x_m}^{x_{m+1}} \Big(P_{-a}(t) - P_{-a}(x_m)\Big)\, dt - x_{m+1}\,.\end{aligned}$$

But, due to (4.20), this implies:

$$\begin{aligned}\theta_{m+1} - \theta_m &= -x_{m+1} + \mathrm{O}\left(n\int_{x_m}^{x_{m+1}} (t - x_m)\, dt\right)\\ &= -x_{m+1} + \mathrm{O}\left(n\,(x_{m+1}-x_m)^2\right) = -x_{m+1} + \mathrm{O}\left(n^{-1}\right).\end{aligned}$$

Since $-\sigma \le -x_{m+1} \le \pi - \sigma$, the variation of $\theta_m$ is thus regular. As a consequence, for a positive proportion $E_n$ of the indices $m \in J_n$, the argument $\theta_m$ will belong to a subarc of $\mathbb{T}$ which lies $\delta$-apart from $\pm\pi/2$, implying $\cos\theta_m \ge \delta$, or else:

$$|E_n| \ge \delta\, n\,,$$

and, for all $m \in E_n$:

$$\mathfrak{Re}\, A_{m,n} \ge \delta\,|A_{m,n}| \ge \delta\, n^{-1/2}.$$

It follows that, for $m \in E_n$:

$$\mathfrak{Re}\, I_{m,n} \ge \delta\, n^{-1/2} - C\, n^{-3/5} \ge \widetilde{\delta}\, n^{-1/2}\,.$$

Since $a_{m,n} = \pi^{-1}\mathfrak{Re}\, I_{m,n}$, that ends the proof of Lemma 4.3. □

We now pass to the proof of Proposition 4.5. The following lemma can be found in [15, Lemma 1, page 47].

**Lemma 4.6.** *Let $F\colon [u,v] \to \mathbb{R}$, with $u < v$, be a $\mathcal{C}^2$- function with $F'' > 0$, and $F'$ not vanishing on $[u,v]$. Let*

$$J = \int_u^v e^{iF(x)}\, dx\,.$$

*Then:*

a) *if $F' > 0$ on $[u,v]$, then $|J| \leq \frac{2}{F'(u)}$ ;*
b) *If $F' < 0$ on $[u,v]$, then $|J| \leq \frac{2}{|F'(v)|}$ .*

*Proof of Proposition 4.5.* Write now the integral $I$ of Proposition 4.5 on $[A,B]$ as $I = I_1 + I_2 + I_3$ with:

$$I_1 = \int_A^{c-\eta} \mathrm{e}^{iF(x)}\, dx\,, \quad I_2 = \int_{c-\eta}^{c+\eta} \mathrm{e}^{iF(x)}\, dx\,, \quad I_3 = \int_{c+\eta}^{B} \mathrm{e}^{iF(x)}\, dx\,.$$

Lemma 4.6 with $u = A$ and $v = c - \eta$ implies:

$$|I_1| \leq \frac{2}{|F'(c-\eta)|} \leq \frac{2}{\eta\lambda_2}\,, \tag{4.22}$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta\, F''(\xi)$$

for some $\xi \in [c-\eta, c]$ so that $F''(\xi) \geq \lambda_2$.

Similarly, Lemma 4.6 with $u = c+\eta$ and $v = B$ implies

$$|I_3| \leq \frac{2}{F'(c+\eta)} \leq \frac{2}{\eta\lambda_2}\,. \tag{4.23}$$

We can now estimate $I_2$. The Taylor formula shows that

$$F(x) = F(c) + \frac{(x-c)^2}{2} F''(c) + R\,,$$

with

$$|R| \leq \frac{|x-c|^3}{6}\,\lambda_3\,.$$

Hence

$$I_2 = \mathrm{e}^{iF(c)} \int_0^\eta 2\exp\left(\frac{i}{2}\, x^2 F''(c)\right) dx + S$$

with

$$|S| \leq \lambda_3 \int_0^\eta \frac{x^3}{3}\, dx = \frac{\eta^4}{12}\,\lambda_3\,.$$

Finally, set

$$K = \int_0^\eta 2 \exp\left(\frac{i}{2}\, x^2 F''(c)\right) dx \, .$$

We make the change of variable $x = \sqrt{\frac{2}{F''(c)}}\,\sqrt{t}$. Recall that $\int_0^\infty \frac{\mathrm{e}^{it}}{\sqrt{t}} dt = \sqrt{\pi}\,\mathrm{e}^{i\pi/4}$ is the classical Fresnel integral, and that an integration by parts gives, for $m > 0$:

$$\left| \int_m^\infty \frac{\mathrm{e}^{it}}{\sqrt{t}}\, dt \right| \leq \frac{2}{\sqrt{m}} \, \cdot$$

Therefore, with $m = \frac{\eta^2}{2}\, F''(c)$:

$$K = \sqrt{\frac{2}{F''(c)}} \int_0^m \frac{\mathrm{e}^{it}}{\sqrt{t}}\, dt = \sqrt{\frac{2\pi}{F''(c)}}\, \mathrm{e}^{i\pi/4} + R_m \, ,$$

with

$$|R_m| \leq C \sqrt{\frac{1}{F''(c)}}\, \frac{1}{\sqrt{m}} \leq \frac{C}{\eta \lambda_2} \, \cdot$$

All in all, we proved that

$$I_2 = \sqrt{\frac{2\pi}{F''(c)}}\, \exp\big[i(F(c) + \pi/4)\big] + O\left(\frac{1}{\eta\lambda_2} + \eta^4 \lambda_3\right) \cdot \tag{4.24}$$

and the same estimate holds for $I$, thanks to (4.22) and (4.23).

We have hence proved Proposition 4.5. □

# 5 Some results on multipliers

The set $\mathcal{M}\big(H^2(\beta)\big)$ of multipliers of $H^2(\beta)$ is by definition the vector space of functions $h$ analytic on $\mathbb{D}$ and such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. When $h \in \mathcal{M}\big(H^2(\beta)\big)$, the operator $M_h$ of multiplication by $h$ is bounded on $H^2(\beta)$ by the closed graph theorem. The space $\mathcal{M}\big(H^2(\beta)\big)$ equipped with the operator norm is a Banach space. We note the obvious property:

$$\mathcal{M}\big(H^2(\beta)\big) \hookrightarrow H^\infty \quad \text{contractively.} \tag{5.1}$$

Indeed, if $h \in \mathcal{M}\big(H^2(\beta)\big)$, we easily get for all $w \in \mathbb{D}$:

$$M_h^*(K_w) = \overline{h(w)} K_w \, ;$$

so that taking norms and simplifying, we are left with $|h(w)| \leq \|M_h\|$, showing that $h \in H^\infty$ with $\|h\|_\infty \leq \|M_h\|$.

**Proposition 5.1.** *We have $\mathcal{M}\big(H^2(\beta)\big) = H^\infty$ isomorphically if and only if $\beta$ is essentially decreasing.*

*Proof.* The sufficient condition is proved in [13, beginning of the proof of Proposition 3.16]. For the necessity, we have $\|M_h\| \approx \|h\|_\infty$ for every $h \in H^\infty$. Now, for $m > n$ (recall that $e_n(z) = z^n$):

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z)\,;$$

so, since $\|M_{e_{m-n}}\| \leq C\, \|e_{m-n}\|_\infty = C$ for some positive constant $C$:

$$\beta_m = \|e_m\|^2 \leq C^2\, \|e_n\|^2 = C^2\, \beta_n\,. \qquad \square$$

In [13, Section 3.6], we gave the following notion of *admissible* Hilbert space of analytic functions.

**Definition 5.2.** *A Hilbert space $H$ of analytic functions on $\mathbb{D}$, containing the constants, and with reproducing kernels $K_a$, $a \in \mathbb{D}$, is said* admissible *if:*

- $(i)$ *$H^2$ is continuously embedded in $H$;*
- $(ii)$ *$\mathcal{M}(H) = H^\infty$;*
- $(iii)$ *the automorphisms of $\mathbb{D}$ induce bounded composition operators on $H$;*
- $(iv)$ $\dfrac{\|K_a\|_H}{\|K_b\|_H} \leq h\bigg(\dfrac{1-|b|}{1-|a|}\bigg)$ *for $a, b \in \mathbb{D}$, where $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an non-decreasing function.*

We proved in that paper that every weighted Hilbert space $H^2(\beta)$ with $\beta$ non-increasing is admissible, under the additional hypothesis that the automorphisms of $\mathbb{D}$ induce bounded composition operators. In view of Theorem 3.3, we get the following result.

**Proposition 5.3.** *Let $\beta$ be essentially decreasing that satisfies the $\Delta_2$-condition. Then $H^2(\beta)$ is admissible.*

Let us give a different proof.

*Proof.* Because $\beta$ is essentially decreasing, item $(i)$ holds, as well as item $(ii)$, by Proposition 5.1. Item $(iii)$ is Theorem 3.3. It remains to show $(iv)$. We may assume that $\beta$ is non-increasing.

Let $0 < s < r < 1$.

Without loss of generality, we may assume that $r, s \geq 1/2$. It is enough to prove:

$$\|K_r\|^2 \leq C\, \|K_{r^2}\|^2 \tag{5.2}$$

for some constant $C > 1$. Indeed, iteration of (5.2) gives:

$$\|K_r\|^2 \leq C^k\, \|K_{r^{2^k}}\|^2$$

and if $k$ is the smallest integer such that $r^{2^k} \leq s$, we have $2^{k-1} \log r > \log s$ and $2^k \leq D \frac{1-s}{1-r}$ where $D$ is a numerical constant. Writing $C = 2^\alpha$ with $\alpha > 1$, we obtain:

$$\left( \frac{\|K_r\|}{\|K_s\|} \right)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \left( \frac{1-s}{1-r} \right)^\alpha .$$

To prove (5.2), we pick some $M > 1$ such that $\beta_{2n} \geq M^{-1} \beta_n$ for all $n \geq 1$ and write $t = r^2$. We have:

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\beta_{2n-1}} ,$$

implying, since $\beta_{2n-1} \geq \beta_{2n} \geq M^{-1} \beta_n$ and $t^{2n-1} \leq 4\, t^{2n}$:

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M \|K_t\|^2 . \qquad \square$$

The notion of admissible Hilbert space $H$ is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{ w \in H \, ; \; w\, (f \circ \varphi) \in H \text{ for all } f \in H \} .$$

As a corollary of [13, Theorem 3.18] we get:

**Corollary 5.4.** *Let $\beta$ be essentially decreasing and satisfying the $\Delta_2$-condition. Then:*

1) $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}\big(H^2(\beta), \varphi\big)$;
2) $\mathcal{M}\big(H^2(\beta), \varphi\big) = H^2(\beta)$ *if and only if* $\|\varphi\|_\infty < 1$;
3) $\mathcal{M}\big(H^2(\beta), \varphi\big) = H^\infty$ *if and only if $\varphi$ is a finite Blaschke product.*

# 6 Miscellaneous remarks

Some of the results of this paper can slightly be improved.

## 6.1 Conditions on the weight $\beta$

First, we say that a sequence $(\beta_n)$ of positive numbers is *slowly oscillating* if there if a function $\rho \colon (0, \infty) \to (0, \infty)$ that is bounded on each compact subset of $(0, \infty)$ for which:

$$\frac{\beta_m}{\beta_n} \leq \rho \left( \frac{m}{n} \right) \cdot$$

This clearly amounts to say that, for some positive constants $C_1 < C_2$, we have:

$$C_1 \leq \frac{\beta_m}{\beta_n} \leq C_2 \quad \text{when } n/2 \leq m \leq 2n .$$

Every essentially decreasing sequence with the $\Delta_2$-condition is slowly oscillating.

**Proposition 6.1.** *The following holds:*

1) *every slowly oscillating sequence has a polynomial minoration;*

2) *there are bounded sequences which are slowly oscillating, but not essentially decreasing.*

*Proof.* 1) is clear, because if $2^j \leq n < 2^{j+1}$, then

$$\beta_n \geq C^{-1}\beta_{2^j} \geq C^{-j-1}\beta_1 \geq C^{-1}\beta_1\, n^{-\alpha}\,,$$

with $\alpha = \log C/\log 2$.

2) We define $\beta_n$ as follows. Let $(a_k)$ be an increasing sequence of positive square integers such that $\lim_{k\to\infty} a_{k+1}/a_k = \infty$, for example $a_k = 4^{k^2}$, and let $b_k = \sqrt{a_k a_{k+1}}$; with our choice, this is an integer and we clearly have $a_k < b_k < a_{k+1}$. We set:

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k \\ \\ (a_k/b_k^2)\, n = (1/a_{k+1})\, n & \text{for } b_k \leq n < a_{k+1}\,. \end{cases}$$

This sequence $(\beta_n)$ is slowly oscillating by construction. Indeed, it suffices to check that for $a_k \leq n/2 < b_k \leq n < a_{k+1}$, the quotient $\beta_m/\beta_n$ remains lower and upper bounded when $n/2 \leq m \leq n$. But for $n/2 \leq m < b_k$, we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn}\,,$$

which is $\leq 2\, b_k^2/n^2 \leq 2$ and $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = 1/4$; and for $b_k \leq m$, we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in [1/2, 1]\,.$$

However, though $(\beta_n)$ is bounded, since $\beta_n \leq 1$ for $a_k \leq n < b_k$ and, for $b_k \leq n < a_{k+1}$,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}}\,(a_{k+1} - 1) \leq 1\,,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}}\,(a_{k+1} - 1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \underset{k\to\infty}{\longrightarrow} \infty\,. \qquad \square$$

By a slight modification (change the value of the constants in the definition of $\beta_n$), we could obtain a sequence which is slowly oscillating, tends to zero, yet again not essentially decreasing.

Now, Theorem 3.3 admits the following variant.

**Theorem 6.2.** *Let $(\beta_n)$ be a sequence of positive numbers which is bounded above and slowly oscillating. Then all symbols that extend analytically in a neighborhood of $\overline{\mathbb{D}}$ induce a bounded composition operator on $H^2(\beta)$.*

It is the case, for example, for finite Blaschke products. The proof follows that of Theorem 3.3, with the help of the following lemma.

**Lemma 6.3.** *Let $(\beta_n)$ be a sequence of positive numbers which is bounded above and slowly oscillating. Let $A = (a_{m,n})_{m,n}$ be the matrix of a bounded operator on $\ell_2$. Assume that, for constants $C_1 < 1$, $C_2 > 1$, and c, b, we have:*

1) $|a_{m,n}| \leq c\,\mathrm{e}^{-bn}$ *for* $m \leq C_1 n$*;*
2) $|a_{m,n}| \leq c\,\mathrm{e}^{-bm}$ *for* $m \geq C_2 n$*.*

*Then the matrix* $\widetilde{A} = \left(a_{m,n}\sqrt{\dfrac{\beta_m}{\beta_n}}\right)$ *also defines a bounded operator on* $\ell_2$*.*

*Sketch of proof.* The matrix $\widetilde{A}$ is Hilbert-Schmidt far from the diagonal since, with $\lambda \geq \beta_n \geq \delta\, n^{-\alpha}$, we have:

$$\sum_{m<C_1 n} |a_{m,n}|^2 \beta_m/\beta_n \leq \sum_{m<C_1 n} \lambda\,\delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n\geq 1} n^{\alpha+1}\mathrm{e}^{-bn} < \infty\,,$$

and

$$\sum_{m>C_2 n} |a_{m,n}|^2 \beta_m/\beta_n \leq \sum_{m>C_2 n} \lambda\,\delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n\geq 1} n^{\alpha}\Bigg(\sum_{m>C_2 n} \mathrm{e}^{-bm}\Bigg).$$

Since $\beta_m/\beta_n$ remains bounded from above and below around the diagonal, the matrix $\widetilde{A}$ behaves like $A$ near the diagonal. □

**Remark.** The proof shows that, instead of 1) and 2), it is enough to have:

$$\sum_{m<C_1 n} n^{\alpha+1} |a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m>C_2 n} m^{\alpha} |a_{m,n}|^2 < \infty\,.$$

Moreover the proof also shows that when $\beta$ is slowly oscillating, if we set $E = \{(m,n)\,;\ C_1 n \leq m \leq C_2 n\}$, then the matrix $\big(\sqrt{\beta_m/\beta_n}\,\mathbb{1}_E(m,n)\big)$ is a Schur multiplier over *all* the bounded matrices, while Kacnel'son's theorem (Theorem 3.5) says that, if $\gamma = (\gamma_n)$ is non-increasing, the matrix $(\gamma_m/\gamma_n)$ is a Schur multiplier of all bounded *lower-triangular* matrices.

*Proof of Theorem 6.2.* We first prove that the assumptions of Lemma 6.3 are satisfied with $a_{m,n} = \widehat{\varphi^n}(m)$. This works nearly as in Lemma 3.4. For every symbol $\varphi$, let $M(r) = \sup_{|z|=r} |\varphi(z)|$ and write $M(1/\mathrm{e}) = \mathrm{e}^{-\delta}$, with $\delta > 0$. Let $A$ be the matrix of $C_\varphi$, with respect to the canonical basis of $H^2$. The Cauchy inequalities give, if $m \leq (\delta/2)n$:

$$|a_{m,n}| \leq [M(1/\mathrm{e})]^n \mathrm{e}^m \leq \mathrm{e}^{m-\delta n} \leq \mathrm{e}^{-\delta/2 n}$$

When $\varphi$ is analytic in a neighborhood of $\overline{D(0,R)}$ with $R > 1$, the Cauchy inequalities give, writing $M(R) = \mathrm{e}^\rho$ and $R = \mathrm{e}^\delta$, for $m \geq C_2 n$, and a suitable constant $C_2$ (take $C_2 = (2\rho)/\delta$ for instance):

$$|a_{m,n}| \leq \frac{[M(R)]^n}{R^m} \leq \mathrm{e}^{n\rho-\delta m} \leq \mathrm{e}^{-(\delta/2)m}\,.$$

We now conclude with Lemma 6.3. □

## 6.2 Singular inner functions

Let $a > 0$ and let $I_a$ be the singular inner function defined by

$$I_a(z) = \exp\Big( - a\,\frac{1+z}{1-z}\Big) = \sum_{m=0}^{\infty} c_m(a) z^m \,. \tag{6.1}$$

If $n$ is a positive integer, we have $I_n(z) = [I_1(z)]^n$ and we write

$$I_n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m \,, \quad \text{with } a_{m,n} = c_m(n)\,.$$

We rely on the following lemma, familiar to experts in orthogonal polynomials and special functions, but maybe not so much as regards the uniformity, essential for our present purposes (see [16] or [12]).

**Lemma 6.4.** *It holds*

$$c_m(n) = c\, n^{1/4} m^{-3/4} \cos(2\sqrt{2\,nm} + \pi/4) + R_m(n) =: M_m(n) + R_m(n)\,, \tag{6.2}$$

*where* $c = \pi^{-1/2} 2^{1/4}$ *and where* $|R_m(n)| \leq K\sqrt{n}\, m^{-5/4}$*, with* $K$ *some numerical constant.*

We use the following [16, p. 253] and [19, p. 198], where $L_m^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree $m$ with parameter $\alpha$.

**Theorem 6.5.** *With the notation of* (6.1)*, we have*

$$c_m(a) = \mathrm{e}^{-a}\, L_m^{(-1)}(2a)\,.$$

*Moreover, we have, uniformly for* $0 < \varepsilon \leq x \leq M < \infty$*:*

$$L_m^{(\alpha)}(x) = \pi^{-1/2} \mathrm{e}^{x/2} x^{-\alpha/2-1/4} m^{\alpha/2-1/4} \cos\big(2\sqrt{mx} - \alpha\pi/2 - \pi/4\big) + R_m(x)\,,$$

*where* $|R_m(x)| \leq K_\alpha\, m^{\alpha/2-3/4}$*, and* $K_\alpha > 0$ *only depending on* $\alpha$*.*

Now, using Theorem 6.5 with $\alpha = -1$ and $a = n$, we get Lemma 6.4. Actually, to get Lemma 6.4, we need some uniformity with respect to $a$ in Theorem 6.5, which is not given by the above statement of that theorem; but provided we change $m^{-5/4}$ into $\sqrt{n}\, m^{-5/4}$ in $R_m$, a careful examination of Fejér's proof of Theorem 6.5 shows that this uniformity holds. Alternatively, write $c_m(a)$ as a Fourier coefficient

$$c_m(n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp\big[i(n \cot x + 2\,mx)\big]\, dx \tag{6.3}$$

and use the previous van der Corput estimates.

Once this lemma is at our disposal, we can prove again the following theorem.

**Theorem 6.6.** *Assume that $\beta$ is a non-increasing sequence and that the composition operator $C_{I_1}$ maps $H^2(\beta)$ to itself. Then $\beta$ satisfies the $\Delta_2$-condition.*

*Proof.* Our assumption implies, with $M = \|C_{I_1}\|^2$:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \leq M \beta_n \, . \tag{6.4}$$

We use Lemma 6.4 with $C^{-2} n \leq m < n$, where $C > 1$ satisfies $C\sqrt{2} < \pi/2$, which is possible since $2\sqrt{2} < \pi$. The term $R_m(n)$ is dominated by $KC^{-1/2} m^{-3/4}$. Let $\theta_m = 2\sqrt{2}\,\sqrt{nm} + \pi/4$ be the argument of the number appearing in (6.2). We have

$$\theta_{m+1} - \theta_m = \frac{2\sqrt{2}\,\sqrt{n}}{\sqrt{m} + \sqrt{m+1}} \, ;$$

hence

$$\sqrt{2} \leq \theta_{m+1} - \theta_m \leq C\,\sqrt{2} < \frac{\pi}{2} \, .$$

The argument $\theta_m$ then varies regularly, and there is a positive constant $\delta$ and a subset $E$ of integers in the interval $[C^{-2} n, n]$ such that $|E| \geq \delta\, n$ and

$$M_m(n) \geq 2\,\delta\, n^{1/4} m^{-3/4} \geq 2\,\delta\, n^{-1/2}$$

for all $m \in E$. Therefore, for $n$ large enough, we have, for all $m \in E$:

$$|a_{m,n}| \geq 2\,\delta\, n^{-1/2} - K\sqrt{n}\; m^{-5/4} \geq 2\,\delta\, n^{-1/2} - KC^{5/2} n^{-3/4} \geq \delta\, n^{-1/2} \, .$$

With this information, (6.4) gives:

$$M\,\beta_n \geq \sum_{m\in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} |E|\, \beta_{\lfloor C^{-2} n \rfloor} \geq \delta^3 \beta_{\lfloor C^{-2} n \rfloor} \, ,$$

where $\lfloor\,.\,\rfloor$ stands for the integer part, and this proves the theorem. □

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