Boundedness of composition operators on general weighted Hardy spaces of analytic functions

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We characterize the (essentially) decreasing sequences of positive numbers $\beta = (\beta_n)$ for which all composition operators on $H^2(\beta)$ are bounded, where $H^2(\beta)$ is the space of analytic functions f in the unit disk such that $\sum_{n=0}^{\infty} |c_n|^2 \beta_n < \infty$ if $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We also give conditions for the boundedness when β is not assumed essentially decreasing.

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1 Introduction

Let $\beta = (\beta_n)_{n \geq 0}$ be a sequence of positive numbers such that

$$\liminf_{n \to \infty} \beta_n^{1/n} \geq 1 \, . \tag{1.1}$$

The associated weighted Hardy space $H^2(\beta)$ is the set of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty \, . \tag{1.2}$$

In view of (1.1), $H^2(\beta)$ is a Hilbert space of analytic functions on $\mathbb{D}$ with the canonical orthonormal basis

$$e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}} \, z^n \, , \quad n \geq 0 \, , \tag{1.3}$$

and the reproducing kernel K_w given for all $w \in \mathbb{D}$ by

$$K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z) \, \overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \, \overline{w}^n \, z^n \, . \tag{1.4}$$

Indeed, we have:

$$\Big|\sum_{k=0}^{\infty} a_n w^n\Big| \leq \Big(\sum_{k=0}^{\infty} \beta_k\, |a_k|^2\Big)^{1/2} \Big(\sum_{k=0}^{\infty} \frac{1}{\beta_k}\, |w|^{2k}\Big)^{1/2} < \infty\,,$$

thanks to (1.1).

Note that (1.1) is necessary for $H^2(\beta)$ to consist of analytic functions in $\mathbb{D}$. Indeed the fact that $\sum_{n\geq 1} \frac{1}{n\sqrt{\beta_n}}\, z^n$ belongs to $H^2(\beta)$ and is analytic in $\mathbb{D}$ implies (1.1).

When $\beta_n \equiv 1$, we recover the usual Hardy space H^2.

Note that H^2 is continuously embedded in $H^2(\beta)$ if and only if β is bounded above. In particular, this is the case when β is non-increasing.

Most of works with weighted Hardy spaces concern the case

$$\beta_n = \int_0^1 t^n\, d\sigma(t) \tag{1.5}$$

where σ is a positive measure on $(0,1)$. More specifically the following definition is often used. Let $G\colon (0,1) \to \mathbb{R}_+$ be an integrable function and let H^2_G be the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|^2_{H^2_G} := \int_{\mathbb{D}} |f(z)|^2\, G(1 - |z|^2)\, dA(z) < \infty\,. \tag{1.6}$$

Such weighted Bergman type spaces are used, for instance, in [11], [10] and in [14]. We have $H^2_G = H^2(\beta)$ with:

$$\beta_n = 2\int_0^1 r^{2n+1} G(1 - r^2)\, dr = \int_0^1 t^n\, G(1 - t)\, dt\,, \tag{1.7}$$

and the sequence $\beta = (\beta_n)_n$ is non-increasing.

Recall that a *symbol* is a (non constant) analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, and the associated composition operator $C_\varphi\colon H^2(\beta) \to \mathcal{H}ol(\mathbb{D})$ is defined (formally) as:

$$C_\varphi(f) = f \circ \varphi\,. \tag{1.8}$$

An important question in the theory is to decide when C_φ is bounded on $H^2(\beta)$, i.e. when $C_\varphi\colon H^2(\beta) \to H^2(\beta)$.

When $\beta_n \equiv 1$, i.e. when $H^2(\beta)$ is the usual Hardy space H^2, it is known ([18, pp. 13–17]) that all symbols generate bounded composition operators. But in Shapiro's presentation, the main point is the case $\varphi(0) = 0$ and a subordination principle for subharmonic functions. The case of automorphisms is claimed simple, using an integral representation for the norm and some change of variable. When β is defined as in (1.5), one disposes of integral representations for the norm in $H^2(\beta)$, and, as in in the Hardy space case, this integral representation rather easily gives the boundedness of C_{T_a} on H^2, where

$$T_a(z) = \frac{a + z}{1 + \bar{a}\, z} \tag{1.9}$$

for $a \in \mathbb{D}$. But the above representation (1.5) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence β, namely its *complete monotony*. When integral representations fail, we have to work with bare hands. If the symbol vanishes at the origin, Kacnel'son's theorem (see Theorem 3.6 below) gives a positive answer when β is essentially decreasing (see [2] or [13, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [8] (see [5, Theorem 6.3]), which itself uses a refinement by Rogosinski of Littlewood's principle ([5, Theorem 6.2]). So that the main issue remains the boundedness of C_{T_a}. A polynomial minoration (see Definition 2.3 below) for β is necessary for any C_{T_a} to be bounded on $H^2(\beta)$ (Proposition 2.5) and we showed in [13, end of Section 3] that for $\beta_n = \exp(-\sqrt{n})$, C_{T_a} is never bounded on $H^2(\beta)$. But this polynomial minoration is not sufficient, as we will see in Theorem 1.1 below, which also evidences the basic role of the maps T_a in the question.

So, our goal in this paper is characterizing the sequences β for which all composition operators act boundedly on the space $H^2(\beta)$, i.e. send $H^2(\beta)$ into itself. Eventually, we will obtain in Theorem 3.4 and Theorem 4.1 the following result, where the Δ_2-condition is defined in (2.2).

Theorem 1.1. *Let β be an essentially decreasing sequence of positive numbers. The following assertions are equivalent:*

1) *all composition operators are bounded on $H^2(\beta)$;*
2) *all maps T_a, for $0 < a < 1$, induce bounded composition operators C_{T_a} on $H^2(\beta)$;*
3) *for some $a \in (0, 1)$, the map T_a induces a bounded composition operator C_{T_a} on $H^2(\beta)$;*
4) *β satisfies the Δ_2-condition.*

Note that, by definition of the norm of $H^2(\beta)$, all rotations R_θ, $\theta \in \mathbb{R}$, induce bounded composition operators on $H^2(\beta)$ and send isometrically $H^2(\beta)$ into itself.

However, we construct a weight β which is not essentially decreasing and for which all composition operators with symbol vanishing at 0 are bounded (Theorem 3.3), though no map T_a with $0 < a < 1$ induces a bounded composition operator (Proposition 4.4).

For spaces of Bergman type $A^2_{\widetilde{G}} := H^2_G$, where $\widetilde{G}(r) = G(1 - r^2)$, defined as the spaces of analytic functions in $\mathbb{D}$ such that $\int_{\mathbb{D}} |f(z)|^2\, \widetilde{G}(|z|)\, dA < \infty$, for a positive non-increasing continuous function $\widetilde{G}$ on $[0, 1)$, Kriete and MacCluer studied in [11] some analogous problems. They proved, in particular [11, Theorem 3] that, for:

$$\widetilde{G}(r) = \exp\Big(- B\, \frac{1}{(1-r)^\alpha}\Big) \,, \qquad B > 0\,,\ 0 < \alpha \leq 2\,,$$

and

$$\varphi(z) = z + t(1-z)^\beta\,, \qquad\qquad 1 < \beta \leq 3\,,\ 0 < t < 2^{1-\beta}\,,$$

then C_φ is bounded on $A^2_{\widetilde{G}}$ if and only if $\beta \geq \alpha + 1$.

Here

$$\beta_n = \int_0^1 t^n \mathrm{e}^{-B/(1-\sqrt{t})^\alpha}\, dt\,;$$

and, since $\beta_n \approx \exp(-c\, n^{\alpha/(\alpha+1)})$, the sequence (β_n) does not satisfy the Δ_2-condition, accordingly to our Theorem 3.4 below.

2 Definitions, notation, and preliminary results

The open unit disk of $\mathbb{C}$ is denoted $\mathbb{D}$ and we write $\mathbb{T}$ its boundary $\partial\mathbb{D}$.

We set $e_n(z) = z^n$, $n \geq 0$.

As said in the introduction, H^2 is continuously embedded in $H^2(\beta)$ when β is non-increasing. In this paper, we need a slightly more general notion.

Definition 2.1. *A sequence of positive numbers $\beta = (\beta_n)$ is said* essentially decreasing *if, for some constant $C \geq 1$, we have, for all $m \geq n \geq 0$:*

$$\beta_m \leq C\, \beta_n\,. \tag{2.1}$$

Note that saying that β is essentially decreasing means that the shift operator on $H^2(\beta)$ is power bounded.

If β is essentially decreasing, and if we set:

$$\widetilde{\beta}_n = \sup_{m\geq n} \beta_m\,,$$

the sequence $\widetilde{\beta} = (\widetilde{\beta}_n)$ is non-increasing and we have $\beta_n \leq \widetilde{\beta}_n \leq C\,\beta_n$. In particular, the space $H^2(\beta)$ is isomorphic to $H^2(\widetilde{\beta})$ and $H^2(\beta)$ is continuously embedded in H^2.

Definition 2.2. *The sequence of positive numbers $\beta = (\beta_n)$ is said to satisfy the* Δ_2-condition *if there is a positive constant $\delta < 1$ such that, for all integers $n \geq 0$:*

$$\beta_{2n} \geq \delta\, \beta_n\,. \tag{2.2}$$

Definition 2.3. *The sequence of positive numbers $\beta = (\beta_n)$ is said to have a* polynomial minoration *if there are positive constants δ and α such that, for all integers $n \geq 1$:*

$$\beta_n \geq \delta\, n^{-\alpha}. \tag{2.3}$$

That means that $H^2(\beta)$ is continuously embedded in the weighted Bergman space $\mathfrak{B}^2_\alpha$ of the analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^2_{\mathfrak{B}^2_\alpha} := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < \infty$$

since $\mathfrak{B}^2_\alpha = H^2(\gamma)$ with $\gamma_n \approx n^{-\alpha}$.

The following simple proposition links those notions.

Proposition 2.4. *Let β be an essentially decreasing sequence of positive numbers. Then if β satisfies the Δ_2-condition, it has a polynomial minoration.*

The converse does not hold.

Proof. 1) Assume $\beta_m \leq C\, \beta_n$ for $m \geq n$ and $\beta_{2p} \geq \mathrm{e}^{-A}\, \beta_p$. Let now n be an integer ≥ 2, and $k \geq 1$ the smallest integer such that $2^k \geq n$, so that $k \leq a \log n$ with a a positive constant. We get:

$$\beta_n \geq C^{-1} \beta_{2^k} \geq C^{-1} \mathrm{e}^{-kA} \beta_1 \geq C^{-1} \beta_1 \mathrm{e}^{-aA \log n} =: \rho\, n^{-\alpha} \, ,$$

with $\rho = C^{-1} \beta_1$ and $\alpha = aA$.

2) Let $\delta > 0$. We set $\beta_0 = \beta_1 = 1$ and for $n \geq 2$:

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)! \, .$$

The sequence β is non-increasing.

For n and k as above, we have:

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta} \, ;$$

hence β has arbitrary polynomial minoration. However we have, for $k \geq 2$:

$$\frac{\beta_{2k!}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \underset{k \to \infty}{\longrightarrow} 0 \, ,$$

so β fails to satisfy the Δ_2-condition. $\square$

For $a \in \mathbb{D}$, we define:

$$T_a(z) = \frac{a + z}{1 + \bar{a}\, z} \, , \quad z \in \mathbb{D} \, . \tag{2.4}$$

Recall that T_a is an automorphism of $\mathbb{D}$ and that $T_a(0) = a$ and $T_a(-a) = 0$.

Though we do not need this, we may remark that $(T_a)_{a \in (-1,1)}$ is a group and $(T_a)_{a \in (0,1)}$ is a semigroup. It suffices to see that $T_a \circ T_b = T_{a * b}$, with:

$$a * b = \frac{a + b}{1 + ab} \, \cdot \tag{2.5}$$

Proposition 2.5. *Let $a \in (0,1)$ and assume that T_a induces a bounded composition operator on $H^2(\beta)$. Then β has a polynomial minoration.*

Remarks. 1) For example, when $\beta_n = \exp\big[- c \big(\log(n+1)\big)^2\big]$, with $c > 0$, no T_a induces a bounded composition operator on $H^2(\beta)$, though all symbols φ with $\varphi(0) = 0$ are bounded, since β is decreasing, as said by the forthcoming Proposition 3.2.

2) For the Dirichlet space $\mathcal{D}^2$, we have $\beta_n = n+1$, but all the maps T_a induce bounded composition operators on $\mathcal{D}^2$ (see [13, Remark before Theorem 3.12]). In this case β has a polynomial minoration though it is not bounded above.

3) However, even for decreasing sequences, a polynomial minoration for β is not enough for some T_a to induce a bounded composition operator. Indeed, we saw in Proposition 2.4 an example of a decreasing sequence β with polynomial minoration, but not sharing the Δ_2-condition, and we will see in Theorem 4.1 that the Δ_2-condition is needed for having some T_a inducing a bounded composition operator.

4) In [7], Eva Gallardo-Gutiérrez and Jonathan Partington give estimates for the norm of C_{T_a}, with $a \in (0,1)$, when C_{T_a} is bounded on $H^2(\beta)$. More precisely, they proved that if β is bounded above and C_{T_a} is bounded, then

$$\|C_{T_a}\| \geq \left(\frac{1+a}{1-a}\right)^{\sigma},$$

where $\sigma = \inf\{s \geq 0\,;\ (1-z)^{-s} \notin H^2(\beta)\}$, and

$$\|C_{T_a}\| \leq \left(\frac{1+a}{1-a}\right)^{\tau},$$

where $\tau = \frac{1}{2} \sup \mathfrak{Re}\, W(A)$, with A the infinitesimal generator of the continuous semigroup (S_t) defined as $S_t = C_{T_{\tanh t}}$, namely $(Af)(z) = f'(z)(1-z^2)$, and $W(A)$ its numerical range.

For $\beta_n = 1/(n+1)^\nu$ with $0 \leq \nu \leq 1$, the two bounds coincide, so they get $\|C_{T_a}\| = \big(\frac{1+a}{1-a}\big)^{(\nu+1)/2}$.

Proof of Proposition 2.5. Since

$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n},$$

we have $\|K_x\| \leq \|K_y\|$ for $0 \leq x \leq y < 1$.

We define by induction a sequence $(u_n)_{n\geq 0}$ with:

$$u_0 = 0 \qquad \text{and} \qquad u_{n+1} = T_a(u_n).$$

Since $T_a(1) = 1$ (recall that $a \in (0,1)$), we have:

$$1 - u_{n+1} = \int_{u_n}^{1} T_a'(t)\, dt = \int_{u_n}^{1} \frac{1-a^2}{(1+at)^2}\, dt\,;$$

hence

$$\frac{1-a}{1+a}\,(1-u_n) \leq 1-u_{n+1} \leq (1-a^2)(1-u_n)\,.$$

Let $0<x<1$. We can find $N \geq 0$ such that $u_N \leq x < u_{N+1}$. Then:

$$1-x \leq 1-u_N \leq (1-a^2)^N\,.$$

On the other hand, since $C_{T_a}^* K_z = K_{T_a(z)}$ for all $z \in \mathbb{D}$, we have:

$$\|K_x\| \leq \|K_{u_{N+1}}\| \leq \|C_{T_a}\|\,\|K_{u_N}\| \leq \|C_{T_a}\|^{N+1}\|K_{u_0}\| = \|C_{T_a}\|^{N+1}\,.$$

Let $s \geq 0$ such that $(1-a^2)^{-s} = \|C_{T_a}\|$. We obtain:

$$\|K_x\| \leq \|C_{T_a}\|\,\frac{1}{(1-x)^s}\,\cdot \tag{2.6}$$

But

$$\|K_x\|^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k}\,;$$

so we get, for any $k \geq 2$:

$$\frac{x^{2k}}{\beta_k} \leq \|C_{T_a}\|^2 \frac{1}{(1-x)^{2s}}\,\cdot$$

Taking $x = 1 - \frac{1}{k}$, we obtain $\beta_k \geq C\,k^{-2s}$. $\square$

Remark. We saw in the proof of Proposition 2.5 that if C_{T_a} is bounded on $H^2(\beta)$ for some $a \in (0,1)$, then the reproducing kernels K_w have, by (2.6), a *slow growth*:

$$\|K_w\| \leq \frac{C}{(1-|w|)^s} \tag{2.7}$$

for positive constants C and s. Actually, we have the following equivalence.

Proposition 2.6. *The sequence β has a polynomial minoration if and only if the reproducing kernels K_w of $H^2(\beta)$ have a slow growth.*

Proof. The sufficiency is easy and seen at the end of the proof of Proposition 2.5. For the necessity, we only have to see that:

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^\alpha |w|^{2n} \leq \frac{C}{(1-|w|^2)^{\alpha+1}}\,\cdot \qquad \square$$

3 Boundedness of composition operators

We study in this section conditions ensuring that all composition operators on $H^2(\beta)$ are bounded.

3.1 Conditions on the weight

We begin with this simple observation.

Proposition 3.1. *If all composition operators, and even if all composition operators with symbol vanishing at* 0*, are bounded on* $H^2(\beta)$*, then the sequence* β *is bounded above.*

Proof. If β is not bounded above, there is a subsequence (β_{n_k}), with $n_k \geq 1$, such that $\beta_{n_k} \geq 4^k$. Then $\varphi(z) = \sum_{k=1}^{\infty} 2^{-k} z^{n_k}$ defines a symbol, since $|\varphi(z)| < 1$, and $\varphi(0) = 0$; but:

$$\|C_\varphi(e_1)\|^2 = \|\varphi\|^2 = \sum_{k=1}^{\infty} 4^{-k} \beta_{n_k} = \infty \, . \qquad \square$$

For symbols vanishing at 0, we have the following characterization.

Proposition 3.2. *The following assertions are equivalent:*

1) *all symbols* φ *such that* $\varphi(0) = 0$ *induce bounded composition operators* C_φ *on* $H^2(\beta)$ *and*

$$\sup_{\varphi(0)=0} \|C_\varphi\| < \infty \, ; \tag{3.1}$$

2) β *is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3.1) is equivalent to:

$$\sup_{\varphi(0)=0} \|f \circ \varphi\| < \infty \quad \text{for all } f \in H^2(\beta) \, .$$

Proof. 2) $\Rightarrow$ 1) We may assume that β is non-increasing. Then the Goluzin-Rogosinski theorem ([5, Theorem 6.3]) gives the result; in fact, writing $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $(C_\varphi f)(z) = \sum_{n=0}^{\infty} d_n z^n$, it says that:

$$\|C_\varphi f\|^2 = |d_0|^2 \beta_0 + \sum_{n=1}^{\infty} |d_n|^2 \beta_n \leq |c_0|^2 \beta_0 + \sum_{n=1}^{\infty} |c_n|^2 \beta_n = \|f\|^2 \, ,$$

leading to C_φ bounded and $\|C_\varphi\| \leq 1$. Alternatively, we can use a result of Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]). This result was also proved by C. Cowen [4, Corollary of Theorem 7].

1) $\Rightarrow$ 2) Set $M = \sup_{\varphi(0)=0} \|C_\varphi\|$. Let $m > n$, and take:

$$\varphi(z) = z \left(\frac{1 + z^{m-n}}{2} \right)^{1/n} .$$

Then $\varphi(0) = 0$ and $[\varphi(z)]^n = \frac{z^n + z^m}{2}$; hence

$$\frac{1}{4}\,(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \leq \|C_\varphi\|^2 \|e_n\|^2 \leq M^2\,\beta_n\,,$$

so β is essentially decreasing. □

For example, let (β_n) such that $\beta_{2k+2}/\beta_{2k+1} \underset{k\to\infty}{\longrightarrow} \infty$ (for instance $\beta_{2k} = 1$ and $\beta_{2k+1} = 1/(k+1)$); if $\varphi(z) = z^2$, then $\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)}$; since $\|z^{2n+1}\|^2 = \beta_{2n+1}$, C_φ is not bounded on $H^2(\beta)$.

A more interesting example is the following. For $0 < r < 1$, let $\beta_n = \pi\, n\, r^{2n}$. This sequence is eventually decreasing, so it is essentially decreasing. The square of the norm $\|f\|^2_{H^2(\beta)}$ is the area of the part of the Riemann surface on which $r\mathbb{D}$ is mapped by f. E. Reich [17], generalizing Goluzin's result [8] (see [5, Theorem 6.3]), proved that for all symbols φ such that $\varphi(0) = 0$, the composition operator C_φ is bounded on $H^2(\beta)$ and

$$\|C_\varphi\| \leq \sup_{n\geq 1} \sqrt{n}\, r^{n-1} \leq \frac{1}{\sqrt{2\,\mathrm{e}}}\,\frac{1}{r\sqrt{\log(1/r)}}\,.$$

For $0 < r < 1/\sqrt{2}$, Goluzin's theorem asserts that $\|C_\varphi\| \leq 1$.

Note that this sequence β does not satisfy the Δ_2-condition since $\beta_{2n}/\beta_n = 2\,r^{2n}$, Theorem 4.1 below states that no composition operator C_{T_a} is bounded.

However that the weight β is essentially decreasing is not necessary for the boundedness of all composition operators C_φ, with symbol φ vanishing at 0. as stated by the following theorem, whose proof will be given in Section 3.3.

Theorem 3.3. *There exists a bounded sequence β, with a polynomial minoration, but which is* not essentially decreasing, *for which every composition operator with symbol vanishing at* 0 *is bounded on $H^2(\beta)$.*

It should be noted that for this weight, the composition operators are not all bounded, as we will see in Proposition 4.4.

3.2 Boundedness of composition operators I

We now have one of the the main results of this section.

Theorem 3.4. *Let $H^2(\beta)$ be a weighted Hardy space with $\beta = (\beta_n)$ essentially decreasing and satisfying the Δ_2-condition. Then all composition operators on $H^2(\beta)$ are bounded.*

For the proof, we need a lemma.

Lemma 3.5. *For $0 < a < 1$, we write:*

$$(T_a z)^n = \sum_{m=0}^{\infty} a_{m,n} z^m \, . \tag{3.2}$$

Then, for every $\alpha > 0$, there exists a positive constant ρ, depending on a, such that:

$$|a_{m,n}| \leq \mathrm{e}^{-\alpha(\rho n - m)} \, . \tag{3.3}$$

Note that this estimate is interesting only when $m < \rho\, n$ since we know that $|a_{m,n}| \leq \|{T_a}^n\|_\infty = 1$ for all m and n.

Proof. For $0 < r < 1$, let:

$$M(r) = \sup_{|z|=r} |T_a(z)| = \sup_{|z|=r} \left| \frac{z+a}{1+az} \right| .$$

We have $M(r) < 1$, so we can write $M(r) = r^\rho$, for some positive $\rho = \rho(a)$.

The Cauchy inequalities give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = r^{\rho n - m} \, ,$$

and we obtain the result by taking $r = \mathrm{e}^{-\alpha}$. $\square$

Proof of Theorem 3.4. We may, and do, assume that β is non-increasing.

Proposition 3.2 gives the result when $\varphi(0) = 0$.

It remains to show that all C_{T_a}, $a \in \mathbb{D}$, are bounded. Indeed, if $a = \varphi(0)$ and $\psi = T_{-a} \circ \varphi$, then $\psi(0) = 0$ and $\varphi = T_a \circ \psi$, so $C_\varphi = C_\psi \circ C_{T_a}$. Moreover, we have only to show that when $a \in [0, 1)$. Indeed, if $a \in \mathbb{D}$ and $a = |a|\, \mathrm{e}^{i\theta}$, we have $T_a = R_\theta \circ T_{|a|} \circ R_{-\theta}$, so $C_{T_a} = C_{R_{-\theta}} \circ C_{T_{|a|}} \circ C_{R_\theta}$.

We consider the matrices

$$A = (a_{m,n})_{m,n\geq 0} \quad \text{and} \quad A_\beta = \left(\sqrt{\frac{\beta_m}{\beta_n}}\, a_{m,n} \right)_{m,n\geq 0} .$$

Since $C_{T_a} e_n = {T_a}^n$, the formula (3.2) shows that A is the matrix of C_{T_a} in H^2 with respect to the basis $(e_n)_{n\geq 0}$. On the other hand, A_β is the matrix of C_{T_a} in $H^2(\beta)$ with respect to the basis $(e_n^\beta)_{n\geq 0}$. We note that $A_\beta = BAB^{-1}$, where B is the diagonal matrix with values $\sqrt{\beta_0}, \sqrt{\beta_1}, \ldots$ on the diagonal.

Since C_{T_a} is a bounded composition operators on H^2, the matrix A defines a bounded operator on ℓ_2. We have to show that A_β also, i.e. $\|A_\beta\| < \infty$.

For that purpose, we split A and A_β into several sub-matrices.

Let N be an integer such that $N \geq 2/\rho$, where ρ is defined in Lemma 3.5 (actually, the proof of that lemma shows that we can take ρ such that $1/\rho$ is an integer, so we could take $N = 2/\rho$). Let $I_0 = [0, N[$ $J_0 = [N, +\infty[$ and for $k = 1, 2, \ldots$:

$$I_k = [N^k, N^{k+1}[\quad \text{and} \quad J_k = [N^{k+1}, +\infty[\,.$$

We define the matrices D_β and R_β, whose entries are respectively:

$$d_{m,n} = \begin{cases} \sqrt{\dfrac{\beta_m}{\beta_n}}\, a_{m,n} & \text{if} \qquad (m,n) \in \displaystyle\bigcup_{k=0}^{\infty} (I_k \times I_k) \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$r_{m,n} = \begin{cases} \sqrt{\dfrac{\beta_m}{\beta_n}}\, a_{m,n} & \text{if} \qquad (m,n) \in \displaystyle\bigcup_{k=0}^{\infty} (I_k \times I_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

We also define the matrix S_β with entries:

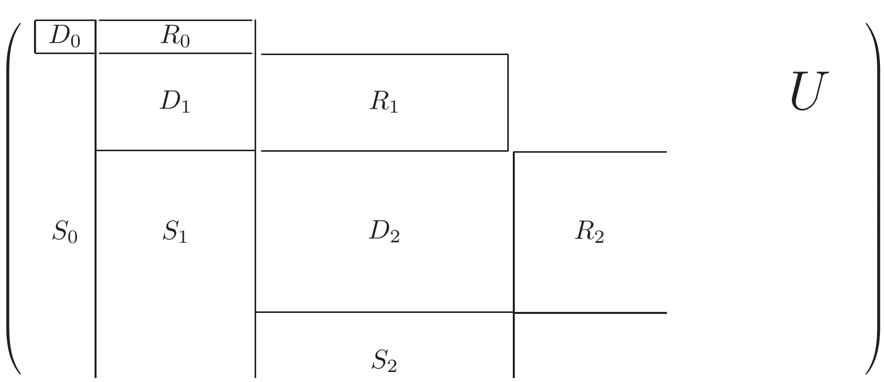
$$s_{m,n} = \begin{cases} \sqrt{\dfrac{\beta_m}{\beta_n}}\, a_{m,n} & \text{if} \qquad (m,n) \in \displaystyle\bigcup_{k=0}^{\infty} (J_k \times I_k) \\ 0 & \text{elsewhere.} \end{cases}$$

Matrices D, R, and S are constructed in the same way from A and we set $U = A - (D + R + S)$.

Now, let H_k be the subspace of the sequences $(x_n)_{n \geq 0}$ in ℓ_2 such that $x_n = 0$ for $n \notin I_k$, i.e. $H_k = \operatorname{span}\{e_k\,;\ k \in I_k\}$, and let P_k be (the matrix of) the orthogonal projection of ℓ_2 with range H_k. We have:

$$D = \sum_{k=0}^{\infty} P_k A P_k \quad \text{and} \quad R = \sum_{k=0}^{\infty} P_k A P_{k+1}\,,$$

where $D_k = P_k A P_k$ is the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times I_k$ and 0 elsewhere, and $R_k = P_k A P_{k+1}$ the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times I_{k+1}$ and 0 elsewhere.



Since the subspaces H_k are orthogonal, the matrices D and R induce bounded operators on ℓ_2, and

$$\|D\| \leq \|A\|\,, \quad \|R\| \leq \|A\|\,. \tag{3.4}$$

Now, for $k \geq 1$, let B_k be the diagonal matrix whose entries are $b_{m,m} = \sqrt{\beta_m}$ if $m \in I_k$ and $b_{m,n} = 0$ otherwise.

Then $P_k D_\beta P_k = P_k B_k D B_k^{-1} P_k$, so

$$\|P_k D_\beta E_k\| \leq \|B_k\|\,\|B_k^{-1}\|\,\|D\| \leq \max_{j\in I_k} \sqrt{\beta_j}\, \max_{j\in I_k} \frac{1}{\sqrt{\beta_j}}\,\|A\|\,.$$

But the weight β satisfies the Δ_2-condition: $\beta_{2l} \geq \delta_0\,\beta_l$, and it follows that for every $l \geq 1$:

$$\beta_{Nl} \geq \delta^2\,\beta_l\,,$$

for some other constant δ, chosen small enough to have $\|P_0 D_\beta P_0\| \leq \delta^{-1}\|A\|$. Since β is non-increasing, we have $\beta_j \geq \delta^2\,\beta_{N^k}$ for $N^k \leq j \leq N^{k+1}$. In particular $\max_{j\in I_k} \sqrt{\beta_j} \leq \delta^{-1} \min_{j\in I_k} \sqrt{\beta_j}$ and

$$\|P_k D_\beta P_k\| \leq \delta^{-1}\|A\|\,.$$

Hence, by orthogonality of the subspaces H_k:

$$\|D_\beta\| \leq \delta^{-1}\|D\|\,. \tag{3.5}$$

In the same way, we have $P_k R_\beta P_k = P_k B_k D B_{k+1}^{-1} P_k$, so:

$$\|P_k R_\beta P_k\| \leq \max_{j\in I_k} \sqrt{\beta_j}\, \max_{j\in I_{k+1}} \frac{1}{\sqrt{\beta_j}}\,\|A\|\,,$$

and, since $n \leq N^{k+2} \leq N^2 m$ when $(m,n) \in I_k \times I_{k+1}$, we get $\max_{j\in I_k} \sqrt{\beta_j} \leq \delta^{-2} \min_{j\in I_{k+1}} \sqrt{\beta_j}$, so

$$\|R_\beta\| \leq \delta^{-2}\|R\|\,. \tag{3.6}$$

Next, consider $U = A - (D + R + S)$; we can compute its Hilbert-Schmidt norm using Lemma 3.5. Note that $u_{m,n} \neq 0$ only (if it happens) when $m \in I_k$ for some $k \geq 0$ and $n \geq \max I_{k+1} > N^{k+2} > Nm$, since $m \in I_k$, so only when $m < \rho\, n/2$. We have, since then $u_{m,n} = a_{m,n}$:

$$\|U\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m<\rho n/2} |a_{m,n}|^2 \leq \sum_{n=0}^{\infty} \sum_{m<\rho n/2} \mathrm{e}^{-2\alpha(\rho n - m)} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2}\,\mathrm{e}^{-\alpha\rho n} < \infty\,.$$

Consequently, with (3.4), we have $\|S\| \leq \|A\| + \|D\| + \|R\| + \|U\| < \infty$.

Now, since S is a lower-triangular matrix and β is non-increasing, we can use the following result of V. È Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]), with $\gamma_j = 1/\sqrt{\beta_j}$.

Theorem 3.6 (V. È Kacnel'son)**.** *Let H be a separable complex Hilbert space and $(e_i)_{i\geq 0}$ a fixed orthonormal basis of H. Let $M\colon H \to H$ be a bounded linear operator. We assume that the matrix of M with respect to this basis is lower-triangular: $\langle M e_j \mid e_i\rangle = 0$ for $i < j$.*

Let $(\gamma_j)_{j\geq 0}$ be a non-decreasing sequence of positive real numbers and Γ the (possibly unbounded) diagonal operator such that $\Gamma(e_j) = \gamma_j e_j$, $j \geq 0$. Then the operator $\Gamma^{-1} M\,\Gamma\colon H \to H$ is bounded and moreover:

$$\|\Gamma^{-1} M\,\Gamma\| \leq \|M\|\,.$$

We get that S_β defines a bounded operator and $\|S_\beta\| \leq \|S\|$.

Further, Proposition 2.4 says that β has a polynomial minoration:

$$\beta_n \geq c\,n^{-\sigma}$$

for positive constants c and σ. Then, if $U_\beta = A_\beta - (D_\beta + R_\beta + S_\beta)$, we have:

$$\|U_\beta\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m<\rho n/2} \frac{|a_{m,n}|^2}{\beta_n} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2}\,\mathrm{e}^{-\alpha\rho n}\,\frac{n^\sigma}{c} < \infty\,.$$

Putting this together with (3.5) and (3.6), we finally obtain that $A_\beta = S_\beta + D_\beta + R_\beta + U_\beta$ is the matrix of a bounded operator, and that ends the proof of Theorem 3.4. $\square$

3.3 Boundedness of composition operators II

In this section, we give a sufficient condition for the boundedness of composition operators with symbol vanishing at 0.

Theorem 3.7. *Let $\beta = (\beta_n)_{n=0}^{\infty}$ be a bounded sequence of positive numbers with a polynomial minoration. Assume that:*

$$\begin{aligned} &\textit{For every } \delta > 0, \textit{ there exists } C = C(\delta) \textit{ such that} \\ &\qquad \beta_m \leq C\,\beta_n \quad \textit{whenever } m > (1+\delta)\,n\,. \end{aligned} \tag{3.7}$$

Then, for all symbols $\varphi\colon \mathbb{D} \to \mathbb{D}$ vanishing at 0, the composition operator C_φ is bounded on $H^2(\beta)$.

To prove Theorem 3.7, we need several lemmas.

Lemma 3.8. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic self-map such that $\varphi(0) = 0$ and $|\varphi'(0)| < 1$. Then there exists $\rho > 0$ such that*

$$|\widehat{\varphi^n}(m)| \leq \exp\Big(-\frac{1}{2}\,[(1+\rho)\,n - m]\Big)\,.$$

Proof. It is the same as that of Lemma 3.5. Since $\varphi(0) = 0$, we can write $\varphi(z) = z\,\varphi_1(z)$. Since $|\varphi'(0)| < 1$, we have $\varphi_1 \colon \mathbb{D} \to \mathbb{D}$. Let $M(r) = \sup_{|z|=r} |\varphi_1(z)|$. The Cauchy inequalities say that $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n / r^m$. We have $M(r) < 1$, so there exists a positive number $\rho = \rho(r)$ such that $M(r) = r^\rho$. We get:

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^\rho}{r^{m-n}} = r^{(1+\rho)\,n - m}\,,$$

and the result follows, by taking $r = \mathrm{e}^{-1/2}$. □

The next lemma is a variant of Theorem 3.6, with the same proof.

Lemma 3.9. *Let $A \colon \ell_2 \to \ell_2$ be a bounded operator represented by the matrix $\big(a_{m,n}\big)_{m,n}$, i.e. $a_{m,n} = \langle A\, e_n, e_m\rangle$.*

Let (d_n) be a sequence of positive numbers such that, for every m and n:

$$d_m < d_n \quad \Longrightarrow \quad a_{m,n} = 0\,. \tag{3.8}$$

Then, D being the diagonal operator with entries d_n, we have:

$$\|D^{-1}AD\| \leq \|A\|\,.$$

For convenience of the reader, we reproduce the proof.

Proof. Let $\mathbb{C}_0$ be the right-half plane $\mathbb{C}_0 = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}$. We set $H_N = \operatorname{span}\{e_n\,;\ n \leq N\}$ and

$$A_N = P_N A J_N\,,$$

where P_N is the orthogonal projection from H onto H_N and J_N the canonical injection from H_N into H. We consider, for $z \in \overline{\mathbb{C}_0}$:

$$A_N(z) = D^{-z} A_N D^z \colon H_N \to H_N\,,$$

where $D^z(e_n) = d_n^z e_n$.

If $\big(a_{m,n}(z)\big)_{m,n}$ is the matrix of $A_N(z)$ on the basis $\{e_n\,;\ n \leq N\}$ of H_N, we clearly have:

$$a_{m,n}(z) = a_{m,n} (d_n/d_m)^z\,.$$

In particular, we have, thanks to (3.8):

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n\,,$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M\,, \qquad \text{for all } z \in \overline{\mathbb{C}_0}\,.$$

Since $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n\leq N} |a_{m,n}^N(z)|^2 \leq (N+1)^2 M^2$, we get:

$$\|A_N(z)\| \leq (N+1)\, M \qquad \text{for all } z \in \overline{\mathbb{C}_0}\,.$$

Let us consider the function $u\colon \overline{\mathbb{C}_0} \to \overline{\mathbb{C}_0}$ defined by:

$$u_N(z) = \|A_N(z)\| \,. \tag{3.9}$$

This function u_N is continuous on $\overline{\mathbb{C}_0}$, bounded above by $(N+1)M$, and subharmonic in $\mathbb{C}_0$. Moreover, thanks to (3.8), the maximum principle gives:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) = \sup_{\partial \mathbb{C}_0} u_N(z) \,.$$

Since $\|D^z\| = \|D^{-z}\| = 1$ for $z \in \partial \mathbb{C}_0$, we have $\|A_N(z)\| \leq \|A_N\|$ for $z \in \partial \mathbb{C}_0$, and we get:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) \leq \|A_N\| \leq \|A\| \,.$$

In particular $u_N(1) \leq \|A\|$, and, letting N going to infinity, we get $\|D^{-1}AD\| \leq \|A\|$. $\square$

Proof of Theorem 3.7. First, if $|\varphi'(0)| = 1$, we have $\varphi(z) = \alpha\, z$ for some α with $|\alpha| = 1$, and the result is trivial.

So, we assume that $|\varphi'(0)| < 1$. Then, by Lemma 3.8, there exists $\rho > 0$ such that, for all m, n:

$$|\widehat{\varphi^n}(m)| \leq \exp\Big(-\frac{1}{2}\, [(1+\rho)\, n - m]\Big) \,.$$

Since $\varphi(0) = 0$, we also know that $\widehat{\varphi^n}(m) = 0$ if $m < n$.

Take $\delta = \rho/2$ and use property (3.7): there exists $C > 0$ such that:

$$\frac{\beta_m}{\beta_n} \leq C \quad \text{when } m \geq (1+\delta)\, n \,.$$

Define now a new sequence $\gamma = (\gamma_n)$ as:

$$\gamma_n = \max \Big\{ \beta_n, \sup_{m > (1+\delta)\, n} \beta_m \Big\} \,.$$

We have:

1) $\beta_n \leq \gamma_n \leq C\, \beta_n$;

2) $\gamma_m \leq \gamma_n \quad$ if $m \geq (1+\delta)\, n$.

Item 1) implies that $H^2(\gamma) = H^2(\beta)$, and we are reduced to prove that $C_\varphi \colon H^2(\gamma) \to H^2(\gamma)$ is bounded.

Let $A = \big(a_{m,n}\big) = \big(\widehat{\varphi^n}(m)\big)$. We have to prove that

$$B = \big(\gamma_m^{1/2} \gamma_n^{-1/2} a_{m,n}\big)_{m,n}$$

represents a bounded operator on ℓ_2.

Define the matrix

$$A_1 = \big(a_{m,n} \mathbb{1}_{\{(m,n)\,;\; m \leq (1+\delta)\, n\}}\big)_{m,n}$$

and set $A_2 = A - A_1$. Define analogously B_1 and $B_2 = B - B_1$.

Then A_1 is a Hilbert-Schmidt operator, because (recall that $a_{m,n} = 0$ if $m < n$)

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{m=1}^{(1+\delta)\, n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)\, n} \exp\big(- [(1+\rho)\, n - m]\big) \\ &\leq \sum_{n=1}^{\infty} (\delta\, n + 1) \exp(-\delta\, n) < \infty\,.\end{aligned}$$

Now, β is bounded above and has a polynomial minoration, so, for some positive constants C_1, C_2, and α, we have:

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)\, n} \frac{\gamma_m}{\gamma_n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)\, n} \frac{C_1}{n^{-\alpha}} \exp(-\delta\, n) \\ &\leq \sum_{n=1}^{\infty} C_2\, n^{\alpha+1} \exp(-\delta\, n) < \infty\,,\end{aligned}$$

meaning that B_1 is a Hilbert-Schmidt operator.

Since A is bounded, it follows that $A_2 = A - A_1$ is bounded. Remark that, writing $A_2 = \big(\alpha_{m,n}\big)_{m,n}$, we have, with $d_n = 1/\sqrt{\gamma_n}$:

$$d_m < d_n \quad \Longrightarrow \quad \gamma_m > \gamma_n \quad \Longrightarrow \quad m < (1+\delta)\, n \quad \Longrightarrow \quad \alpha_{m,n} = 0\,.$$

Hence we can apply Lemma 3.9 to the matrix A_2, and it ensues that B_2 is bounded, and therefore that $B = B_1 + B_2$ is bounded as well, as wanted. □

We now can prove Theorem 3.3, as a corollary of Theorem 3.7.

Proof of Theorem 3.3. Define $\beta_n = 1$ for $n \leq 3!$, and, for $k \geq 3$:

$$\begin{cases} \beta_n = \dfrac{1}{k!} & \text{for } k! < n \leq (k+1)! - 2 \text{ and for } n = (k+1)! \\[2ex] \beta_n = \dfrac{1}{(k+1)!} & \text{for } n = (k+1)! - 1\,. \end{cases}$$

Note that, for $m > n$, we have $\beta_m > \beta_n$ only if $n = (k+1)! - 1$ and $m = (k+1)! = n + 1$, for some $k \geq 3$.

However, β is not essentially decreasing since, for every $k \geq 3$, we have $\beta_{n+1}/\beta_n = k + 1$ if $n = (k+1)! - 1$.

The sequence β has a polynomial minoration because $\beta_n \geq 1/(2\,n)$ for all $n \geq 1$. In fact, for $k \geq 3$, we have $\beta_n \geq (k+1)/n \geq 1/n$ if $k! < n \leq (k+1)!-2$ or if $n = (k+1)!$; and for $n = n = (k+1)!-1$, we have $n\,\beta_n = [(k+1)!-1]/(k+1)! \geq 1/2$.

Now, it remains to check (3.7) in order to apply Theorem 3.7 and finish the proof of Theorem 3.3. Note first that we have $\beta_m/\beta_n \leq 1$ if $m \geq n+2$. Next, for given $\delta > 0$, there exists an integer N such that $(1+\delta)\,n \geq n+2$ for every $n \geq N$, so $\beta_m/\beta_n \leq 1$ if $m \geq (1+\delta)\,n$ and $n \geq N$. It suffices to take $C = \max_{1\leq n\leq N} \beta_{n+1}/\beta_n$ to obtain (3.7). □

4 Necessity of the Δ_2-condition

In this section, we will show that, for essentially decreasing sequences β, the Δ_2-condition is necessary for having boundedness of composition operators on $H^2(\beta)$.

Theorem 4.1. *Let β be an essentially decreasing sequence and assume that, for some $a \in (0,1)$, T_a induces a bounded composition operator on $H^2(\beta)$. Then β satisfies the Δ_2-condition.*

In order to prove Theorem 4.1, we need several preliminary lemmas. The first one is standard, but we give it for convenience.

Lemma 4.2. *Let $a \in (0,1)$ and let*

$$P_{-a}(x) = \frac{1-a^2}{1+2\,a\cos x + a^2}$$

be the Poisson kernel at the point $-a$. Then, for all $x \in [-\pi,\pi]$:

$$T_a(\mathrm{e}^{ix}) = \exp\big[i\,h_a(x)\big]\,, \tag{4.1}$$

where

$$h_a(x) = \int_0^x P_{-a}(t)\,dt\,. \tag{4.2}$$

Proof. For $t \in [-\pi,\pi]$, write:

$$\psi(t) := \frac{\mathrm{e}^{it}+a}{1+a\,\mathrm{e}^{it}} = \exp(i\,u(t))\,,$$

with u a real-valued, $\mathcal{C}^1$ function on $[-\pi,\pi]$ such that $u(0)=0$. This is possible since $|\psi(\mathrm{e}^{it})| = 1$ and $\psi(0) = 1$. Differentiating both sides with respect to t, we get:

$$i\,\mathrm{e}^{it}\frac{1-a^2}{(1+a\,\mathrm{e}^{it})^2} = i\,u'(t)\,\frac{\mathrm{e}^{it}+a}{1+a\,\mathrm{e}^{it}}\,.$$

This implies

$$u'(t) = \frac{1-a^2}{|1+a\,\mathrm{e}^{it}|^2} = P_{-a}(t)\,,$$

and the result follows since $u(0) = 0 = h_a(0)$. □

4.1 Main lemma

In this section, we state and prove the following lemma. The proof of this lemma is delicate, uses Lemma 4.2 and a van der Corput type estimate, inspired from [21, pp. 72–73]. We thank R. Zarouf [22] for interesting recent informations in this respect, related to his joint work with O. Szehr on the Schäffer problem (see [20]; see also the forthcoming thesis of K. Fouchet [6] and paper [1]). A similar estimate on Taylor coefficients of high powers of Blaschke factors is needed by these authors, even if they were primarily concerned with upper bounds.

Recall that we have set:

$$[T_a(z)]^n = \sum_{m=0}^{\infty} a_{m,n} z^m \, . \tag{4.3}$$

Lemma 4.3. *Let $a \in (0,1)$. We set:*

$$\tau = \frac{1+a}{1-a} > 1 \tag{4.4}$$

and write:

$$\tau^{-1} = 1 - 3\mu \, , \tag{4.5}$$

with $\mu = \mu_a \in (0, 1/3)$. For every fixed positive integer n, let:

$$J_n = [(1-2\mu)\, n, (1-\mu)\, n] \, . \tag{4.6}$$

Then, there exists $\delta = \delta_a > 0$ such that, for every n large enough, there exists a set of indices $E \subseteq J_n$ with cardinality $|E| \geq \delta n$ and such that:

$$m \in E \quad \Longrightarrow \quad |a_{m,n}| \geq \delta\, n^{-1/2} \, . \tag{4.7}$$

As a corollary of this lemma, we have the following result.

Proposition 4.4. *For the weight β constructed in the proof of Theorem 3.3, no automorphism T_a with $0 < a < 1$ can be bounded.*

Proof. When T_a is bounded, we have, with $M = \|C_{T_a}\|$, for every n:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|{T_a}^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n \, . \tag{4.8}$$

For the weight β constructed in the proof of Theorem 3.3, we are going to see that this condition is not satisfied for $n = (k+1)! + 1 := n_k$.

In fact, by Lemma 4.3, there exist constants $0 < \alpha_1 < \alpha_2 < 1$ and $\gamma, \delta > 0$, depending only on a, and a set E_k of integers such that $E_k \subseteq [\alpha_1 n_k, \alpha_2 n_k]$ and $|E_k| \geq \gamma\, n_k$ for which

$$|a_{m,n}| \geq \delta\, n_k^{-1/2} \quad \text{for } m \in E_k \, .$$

For k large enough, we have $[\alpha_1 n_k, \alpha_2 n_k] \subseteq [k!+1, (k+1)!-1] \setminus \{(k+1)!-1\}$; hence the left-hand side of (4.8) is larger than

$$\frac{\delta^2}{k!} \sum_{m \in E_k} \frac{1}{n_k} \geq \frac{\delta^2}{k!} \gamma \, .$$

Since the right-hand side of (4.8) is $M^2/(k+1)!$, that is much smaller, T_a cannot be bounded. $\square$

To prove Lemma 4.3, we will use a variant of [21, Lemma 4.6 p. 72] on the stationary phase method. It is observed in [21, p. 90] that some error term $O\left(\lambda_2^{-4/5} \lambda_3^{1/5}\right)$ to come can be replaced by $O\left(\lambda_2^{-1} \lambda_3^{1/3}\right)$, but this refinement is not needed here. However, a careful reading of the proof in [21, p. 72] gives the version below, which allows the derivative F' of F to vanish at some point, as occurs in our situation. For sake of completeness, we will give a proof, however postponed.

Theorem 4.5 (Stationary phase)**.** *Let F be real function on the interval $[A, B]$, with continuous derivatives up to the third order and $F'' > 0$ throughout $]A, B[$. Let c be the unique point in $]A, B[$ where $F'(c) = 0$. Assume that, for some positive numbers λ_2, λ_3, and η, the following assertions hold:*

1) $[c - \eta, c + \eta] \subseteq [A, B]$*;*
2) $F''(x) \geq \lambda_2$ *for all* $x \in [c - \eta, c + \eta]$*;*
3) $|F'''(x)| \leq \lambda_3$ *for all* $x \in [A, B]$*.*

Then:

$$\int_A^B \mathrm{e}^{iF(x)} \, dx = \sqrt{2\pi} \, \frac{\mathrm{e}^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta \lambda_2} + \eta^4 \lambda_3\right), \tag{4.9}$$

where the O involves an absolute constant.

Proof of Lemma 4.3. We turn to the problem of bounding $a_{m,n}$ from below, *in the case $m \in J$, and only in that case.* Since $\inf_{[0,\pi]} P_{-a} = \tau^{-1} < \sup_{[0,\pi]} P_{-a} = \tau$, there exists a unique point $x_m = x_{m,n} \in [0, \pi]$ such that

$$n P_{-a}(x_m) - m = n \, \frac{(1 - a^2)}{1 + 2\, a \cos x_m + a^2} - m = 0 \, ,$$

or else:

$$\cos x_m = \frac{n}{m} \, \frac{1 - a^2}{2\, a} - \frac{1 + a^2}{2\, a} \, \cdot \tag{4.10}$$

The point is that if $m \in J$, x_m can approach neither 0 nor π, so that $\sin x_m \geq \delta_a > 0$; in fact, otherwise m/n approaches $\tau^{\pm 1}$, which is impossible by definition of J. With h_a the function of Lemma 4.2, the Fourier formulas give, since $a_{m,n}$ is real, or since $h_a(x) - mx$ is odd:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp i[nh_a(x) - mx]\, dx = 2\, \Re\mathrm{e}\, I_{m,n}\,,$$

where

$$I_{m,n} = \int_{0}^{\pi} \exp i[nh_a(x) - mx]\, dx\,. \tag{4.11}$$

Write:

$$I_{m,n} = \int_{0}^{\pi} \exp[i\, F(x)]\, dx\,, \tag{4.12}$$

with:

$$F(x) = nh_a(x) - mx = n \int_0^x P_{-a}(t)\, dt - mx\,. \tag{4.13}$$

We have:

$$F'(x) = n\, P_{-a}(x) - m\,. \tag{4.14}$$

We will now proceed in two steps, first giving good lower bounds for $|I_{m,n}|$, then showing that the argument of $I_{m,n}$ is often far from $\pi/2$ mod. π. Then, we will be done.

First step. We will prove that:

$$I_{m,n} = \sqrt{2\pi}\, n^{-1/2}\, \frac{\mathrm{e}^{i\,(F(x_m)+\pi/4)}}{\sqrt{|h_a''(x_m)|}} + O\,(n^{-3/5})\,, \tag{4.15}$$

where the O only depends on a.

Note that $3/5 > 1/2$ and $F'' = n\, h_a''$.

To get (4.15), we will show that Theorem 4.5 is applicable with:

$$[A, B] = [0, \pi]\,, \quad c = x_m\,, \quad \lambda_2 = \kappa_0\, n\,, \quad \lambda_3 = C_0 n\,, \quad \eta = (\lambda_2\lambda_3)^{-1/5}\,.$$

The parameter η is chosen in order to make both error terms in Theorem 4.5 equal: $\frac{1}{\eta\lambda_2} = \eta^4\lambda_3$; so:

$$\eta = \kappa\, n^{-2/5}$$

and

$$\frac{1}{\eta\lambda_2} + \eta^4\lambda_3 = \widetilde{\kappa}\, n^{-3/5} = O\,(n^{-3/5}) \tag{4.16}$$

(with $\kappa = (\kappa_0 C_0)^{-1/5}$ and $\widetilde{\kappa} = 2/\kappa_0\kappa$).

The slight technical difficulty encountered here is that $F''(x)$ vanishes at 0 and π. But Theorem 4.5 covers this case. We have

$$F''(x) = nP'_{-a}(x) = 2\,a(1-a^2)\,\frac{\sin x}{(1+2a\cos x+a^2)^2}\,n\,,$$

and there are some positive κ_0 and σ such that

$$F''(x) \geq \kappa_0 n = \lambda_2 \quad \text{for } x \in [\sigma, \pi-\sigma]\,. \tag{4.17}$$

Now (for n large enough), $[x_m-\eta, x_m+\eta] \subseteq [\sigma, \pi-\sigma]$. Hence the assumptions 1) and 2) of Theorem 4.5 are satisfied.

Finally, since $F = nh_a$, and h_a is $\mathcal{C}^\infty$ on $\mathbb{R}$, we have, for all $x \in [0,\pi]$:

$$|F'''(x)| \leq C_0\,n = \lambda_3\,,$$

and assertion 3) of Theorem 4.5 holds.

With (4.16) this ends the proof of (4.15), once we remarked that $nh''_a(x_m) = F''(x_m)$.

Note that, since $|h''_a(x_m)| \leq M_a$, we get that $|I_{m,n}| \geq \delta\,n^{-1/2}$ when $m \in J$.

Second step. The mean-value theorem gives, for $m \in J$:

$$|\sin x_m| \geq \delta \quad \text{and} \quad x_{m+1} - x_m \approx \cos x_m - \cos x_{m+1}\,. \tag{4.18}$$

We also have, for $x \in \mathcal{J} = [1-2\mu, 1-\mu]$, with another constant δ:

$$\delta \leq P'_{-a}(x) = 2\,a(1-a^2)\,\frac{\sin x}{(1+2a\cos x+a^2)^2} \leq \delta^{-1}\,. \tag{4.19}$$

We now claim that

$$x_{m+1} - x_m \approx n^{-1} \quad \text{for } m \in J\,. \tag{4.20}$$

Indeed, since $m \in J$, we have, using (4.10):

$$\cos x_m - \cos x_{m+1} = \frac{1-a^2}{2a}\,\frac{n}{m(m+1)} \approx \frac{n}{m^2} \approx n^{-1}\,.$$

In view of (4.18), this proves (4.20).

Now, according to (4.15), when $m \in J$, the main term in $I_{m,n}$ is

$$A_{m,n} := n^{-1/2}\,\frac{\sqrt{2\pi}}{\sqrt{|h''_a(x_m)|}}\,\mathrm{e}^{i(F(x_m)+\pi/4)}\,,$$

and its argument θ_m is $F(x_m) + \pi/4$. Going from m to $m+1$, the variation $F(x_{m+1}) - F(x_m)$ of this argument is

$$\theta_{m+1} - \theta_m = n\int_{x_m}^{x_{m+1}} \Big(P_{-a}(t) - \frac{m}{n}\Big)\,dt = n\int_{x_m}^{x_{m+1}} \Big(P_{-a}(t) - P_{-a}(x_m)\Big)\,dt\,.$$

But, due to (4.19), this implies:

$$\theta_{m+1} - \theta_m \approx n \int_{x_m}^{x_{m+1}} (t - x_m)\, dt \approx n\, (x_{m+1} - x_m)^2 \approx n^{-1}\,.$$

The variation of θ_m is thus regular, like that of the argument of n-th roots of unity. As a consequence, for a positive proportion E of the indices $m \in J$, the argument θ_m will belong to a subarc of $\mathbb{T}$ which lies δ-apart from $\pm\pi/2$, implying $\cos\theta_m \geq \delta$, or else:

$$|E| \geq \delta\, n\,,$$

and, for all $m \in E$:

$$\mathfrak{Re}\, A_{m,n} \geq \delta |A_{m,n}| \geq \delta\, n^{-1/2}.$$

It follows that, for $m \in E$:

$$\mathfrak{Re}\, I_{m,n} \geq \delta\, n^{-1/2} - C\, n^{-3/5} \geq \widetilde{\delta}\, n^{-1/2}\,.$$

Since $a_{m,n} = \pi^{-1}\mathfrak{Re}\, I_{m,n}$, that ends the proof of Lemma 4.3. □

4.2 Proof of Theorem 4.5

The following lemma can be found in [15, Lemma 1, page 47].

Lemma 4.6. *Let $F\colon [u,v] \to \mathbb{R}$, with $u < v$, be a C^2- function with $F'' > 0$, and F' not vanishing on $[u,v]$. Let*

$$J = \int_u^v e^{iF(x)}\, dx\,.$$

Then:

a) *if $F' > 0$ on $[u,v]$, then $|J| \leq \frac{2}{F'(u)}$;*
b) *If $F' < 0$ on $[u,v]$, then $|J| \leq \frac{2}{|F'(v)|}$.*

Write now the integral I of Theorem 4.5 on $[A,B]$ as $I = I_1 + I_2 + I_3$ with:

$$I_1 = \int_A^{c-\eta} \mathrm{e}^{iF(x)}\, dx\,, \quad I_2 = \int_{c-\eta}^{c+\eta} \mathrm{e}^{iF(x)}\, dx\,, \quad I_3 = \int_{c+\eta}^{B} \mathrm{e}^{iF(x)}\, dx\,.$$

Lemma 4.6 with $u = A$ and $v = c - \eta$ implies:

$$|I_1| \leq \frac{2}{|F'(c-\eta)|} \leq \frac{2}{\eta\lambda_2}\,, \tag{4.21}$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta\, F''(\xi)$$

for some $\xi \in [c-\eta, c]$ so that $F''(\xi) \geq \lambda_2$.

Similarly, Lemma 4.6 with $u = c + \eta$ and $v = B$ implies

$$|I_3| \leq \frac{2}{F'(c+\eta)} \leq \frac{2}{\eta \lambda_2} \cdot \tag{4.22}$$

We can now estimate I_2. The Taylor formula shows that

$$F(x) = F(c) + \frac{(x-c)^2}{2} F''(c) + R\,,$$

with

$$|R| \leq \frac{|x-c|^3}{6}\, \lambda_3\,.$$

Hence

$$I_2 = \mathrm{e}^{iF(c)} \int_0^\eta 2 \exp\left(\frac{i}{2}\, x^2 F''(c)\right) dx + S$$

with

$$|S| \leq \lambda_3 \int_0^\eta \frac{x^3}{3}\, dx = \frac{\eta^4}{12}\, \lambda_3\,.$$

Finally, set

$$K = \int_0^\eta 2 \exp\left(\frac{i}{2}\, x^2 F''(c)\right) dx\,.$$

We make the change of variable $x = \sqrt{\frac{2}{F''(c)}}\, \sqrt{t}$. Recall that $\int_0^\infty \frac{\mathrm{e}^{it}}{\sqrt{t}} dt = \sqrt{\pi}\, \mathrm{e}^{i\pi/4}$ is the classical Fresnel integral, and that an integration by parts gives, for $m > 0$:

$$\left| \int_m^\infty \frac{\mathrm{e}^{it}}{\sqrt{t}}\, dt \right| \leq \frac{2}{\sqrt{m}} \cdot$$

Therefore, with $m = \frac{\eta^2}{2}\, F''(c)$:

$$K = \sqrt{\frac{2}{F''(c)}} \int_0^m \frac{\mathrm{e}^{it}}{\sqrt{t}}\, dt = \sqrt{\frac{2\pi}{F''(c)}}\, \mathrm{e}^{i\pi/4} + R_m\,,$$

with

$$|R_m| \leq C \sqrt{\frac{1}{F''(c)}}\, \frac{1}{\sqrt{m}} \leq \frac{C}{\eta \lambda_2} \cdot$$

All in all, we proved that

$$I_2 = \sqrt{\frac{2\pi}{F''(c)}}\, \exp\big[i(F(c) + \pi/4)\big] + O\left(\frac{1}{\eta\lambda_2} + \eta^4 \lambda_3\right) \cdot \tag{4.23}$$

and the same estimate holds for I, thanks to (4.21) and (4.22).

We have hence proved Theorem 4.5. ☐

4.3 Proof of Theorem 4.1

We may, and do, assume that β is non-increasing.

Set $M = \|C_{T_a}\|$. We have:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|{T_a}^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n \, ;$$

so, by Lemma 4.3, since β is non-increasing and $E \subseteq J = [(1-2\mu)n, (1-\mu)n]$:

$$M^2 \beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 \, n^{-1} \sum_{m \in E} \beta_m \geq \delta^2 \, n^{-1} |E| \, \beta_{(1-\mu)n} \, ,$$

where we have set, for x not an integer, $\beta_x = \beta_k$ with k the least integer greater than x.

That implies, for all integers $n \geq 1$:

$$\beta_n \geq (\delta^2/M^2) \, n^{-1} (\delta \, n) \, \beta_{(1-\mu)n} \geq c \, \beta_{(1-\mu)n} \, .$$

Let $r \geq 1$ such that $(1-\mu)^r \leq 1/2$; we have:

$$\beta_n \geq c^r \, \beta_{(1-\mu)^r n} \geq c^r \, \beta_{n/2} \, ,$$

so β satisfies the Δ_2-condition. $\square$

5 Some results on multipliers

The set $\mathcal{M}\big(H^2(\beta)\big)$ of multipliers of $H^2(\beta)$ is by definition the vector space of functions h analytic on $\mathbb{D}$ and such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. When $h \in \mathcal{M}\big(H^2(\beta)\big)$, the operator M_h of multiplication by h is bounded on $H^2(\beta)$ by the closed graph theorem. The space $\mathcal{M}\big(H^2(\beta)\big)$ equipped with the operator norm is a Banach space. We note the obvious property:

$$\mathcal{M}\big(H^2(\beta)\big) \hookrightarrow H^\infty \quad \text{contractively.} \tag{5.1}$$

Indeed, if $h \in \mathcal{M}\big(H^2(\beta)\big)$, we easily get for all $w \in \mathbb{D}$:

$$M_h^*(K_w) = \overline{h(w)} K_w \, ;$$

so that taking norms and simplifying, we are left with $|h(w)| \leq \|M_h\|$, showing that $h \in H^\infty$ with $\|h\|_\infty \leq \|M_h\|$.

Proposition 5.1. *We have $\mathcal{M}\big(H^2(\beta)\big) = H^\infty$ isomorphically if and only if β is essentially decreasing.*

Proof. The sufficient condition is proved in [13, beginning of the proof of Proposition 3.16]. For the necessity, we have $\|M_h\| \approx \|h\|_\infty$ for every $h \in H^\infty$. Now, for $m > n$ (recall that $e_n(z) = z^n$):

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z) \, ;$$

so, since $\|M_{e_{m-n}}\| \leq C\, \|e_{m-n}\|_\infty = C$ for some positive constant C:

$$\beta_m = \|e_m\|^2 \leq C^2\, \|e_n\|^2 = C^2\, \beta_n\,. \qquad \square$$

In [13, Section 3.6], we gave the following notion of *admissible* Hilbert space of analytic functions.

Definition 5.2. *A Hilbert space H of analytic functions on $\mathbb{D}$, containing the constants, and with reproducing kernels K_a, $a \in \mathbb{D}$, is said* admissible *if:*

- (i) *H^2 is continuously embedded in H;*
- (ii) *$\mathcal{M}(H) = H^\infty$;*
- (iii) *the automorphisms of $\mathbb{D}$ induce bounded composition operators on H;*
- (iv) $\dfrac{\|K_a\|_H}{\|K_b\|_H} \leq h\Big(\dfrac{1-|b|}{1-|a|}\Big)$ *for $a, b \in \mathbb{D}$, where $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an non-decreasing function.*

We proved in that paper that every weighted Hilbert space $H^2(\beta)$ with β non-increasing is admissible, under the additional hypothesis that the automorphisms of $\mathbb{D}$ induce bounded composition operators. In view of Theorem 3.4, we get the following result.

Proposition 5.3. *Let β be essentially decreasing that satisfies the Δ_2-condition. Then $H^2(\beta)$ is admissible.*

Let us give a different proof.

Proof. Because β is essentially decreasing, item (i) holds, as well as item (ii), by Proposition 5.1. Item (iii) is Theorem 3.4. It remains to show (iv). We may assume that β is non-increasing.

Let $0 < s < r < 1$.

Without loss of generality, we may assume that $r, s \geq 1/2$. It is enough to prove:

$$\|K_r\|^2 \leq C\, \|K_{r^2}\|^2 \tag{5.2}$$

for some constant $C > 1$. Indeed, iteration of (5.2) gives:

$$\|K_r\|^2 \leq C^k\, \|K_{r^{2^k}}\|^2$$

and if k is the smallest integer such that $r^{2^k} \leq s$, we have $2^{k-1} \log r > \log s$ and $2^k \leq D\, \frac{1-s}{1-r}$ where D is a numerical constant. Writing $C = 2^\alpha$ with $\alpha > 1$, we obtain:

$$\bigg(\frac{\|K_r\|}{\|K_s\|}\bigg)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \bigg(\frac{1-s}{1-r}\bigg)^\alpha\,.$$

To prove (5.2), we pick some $M > 1$ such that $\beta_{2n} \geq M^{-1} \beta_n$ for all $n \geq 1$ and write $t = r^2$. We have:

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^\infty \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^\infty \frac{t^{2n-1}}{\beta_{2n-1}}\,,$$

implying, since $\beta_{2n-1} \geq \beta_{2n} \geq M^{-1}\beta_n$ and $t^{2n-1} \leq 4\, t^{2n}$:

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M \|K_t\|^2 \,. \qquad \square$$

The notion of admissible Hilbert space H is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{w \in H\, ;\ w\,(f \circ \varphi) \in H \text{ for all } f \in H\} \,.$$

As a corollary of [13, Theorem 3.18] we get:

Corollary 5.4. *Let β be essentially decreasing and satisfying the Δ_2-condition. Then:*

1) $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}\big(H^2(\beta), \varphi\big)$*;*
2) $\mathcal{M}\big(H^2(\beta), \varphi\big) = H^2(\beta)$ *if and only if* $\|\varphi\|_\infty < 1$*;*
3) $\mathcal{M}\big(H^2(\beta), \varphi\big) = H^\infty$ *if and only if φ is a finite Blaschke product.*

6 Miscellaneous remarks

Some of the results of this paper can slightly be improved.

6.1 Conditions on the weight β

First, we say that a sequence (β_n) of positive numbers is *slowly oscillating* if there if a function $\rho\colon (0, \infty) \to (0, \infty)$ that is bounded on each compact subset of $(0, \infty)$ for which:

$$\frac{\beta_m}{\beta_n} \leq \rho\left(\frac{m}{n}\right) \cdot$$

This clearly amounts to say that, for some positive constants $C_1 < C_2$, we have:

$$C_1 \leq \frac{\beta_m}{\beta_n} \leq C_2 \quad \text{when } n/2 \leq m \leq 2n \,.$$

Every essentially decreasing sequence with the Δ_2-condition is slowly oscillating.

Proposition 6.1. *The following holds:*

1) *every slowly oscillating sequence has a polynomial minoration;*

2) *there are bounded sequences which are slowly oscillating, but not essentially decreasing.*

Proof. 1) is clear, because if $2^j \leq n < 2^{j+1}$, then

$$\beta_n \geq C^{-1}\beta_{2^j} \geq C^{-j-1}\beta_1 \geq C^{-1}\beta_1\, n^{-\alpha} \,,$$

with $\alpha = \log C / \log 2$.

2) We define β_n as follows. Let (a_k) be an increasing sequence of positive square integers such that $\lim_{k\to\infty} a_{k+1}/a_k = \infty$, for example $a_k = 4^{k^2}$, and let $b_k = \sqrt{a_k a_{k+1}}$; with our choice, this is an integer and we clearly have $a_k < b_k < a_{k+1}$. We set:

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k \\ \\ (a_k/b_k^2)\, n = (1/a_{k+1})\, n & \text{for } b_k \leq n < a_{k+1}\,. \end{cases}$$

This sequence (β_n) is slowly oscillating by construction. Indeed, it suffices to check that for $a_k \leq n/2 < b_k \leq n < a_{k+1}$, the quotient β_m/β_n remains lower and upper bounded when $n/2 \leq m \leq n$. But for $n/2 \leq m < b_k$, we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn}\,,$$

which is $\leq 2\, b_k^2/n^2 \leq 2$ and $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = 1/4$; and for $b_k \leq m$, we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in [1/2, 1]\,.$$

However, though (β_n) is bounded, since $\beta_n \leq 1$ for $a_k \leq n < b_k$ and, for $b_k \leq n < a_{k+1}$,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}}\,(a_{k+1} - 1) \leq 1\,,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}}\,(a_{k+1} - 1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \underset{k\to\infty}{\longrightarrow} \infty\,. \qquad \square$$

By a slight modification (change the value of the constants in the definition of β_n), we could obtain a sequence which is slowly oscillating, tends to zero, yet again not essentially decreasing.

Now, Theorem 3.4 admits the following variant.

Theorem 6.2. *Let (β_n) be a sequence of positive numbers which is bounded above and slowly oscillating. Then all symbols that extend analytically in a neighborhood of $\overline{\mathbb{D}}$ induce a bounded composition operator on $H^2(\beta)$.*

It is the case, for example, for finite Blaschke products.

The proof follows that of Theorem 3.4, with the help of the following lemma.

Lemma 6.3. *Let (β_n) be a sequence of positive numbers which is bounded above and slowly oscillating. Let $A = (a_{m,n})_{m,n}$ be the matrix of a bounded operator on ℓ_2. Assume that, for constants $C > 1$, and a, b, we have:*

1) $|a_{m,n}| \leq a\,\mathrm{e}^{-bn}$ *for* $m \leq C^{-1} n$;

2) $|a_{m,n}| \leq a\,\mathrm{e}^{-bm}$ *for* $m \geq C\, n$.

Then the matrix $\widetilde{A} = \left(a_{m,n}\sqrt{\frac{\beta_m}{\beta_n}}\right)$ *also defines a bounded operator on* ℓ_2.

Sketch of proof. The matrix $\widetilde{A}$ is Hilbert-Schmidt far the diagonal since, with $\lambda \geq \beta_n \geq \delta\, n^{-\alpha}$, we have:

$$\sum_{m<C^{-1}n} |a_{m,n}|^2 \beta_m/\beta_n \leq \sum_{m<C^{-1}n} \lambda\, \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n\geq 1} n^{\alpha+1} \mathrm{e}^{-bn} < \infty\,,$$

and

$$\sum_{m>Cn} |a_{m,n}|^2 \beta_m/\beta_n \leq \sum_{m>Cn} \lambda\, \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n\geq 1} n^\alpha \bigg(\sum_{m>Cn} \mathrm{e}^{-bm}\bigg).$$

Since β_m/β_n remains bounded from above and below around the diagonal, the matrix $\widetilde{A}$ behaves like A near the diagonal. $\square$

Remark. The proof shows that, instead of 1) and 2), it is enough to have:

$$\sum_{m<C^{-1}n} n^{\alpha+1} |a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m>Cn} m^\alpha |a_{m,n}|^2 < \infty\,.$$

Moreover the proof also shows that when β is slowly oscillating, if we set $E = \{(m,n)\,;\ C^{-1}n \leq m \leq C\,n\}$, then the matrix $\big(\sqrt{\beta_m/\beta_n}\, \mathbb{1}_E(m,n)\big)$ is a Schur multiplier over *all* the bounded matrices, while Theorem 3.6 says that, if $\gamma = (\gamma_n)$ is non-increasing, the matrix (γ_m/γ_n) is a Schur multiplier of all bounded *lower-triangular* matrices.

Proof of Theorem 6.2. For every symbol φ, let $M(r) = \sup_{|z|=r} |\varphi(z)|$ and write $M(1/\mathrm{e}) = \mathrm{e}^{-\delta}$, with $\delta > 0$. Let A be the matrix of C_φ, with respect to the canonical basis of H^2. The Cauchy inequalities give, if $m \leq (\delta/2)n$:

$$|a_{m,n}| \leq [M(1/\mathrm{e})]^n \mathrm{e}^m \leq \mathrm{e}^{m-\delta n} \leq \mathrm{e}^{-\delta/2n}$$

When φ is analytic in a neighborhood of $\overline{D(0,R)}$ with $R > 1$, the Cauchy inequalities give, writing $M(R) = \mathrm{e}^\rho$ and $R = \mathrm{e}^\delta$, for $m \geq Cn$, and a suitable constant C (take $C = 2\rho^\delta$ for instance):

$$|a_{m,n}| \leq \frac{[M(R)]^n}{R^m} \leq \mathrm{e}^{n\rho-\delta m} \leq \mathrm{e}^{-(\delta/2)m}\,.$$

We conclude with Lemma 6.3. $\square$

The proof of Theorem 4.1 actually gives a little bit more than its statement.

Let (β_n) be a sequence of positive numbers with limit zero. We say that (β_n) is *mean*-Δ_2 if, denoting by (β_n^*) its non-increasing rearrangement, there is a positive constant δ such that:

$$\beta_{2n} \geq \delta\, \frac{1}{n} \sum_{j=1}^{n} \beta_j^*\,. \tag{6.1}$$

This condition is less restrictive than being slowly oscillating. Then, with the same proof, we have this variant of Theorem 4.1.

Theorem 6.4. *Let $\beta = (\beta_n)$ be a bounded sequence of positive numbers. If some automorphism T_a induces a bounded composition operator on $H^2(\beta)$, then β is mean-Δ_2.*

Sketch of proof. We proved that, for some $E \subseteq [n/2, n]$ with $|E| \gtrsim n$, we have $\beta_{2n} \geq \delta\, \beta_j$ for $j \in E$. Since

$$\sum_{j=1}^{n} \beta_j^* = \max_{|E|=n} \sum_{j \in E} \beta_j \,,$$

we get (6.1). $\square$

6.2 Singular inner functions

Let $a > 0$ and let I_a be the singular inner function defined by

$$I_a(z) = \exp\left(- a\, \frac{1+z}{1-z} \right) = \sum_{m=0}^{\infty} c_m(a) z^m \,. \tag{6.2}$$

If n is a positive integer, we have $I_n(z) = [I_1(z)]^n$ and we write

$$I_n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m \,, \quad \text{with } a_{m,n} = c_m(n) \,.$$

We rely on the following lemma, familiar to experts in orthogonal polynomials and special functions, but maybe not so much as regards the uniformity, essential for our present purposes (see [16] or [12]).

Lemma 6.5. *It holds*

$$c_m(n) = c\, n^{1/4} m^{-3/4} \cos(2\sqrt{2\, nm} + \pi/4) + R_m(n) =: M_m(n) + R_m(n) \,, \tag{6.3}$$

where $c = \pi^{-1/2} 2^{1/4}$ and where $|R_m(n)| \leq K \sqrt{n}\, m^{-5/4}$, with K some numerical constant.

We use the following [16, p. 253] and [19, p. 198], where $L_m^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree m with parameter α.

Theorem 6.6. *With the notation of* (6.2)*, we have*

$$c_m(a) = \mathrm{e}^{-a}\, L_m^{(-1)}(2a) \,.$$

Moreover, we have, uniformly for $0 < \varepsilon \leq x \leq M < \infty$:

$$L_m^{(\alpha)}(x) = \pi^{-1/2} \mathrm{e}^{x/2} x^{-\alpha/2 - 1/4} m^{\alpha/2 - 1/4} \cos\left(2\sqrt{mx} - \alpha\pi/2 - \pi/4\right) + R_m(x) \,,$$

where $|R_m(x)| \leq K_\alpha\, m^{\alpha/2 - 3/4}$, and $K_\alpha > 0$ only depending on α.

Now, using Theorem 6.6 with $\alpha = -1$ and $a = n$, we get Lemma 6.5. Actually, to get Lemma 6.5, we need some uniformity with respect to a in Theorem 6.6, which is not given by the above statement of that theorem; but provided we change $m^{-5/4}$ into $\sqrt{n}\, m^{-5/4}$ in R_m, a careful examination of Fejér's proof of Theorem 6.6 shows that this uniformity holds. Alternatively, write $c_m(a)$ as a Fourier coefficient

$$c_m(n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp\big[i(n \cot x + 2\, mx)\big]\, dx \tag{6.4}$$

and use the previous van der Corput estimates.

Once this lemma is at our disposal, we can prove again the following theorem.

Theorem 6.7. *Assume that β is a non-increasing sequence and that the composition operator C_{I_1} maps $H^2(\beta)$ to itself. Then β satisfies the Δ_2-condition.*

Proof. Our assumption implies, with $M = \|C_{I_1}\|^2$:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \leq M \beta_n\,. \tag{6.5}$$

We use Lemma 6.5 with $C^{-2} n \leq m < n$, where $C > 1$ satisfies $C\sqrt{2} < \pi/2$, which is possible since $2\sqrt{2} < \pi$. The term $R_m(n)$ is dominated by $KC^{-1/2} m^{-3/4}$. Let $\theta_m = 2\sqrt{2}\,\sqrt{nm} + \pi/4$ be the argument of the number appearing in (6.3). We have

$$\theta_{m+1} - \theta_m = \frac{2\sqrt{2}\,\sqrt{n}}{\sqrt{m} + \sqrt{m+1}}\,;$$

hence

$$\sqrt{2} \leq \theta_{m+1} - \theta_m \leq C\,\sqrt{2} < \frac{\pi}{2}\,.$$

The argument θ_m then varies not too fast, and there is a positive constant δ and a subset E of integers in the interval $[C^{-2}n, n]$ such that $|E| \geq \delta\, n$ and

$$M_m(n) \geq 2\,\delta\, n^{1/4} m^{-3/4} \geq 2\,\delta\, n^{-1/2}$$

for all $m \in E$. Therefore, for n large enough, we have, for all $m \in E$:

$$|a_{m,n}| \geq 2\,\delta\, n^{-1/2} - K\sqrt{n}\; m^{-5/4} \geq 2\,\delta\, n^{-1/2} - KC^{5/2} n^{-3/4} \geq \delta\, n^{-1/2}\,.$$

With this information, (6.5) gives:

$$M\,\beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} |E|\, \beta_{\lfloor C^{-2} n \rfloor} \geq \delta^3 \beta_{\lfloor C^{-2} n \rfloor}\,,$$

where $\lfloor\,.\,\rfloor$ stands for the integer part, and this proves the theorem. $\square$

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Pascal Lefèvre
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) UR 2462, & Fédération Mathématique des Hauts-de-France FR 2037 CNRS, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
pascal.lefevre@univ-artois.fr

Daniel Li
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) UR 2462, & Fédération Mathématique des Hauts-de-France FR 2037 CNRS, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
daniel.li@univ-artois.fr

Hervé Queffélec
Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération Mathématique des Hauts-de-France FR 2037 CNRS, F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille.fr

Luis Rodríguez-Piazza
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n 41 012 SEVILLA, SPAIN
piazza@us.es