REFINEMENT INVARIANCE OF INTERSECTION (CO)HOMOLOGIES

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Abstract. We prove the refinement invariance of several intersection (co)homologies existing in the literature: Borel-Moore, Blown-up, the classical one, ... These (co)homo-1logies have been introduced in order to establish the Poincaré Duality in various contexts. In particular, we retrieve the classical topological invariance of the intersection homology as well as several refinement invariance results already known.

Let us consider a topological space X supporting two stratifications $\mathcal{S}$ and $\mathcal{T}$. We say that $(X, \mathcal{S})$ is a *refinement* of $(X, \mathcal{T})$ if each stratum of $\mathcal{T}$ is a union of strata of $\mathcal{S}$. In this work we answer the following question about the invariance property of the intersection homology:

Can we find two perversities $\overline{p}$ and $\overline{q}$ such that
the identity $I\colon X \to X$ induces the isomorphism

$$H_*^{\overline{p}}(X, \mathcal{S}) \cong H_*^{\overline{q}}(X, \mathcal{T})? \tag{1}$$

For pseudomanifolds and using the original *Goresky-MacPherson* perversities, an answer comes directly from the topological invariance of the intersection homology [15, Corollary pag. 148] (see also [19, Theorem 9]): it suffices to take $\overline{p} = \overline{q}$. In other words, the intersection homology does not depend on the chosen stratification. We work in a more general setting.

• **Spaces**. We do not work with pseudomanifolds, but with the more general notion of CS-set (cf. Section 3). They are locally cone-like spaces, but their links are not necessarily pseudomanifolds.

• **Perversities** (cf. Paragraph 1.3). We deal with the more general notion of perversity introduced by MacPherson in [21]: the *M-perversities*. This kind of perversity $\overline{p}$ associates a number $\overline{p}(S) \in \overline{\mathbb{Z}} = \mathbb{Z} \sqcup \{-\infty, \infty\}$ to any stratum S of the CS-set, while a *classical perversity* $\overline{p}$ associates a number $\overline{p}(\operatorname{codim} S)$ to the codimension of the stratum. An M-perversity strongly depends on the stratification and so the topological invariance of the related intersection homology does not apply.

• **(Co)homologies** (cf. Section 1). We consider not only the intersection homology $H_*^{\overline{p}}$, but also the following:

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+ **Intersection cohomologies $H^*_{\overline{p}}, H^*_{\overline{p},c}$.** The intersection cohomology $H^*_{\overline{p}}$ is the cohomology of the complex defined by using the functor Hom over intersection chains. The version with compact supports is $H^*_{\overline{p},c}$.

+ **Tame intersection homology $\mathfrak{H}^{\overline{p}}_*$.** A variation of the intersection homology avoiding intersection chains who live in singular strata. We have $\mathfrak{H}^{\overline{p}}_* = H^{\overline{p}}_*$ when $\overline{p}$ is smaller than the top perversity $\overline{t}$. This homology is isomorphic to the blown-up intersection cohomology with compact supports $\mathscr{H}^*_{\overline{p},c}$ through the Poincaré duality [5] when one works with pseudomanifolds.

+ **Tame intersection cohomologies $\mathfrak{H}^*_{\overline{p}}, \mathfrak{H}^*_{\overline{p},c}$.** The tame intersection cohomology $\mathfrak{H}^*_{\overline{p}}$ is the cohomology of the complex defined by using the functor Hom over tame intersection chains. The version with compact supports is $\mathfrak{H}^*_{\overline{p},c}$. We have $\mathfrak{H}^*_{\overline{p}} = H^*_{\overline{p}}$ and $\mathfrak{H}^*_{\overline{p},c} = H^*_{\overline{p},c}$ when $\overline{p} \leqslant \overline{t}$. The cohomology $\mathfrak{H}^*_{\overline{p},c}$ is dual to $\mathfrak{H}^{D\overline{p}}_{n-*}$ through the Poincaré duality [14, 8] when the coefficient is a field[1] and one works with pseudomanifolds. The cohomology $\mathfrak{H}^*_{\overline{p}}$ is isomorphic to the de Rham intersection cohomology through the de Rham duality [22]. Here, we use real coefficients.

+ **Borel-Moore intersection homology $H^{BM,\overline{p}}_*$.** Similar to the intersection homology using locally finite chains instead of finite chains.

+ **Borel-Moore tame intersection homology $\mathfrak{H}^{BM,\overline{p}}_*$.** Similar to $\mathfrak{H}^*_{\overline{p}}$ using locally finite chains instead of finite chains. We have $\mathfrak{H}^{BM,\overline{p}}_* = H^{BM,\overline{p}}_*$ when $\overline{p} \leqslant \overline{t}$.

+ **Blown-up intersection cohomologies $\mathscr{H}^*_{\overline{p}}, \mathscr{H}^*_{\overline{p},c}$.** These cohomologies are defined by using simplicial cochains, inspired by Dennis Sullivan's approach to rational homotopy theory. The compact supports version $\mathscr{H}^*_{\overline{p},c}$ (resp. closed supports version $\mathscr{H}^*_{\overline{p}}$) is isomorphic to the tame intersection homology $\mathfrak{H}^{\overline{p}}_{n-*}$ (resp. Borel-Moore tame intersection homology $\mathfrak{H}^{BM,\overline{p}}_{n-*}$) through the Poincaré duality [3] (resp. cf. [23]) when one works with pseudomanifolds. We have $\mathscr{H}^*_{\overline{p}} = \mathfrak{H}^*_{D\overline{p}}$ and $\mathscr{H}^*_{\overline{p},c} = \mathfrak{H}^*_{D\overline{p},c}$ when the coefficient ring is a field[1] (cf. [3, Theorem F] and [5, Corollary 13.1]).

In this paper, we give two answers to the question (1).

– *Pull-back.* Given a perversity $\overline{q}$ on $(X, \mathcal{T})$ we take $\overline{p}$ the pull-back perversity $I^\star\overline{q}$ on $(X, \mathcal{S})$ (cf. Paragraph 1.3). We prove

$$H^{I^\star\overline{q}}_*(X, \mathcal{S}) \cong H^{\overline{q}}_*(X, \mathcal{T}),$$

and similarly for the other (co)homologies used in this work (cf. Theorem B).

– *Push-forward.* Given a perversity $\overline{p}$ on $(X, \mathcal{S})$ we take $\overline{q}$ the push-forward perversity $I_\star\overline{p}$ on $(X, \mathcal{T})$ (cf. Paragraph 1.3). We prove

$$H^{\overline{p}}_*(X, \mathcal{S}) \cong H^{I_\star\overline{p}}_*(X, \mathcal{T}),$$

and similarly for the other (co)homologies used in this work (cf. Theorem A). In this case, we need the following conditions on $\overline{p}$:

(K1) $$\overline{p}(Q) \leqslant \overline{p}(S) \leqslant \overline{p}(Q) + \overline{t}(S) - \overline{t}(Q),$$

[1] In fact, following [17], *locally $\overline{p}$-torsion free.*

for any strata $S, Q \in \mathcal{S}$ with $S \subset \overline{Q}$ and $S, Q \subset T$ for some stratum $T \in \mathcal{T}$, and

$$\text{(K2)} \qquad \overline{p}(Q) = \overline{p}(S),$$

for any strata $S, Q \in \mathcal{S}$ with $\dim S = \dim Q$ and $S, Q \subset T$ for some stratum $T \in \mathcal{T}$[2].

These results encompass some other already known results about invariance of intersection (co)homology:

- The topological invariance of $H_*^{\overline{p}}$ for pseudomanifolds [15] and CS-sets [19, 13].
- The topological invariance of $\mathfrak{H}_*^{\overline{p}}$ and $\mathfrak{H}_*^{BM,\overline{p}}$ for pseudomanifolds [9, 11].
- The topological invariance of $\mathscr{H}_{\overline{p}}^*$ and $\mathscr{H}_{\overline{p},c}^*$ for CS-sets [3, 5].
- The refinement invariance of $H_*^{\overline{p}}$ for PL-pseudomanifolds [24] using M-perversities.
- The refinement invariance of $H_*^{\overline{p}}$ and $\mathfrak{H}_*^{\overline{p}}$ for CS-sets [6] using M-perversities.

Recently, the topological invariance of the intersection homology has been extended to the more general setting of the torsion sensitive intersection homology (cf. [12]).

We end this introduction by giving an idea of the proof of Theorems A and B. The original proof of the classical topological invariance of the intersection homology given by King in [19] uses the intrinsic stratification $\mathcal{S}^*$. He proves that the identity map $I\colon X \to X$ induces an isomorphism between the intersection homology of $(X, \mathcal{S})$ and that of $(X, \mathcal{S}^*)$. This gives the topological invariance since $\mathcal{S}^* = \mathcal{T}^*$.

The proof uses the Mayer-Vietoris technique in order to reduce the question to a local one. Near a point x of X the identity $I\colon (X, \mathcal{S}) \to (X, \mathcal{S}^*)$ becomes the stratified map

$$h\colon \mathring{\mathtt{c}}(S^m * L) \to B^{m+1} \times \mathring{\mathtt{c}}L \tag{2}$$

(cf. (10)). Here, $B^{m+1} = \{z \in \mathbb{R}^{m+1} \mid ||z|| < 1\}$ and L denotes the link of x. Using the usual local calculations of intersection homology one proves that h is a quasi-isomorphism for this homology.

In our context, we don't know wether the the identity map $I\colon (X, \mathcal{S}) \to (X, \mathcal{T})$ has the nice local description (2). We proceed in a different way. We construct a finite sequence of CS-sets

$$(X, \mathcal{S}) = (X, \mathcal{R}_0) \xrightarrow{I} (X, \mathcal{R}_1) \xrightarrow{I} \cdots (X, \mathcal{R}_{\ell-1}) \xrightarrow{I} (X, \mathcal{R}_\ell) = (X, \mathcal{T}), \tag{3}$$

where each step is a refinement having the (2)-local description (called *simple refinement*). Now, we can follow the procedure of [19] in order to get the isomorphism between the intersection homologies of $(X, \mathcal{S})$ and $(X, \mathcal{T})$.

The construction of this sequence uses the fact that any stratum of $S \in \mathcal{S}$ is included in a stratum $T \in \mathcal{T}$. This gives the following dichotomy: S is a *source stratum* if $\dim S = \dim T$ and S is a *virtual stratum* if $\dim S < \dim T$. Somehow, the virtual strata of $\mathcal{S}$ disappear in $\mathcal{T}$ while source strata become larger. The first step in the construction of the above sequence is to *eliminate* the maximal virtual strata of $\mathcal{S}$. In this way, we obtain in the CS-set $(X, \mathcal{R}_1)$. We continue applying this principle to the refinement $(X, \mathcal{R}_1) \xrightarrow{I} (X, \mathcal{T})$ and we eventually get (3).

What do we mean by "eliminate"? Let us suppose that $S \in \mathcal{S}$ is the unique virtual stratum. There exist two source strata $R_0, R_1 \in \mathcal{S}$ with $T = S \cup R_0 \cup R_1$ (maybe

[2] Remark 4.9 gives a relation between classical perversities and those verifying (K1), (K2).

$R_0 = R_1$). We replace the strata S, R_0, R_1 of $\mathcal{S}$ by the stratum T in order to get $\mathcal{R}_1$, that is, $\mathcal{R}_1 = \{Q \in \mathcal{S} \mid Q \neq S, R_0, R_1\} \cup \{T\}$ (cf. Example 2.7 for a richer situation). A similar phenomenon appears in (2), relatively to $S = \{\text{apex of } \mathring{\mathtt{c}}(S^m * L)\}$ and $T = B^{m+1} \times \{\text{apex of } \mathring{\mathtt{c}}L\}$.

Exceptional strata are singular strata of $\mathcal{S}$ included in regular strata1 of $\mathcal{T}$. For example, the apex of the open cone of the sphere S^0 (which is indeed an open interval) is an exceptional stratum if we take $\mathcal{T}$ the one-stratum stratification of that interval. This example is a limit case for the refinement invariance results we establish in this work (see Remark 4.7).

For a topological space X, we denote by $\mathtt{c}X = X \times [0,1]/(X \times \{0\})$ the *cone* on X and $\mathring{\mathtt{c}}X = X \times [0,1[/(X \times \{0\})$ the *open cone* on X. A point of the cone is denoted by $[x,t]$. The apex of the cone is $\mathtt{v} = [-,0]$.
We shall write $B^m = \{z \in \mathbb{R}^m \mid ||z|| < 1\}$ and $D^m = \{z \in \mathbb{R}^m \mid ||z|| \leqslant 1\}$, $m \in \mathbb{N}$.

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Contents

1. Intersection homologies and cohomologies (filtered spaces)

We present the homologies and cohomologies studied in this work. We review their main computational properties which we are going to use in the proof of Theorems A and B.

1.1. **Filtered spaces.** A *filtered space* is a Hausdorff topological space endowed with a filtration by closed sub-spaces

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subsetneq X_n = X.$$

The *formal dimension* of X is $\dim X = n$. Any non-empty connected component S of a $X_i \backslash X_{i-1}$ is a *stratum*. We say that i is the *formal dimension* of S, written $i = \dim S$. We denote by $\mathcal{S}$ the family of strata. In order to avoid confusion we also write $(X, \mathcal{S})$ the filtered space. The n-dimensional strata are the *regular strata*, other strata are *singular strata*. The family of singular strata is denoted by $\mathcal{S}^{sing}$. Their union is the singular part Σ.

A continuous map $f\colon (X,\mathcal{S}) \to (Y,\mathcal{T})$ between two filtered spaces is a *stratified map* if for each $S \in \mathcal{S}$ there exists $S^f \in \mathcal{T}$ with $f(S) \subset S^f$ and $\operatorname{codim} S^f \leqslant \operatorname{codim} S$. The map f is a *stratified homeomorphism* if f is a homeomorphism and f^{-1} is a stratified map.

1.2. **Examples.** Unless expressly stated otherwise, a manifold M is endowed with the filtration $\emptyset \subseteq M$. The associated filtration is denoted by $\mathcal{I} = \{M_{cc}\}$, where *cc* denotes connected component. Consider $(X, \mathcal{S})$ a filtered space.

+ An open subset $U \subset X$ inherits the *induced filtration* $U_i = U \cap X_i$. The associated *induced stratification* is $\mathcal{S}_U = \{(S \cap U)_{cc} \mid S \in \mathcal{S}\}$. We write $(U, \mathcal{S})$ instead of $(U, \mathcal{S}_U)$.

+ Given an m-dimensional manifold M the product $M \times X$ inherits the *product filtration* $(M \times X)_i = M \times X_{i-m}$. The associated *product stratification* is $\mathcal{I} \times \mathcal{S} = \{M_{cc} \times S \mid S \in \mathcal{S}\}$.

+ The cone $\mathring{\mathtt{c}}X$ inherits the *cone filtration* $\mathring{\mathtt{c}}X_i = \mathring{\mathtt{c}}X_{i-1}$, with the convention $\mathring{\mathtt{c}}\emptyset = \{\mathtt{v}\}$. The associated *cone stratification* is $\mathring{\mathtt{c}}\mathcal{S} = \{\{\mathtt{v}\}\} \sqcup \{S \times]0,1[, S \in \mathcal{S}\}$.

+ Let $m \in \mathbb{N}$. We consider the join $S^m * X = D^{m+1} \times X / \sim$, where the equivalence relation is generated by $(z,x) \sim (z,x')$ if $||z|| = 1$. An element of $S^m * X$ is denoted by $[z,t]$. We identify S^m with $\{[z,x] \ / \ ||z|| = 1\}$ and X with $\{[0,x] \ / \ x \in X\}$. The join $S^m * X$ is endowed with the *join filtration* $S^m \subset S^m * X_0 \cdots \subset S^m * X_{n-1} \subsetneq S^m * X_n$. The associated *join stratification* is $\mathcal{S}_{\star m} = \{S^m, B^{m+1} \times S \mid S \in \mathcal{S}\}$.

1.3. **Perversities.** Consider $(X, \mathcal{S})$ a filtered space. A *M-perversity*, or simply *perversity*, on $(X, \mathcal{S})$ is a map $\overline{p} \colon \mathcal{S} \to \overline{\mathbb{Z}} = \mathbb{Z} \sqcup \{-\infty, \infty\}$ verifying $\overline{p}(S) = 0$ for any regular stratum [21].

The *top perversity* is the perversity defined by $\overline{t}(S) = \operatorname{codim} S - 2$ for each singular stratum S. The *dual perversity* of $\overline{p}$ is the perversity $D\overline{p}$ defined by $D\overline{p} = \overline{t} - \overline{p}$.

A perverse space is a triple $(X, \mathcal{S}, \overline{p})$ where $(X, \mathcal{S})$ is a filtered space and $\overline{p}$ is a perversity on $(X, \mathcal{S})$. Given a stratified map $f \colon (X, \mathcal{S}) \to (Y, \mathcal{T})$, a perversity $\overline{q}$ on $(Y, \mathcal{T})$ and a perversity $\overline{p}$ on $(X, \mathcal{S})$, we define

- the *pull-back perversity* $f^\star \overline{q}$ on $(X, \mathcal{S})$ by: $f^\star \overline{q}(S) = \overline{q}(S^f)$ for each $S \in \mathcal{S}^{sing}$,
- the *push-forward perversity* $f_\star \overline{p}$ on $(X, \mathcal{T})$ by: $f_\star \overline{p}(T) = \min\{\overline{p}(Q) \mid Q^f = T\}$ for each $T \in \mathcal{T}^{sing}$, with $\inf \emptyset = \infty$.

Notice that $f^\star f_\star \overline{p} \leqslant \overline{p}$.

We make a quick reminder of the intersection homologies/cohomologies deployed in this work. They have been mainly introduced in order to study the Poincaré duality of the intersection (co)homology in different contexts.

1.4. **Tame and intersection (co)homologies.** (cf. [5, 6]). We fix an n-dimensional perverse space $(X, \mathcal{S}, \overline{p})$. Tame intersection homology is a variant of the classic intersection homology (cf. [15, 16, 19]). When the perversity $\overline{p}$ is greater than the top perversity it is possible to have a $\overline{p}$-intersection chains contained in the singular part Σ of X. This fact prevents the Poincaré duality and the de Rham Theorem. For this reason the tame intersection homology was introduced (cf. [6, 5, 8, 22]).

A *filtered simplex* is a singular simplex $\sigma \colon \Delta \to X$ where the euclidean simplex Δ is endowed with a filtration $\Delta = \Delta_0 * \Delta_1 * \cdots * \Delta_n$, called *$\sigma$-decomposition*, verifying $\sigma^{-1} X_i = \Delta_0 * \Delta_1 * \cdots * \Delta_i$, for each $i \in \{0, \ldots, n\}$. A factor Δ_i can be empty with the convention $\emptyset * Y = Y$. The filtered simplex σ is a *regular simplex* when $\operatorname{Im} \sigma \not\subset \Sigma$, that is, $\Delta_n \neq \emptyset$.

We decompose the boundary of a filtered simplex $\Delta = \Delta_0 * \cdots * \Delta_n$ as $\partial \Delta = \partial_{\rm reg} \Delta + \partial_{\rm sing} \Delta$, where $\partial_{\rm reg} \Delta$ contains all the regular simplices.

The *perverse degree* of the filtered simplex σ relatively to a stratum $S \in \mathcal{S}_{\mathcal{F}}$ is

$$\|\sigma\|_S = \begin{cases} -\infty, & \text{if } S \cap \operatorname{Im}\sigma = \emptyset, \\ \dim(\Delta_0 * \cdots * \Delta_{\dim S}), & \text{otherwise.} \end{cases}$$

A filtered simplex $\sigma \colon \Delta \to X$ is *$\overline{p}$-allowable* if $\|\sigma\|_S \leqslant \dim \Delta - \operatorname{codim} S + \overline{p}(S)$, for each $S \in \mathcal{S}$. Moreover, if $\operatorname{Im}\sigma \not\subset \Sigma$ then the simplex σ is said to be *$\overline{p}$-tame.*

The chain complex $C^{\overline{p}}_*(X;\mathcal{S})$ is the G-module formed of the linear combinations $\xi = \sum_{j\in J} n_j\sigma_j$, where each σ_j is $\overline{p}$-allowable, and such that $\partial\xi = \sum_{\ell\in L} n_\ell\tau_\ell$, where the simplices τ_ℓ are $\overline{p}$-allowable. We call $(C^{\overline{p}}_*(X;\mathcal{S}),\partial)$ the *$\overline{p}$-intersection complex* and its homology, $H^{\overline{p}}_*(X;\mathcal{S})$, the *$\overline{p}$-intersection homology.* This designation is justified since this homology matches with the usual intersection homology (cf. [6, Theorem A]).

The chain complex $\mathfrak{C}^{\overline{p}}_*(X;\mathcal{S})$ is the G-module formed of the linear combinations $\xi = \sum_{j\in J} n_j\sigma_j$, with each σ_j is $\overline{p}$-tame, and such that $\partial_{\mathrm{reg}}\xi = \sum_{\ell\in L} n_\ell\tau_\ell$, where the simplices τ_ℓ are $\overline{p}$-tame. We call $(\mathfrak{C}^{\overline{p}}_*(X;\mathcal{S}),\mathfrak{d} = \partial_{\mathrm{reg}})$ the *tame $\overline{p}$-intersection complex* and its homology, $\mathfrak{H}^{\overline{p}}_*(X;\mathcal{S})$, the *tame $\overline{p}$-intersection homology.* This designation is justified since this homology matches with the usual tame intersection homology (cf. [6, Theorem B]). We have $H^{\overline{p}}_*(X;\mathcal{S}) = \mathfrak{H}^{\overline{p}}_*(X;\mathcal{S})$ when $\overline{p} \leqslant \overline{t}$ (cf. [6, Remark 3.9]).

Associated cohomology is defined by using the functor Hom, as usual in algebraic topology. We put the dual complexes $C^*_{\overline{p}}(X;\mathcal{S}) = \hom(C^{\overline{p}}_*(X;\mathcal{S});\mathcal{S})$ and $\mathfrak{C}^*_{\overline{p}}(X;\mathcal{S}) = \hom(\mathfrak{C}^{\overline{p}}_*(X;\mathcal{S});\mathcal{S})$ endowed with the dual differential d. Their cohomologies are the *$\overline{p}$-intersection cohomology* $H^*_{\overline{p}}(X;\mathcal{S})$ and the *$\overline{p}$-tame intersection cohomology* $\mathfrak{H}^*_{\overline{p}}(X;\mathcal{S})$.

Let $U \subset X$ be an open subset . The relative homologies $H^{\overline{p}}_*(X,U;\mathcal{S})$ and $\mathfrak{H}^{\overline{p}}_*(X,U;\mathcal{S})$ are defined from quotient complexes $C^{\overline{p}}_*(X,U;\mathcal{S}) = C^{\overline{p}}_*(X;\mathcal{S})/C^{\overline{p}}_*(U;\mathcal{S})$ and $\mathfrak{C}^{\overline{p}}_*(X,U;\mathcal{S}) = \mathfrak{C}^{\overline{p}}_*(X;\mathcal{S})/\mathfrak{C}^{\overline{p}}_*(U;\mathcal{S})$ ([6, Definition 4.5]). The relative cohomologies $H^*_{\overline{p}}(X,U;\mathcal{S})$ and $\mathfrak{H}^*_{\overline{p}}(X,U;\mathcal{S})$ are defined by using the functor Hom (cf. [8, Definition 7.1.1]).

The *(tame) intersection cohomology with compact supports* are defined by

$$H^*_{\overline{p},c}(X;\mathcal{S}) = \varinjlim_{K\subset X} H^*_{\overline{p}}(X,X\backslash K;\mathcal{S}) \qquad \text{and} \qquad \mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}) = \varinjlim_{K\subset X} \mathfrak{H}^*_{\overline{p}}(X,X\backslash K;\mathcal{S}), \quad (4)$$

where K runs over the compact subsets of X (cf. [14, Definition 6.1]).

1.5. **Main properties for (tame) intersection (co)homology.** We group here the main properties of the (tame) intersection homology. We fix a perverse set $(X,\mathcal{S},\overline{p})$.

a. **Mayer-Vietoris.** Associated to an open cover $\{U,V\}$ of X we have the long exact sequences

$$\cdots \to H^{\overline{p}}_{k+1}(X;\mathcal{S}) \to H^{\overline{p}}_k(U\cap V;\mathcal{S}) \to H^{\overline{p}}_k(U;\mathcal{S}) \oplus H^{\overline{p}}_k(V;\mathcal{S}) \to H^{\overline{p}}_k(X;\mathcal{S}) \to \cdots,$$

$$\cdots \to \mathfrak{H}^{\overline{p}}_{k+1}(X;\mathcal{S}) \to \mathfrak{H}^{\overline{p}}_k(U\cap V;\mathcal{S}) \to \mathfrak{H}^{\overline{p}}_k(U;\mathcal{S}) \oplus \mathfrak{H}^{\overline{p}}_k(V;\mathcal{S}) \to \mathfrak{H}^{\overline{p}}_k(X;\mathcal{S}) \to \cdots,$$

(cf. [6, Proposition 4.1]).

b. **Local calculations.** We have the isomorphisms $H^{\overline{p}}_*(\mathbb{R}^m \times X,\mathcal{I}\times\mathcal{S}) = H^{\overline{p}}_*(X,\mathcal{S})$ and $\mathfrak{H}^{\overline{p}}_*(\mathbb{R}^m \times X,\mathcal{I}\times\mathcal{S}) = \mathfrak{H}^{\overline{p}}_*(X,\mathcal{S})$ where the isomorphism comes from the canonical projection pr: $\mathbb{R}^m \times X \to X$. (cf. [6, Corollary 3.14]). If L is compact, we have the isomorphisms

$$H_k^{\overline{p}}(\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{S}) = \begin{cases} H_k^{\overline{p}}(L,\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathtt{v}), \\ 0 & \text{if } 0 \neq k > D\overline{p}(\mathtt{v}), \\ G & \text{if } 0 = k > D\overline{p}(\mathtt{v}), \end{cases}$$

$$\mathfrak{H}_k^{\overline{p}}(\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{S}) = \begin{cases} \mathfrak{H}_k^{\overline{p}}(L;\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathtt{v}), \\ 0 & \text{if } k > D\overline{p}(\mathtt{v}) \end{cases}$$

where the isomorphisms (first line) come from the inclusion $\iota\colon L \to \mathbb{R}^m \times \mathring{\mathtt{c}}L$ defined by $\iota(x) = (0, [x, 1/2])$ (cf. [6, Proposition 5.1]).

c. **Join. Suspension** Using the above calculation (see also [7, Lemma 3.6]), one gets:

$$H_k^{\overline{p}}(S^m * X, \mathcal{S}_{\star m}) = \begin{cases} H_k^{\overline{p}}(X,\mathcal{S}) & \text{if } k \leqslant D\overline{p}(S^m), \\ G & \text{if } 0 = k > D\overline{p}(S^m), \\ 0 & \text{if } D\overline{p}(S^m) < k \leqslant D\overline{p}(S^m) + m + 1, k \neq 0 \\ \widetilde{H}_{k-m-1}^{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(S^m) + m + 2, k \neq 0 \end{cases}$$

$$\mathfrak{H}_k^{\overline{p}}(S^m * X, \mathcal{S}_{\star m}) = \begin{cases} \mathfrak{H}_k^{\overline{p}}(X;\mathcal{S}) & \text{if } k \leqslant D\overline{p}(S^m), , \\ 0 & \text{if } D\overline{p}(S^m) < k \leqslant D\overline{p}(S^m) + m + 1, \\ \mathfrak{H}_{k-m-1}^{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(S^m) + m + 2, \end{cases}$$

where the isomorphism comes from the inclusion $\iota\colon X \to S^m * X$ defined by $\iota(x) = [0, x]$ Let us look at the case $m = 0$, that is, the suspension ΣX. Previous calculations suppose that the perversity $\overline{p}$ takes the same value at the north pole $\mathtt{n}$ and at the south pole $\mathtt{s}$. In the general case, if $\overline{p}(\mathtt{s}) \geqslant \overline{p}(\mathtt{n})$, we have

$$H_k^{\overline{p}}\big(S^0 * X, \mathcal{S}_{\star 0}\big) = \begin{cases} H_k^{\overline{p}}(X,\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathtt{s}), \\ G & \text{if } 0 = k > D\overline{p}(\mathtt{s}), \\ 0 & \text{if } D\overline{p}(\mathtt{s}) < k \leqslant D\overline{p}(\mathtt{n}) + 1, k \neq 0 \\ \widetilde{H}_{k-1}^{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(\mathtt{n}) + 2, k \neq 0 \end{cases}$$

$$\mathfrak{H}_k^{\overline{p}}\big(S^0 * X, \mathcal{S}_{\star 0}\big) = \begin{cases} \mathfrak{H}_k^{\overline{p}}(X;\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathtt{s}), , \\ 0 & \text{if } D\overline{p}(\mathtt{s}) < k \leqslant D\overline{p}(\mathtt{n}) + 1, \\ \mathfrak{H}_{k-1}^{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(\mathtt{n}) + 2, \end{cases}$$

d. **Relative homologies**. Let U be an open subset of X. We have the associated long exact sequences for homology

$$\cdots \to H_k^{\overline{p}}(U;\mathcal{S}) \to H_k^{\overline{p}}(X;\mathcal{S}) \to H_k^{\overline{p}}(X,U;\mathcal{S}) \to H_{k-1}^{\overline{p}}(U;\mathcal{S}) \to \cdots,$$

$$\cdots \to \mathfrak{H}_k^{\overline{p}}(U;\mathcal{S}) \to \mathfrak{H}_k^{\overline{p}}(X;\mathcal{S}) \to \mathfrak{H}_k^{\overline{p}}(X,U;\mathcal{S}) \to \mathfrak{H}_{k-1}^{\overline{p}}(U;\mathcal{S}) \to \cdots$$

(cf. [6, Definition 4.5]) and for cohomology

$$\cdots \to H^k_{\overline{p}}(X,U;\mathcal{S}) \to H^k_{\overline{p}}(X;\mathcal{S}) \to H^k_{\overline{p}}(U;\mathcal{S}) \to H^{k+1}_{\overline{p}}(X,U;\mathcal{S}) \to \cdots$$

$$\cdots \to \mathfrak{H}^k_{\overline{p}}(X,U;\mathcal{S}) \to \mathfrak{H}^k_{\overline{p}}(X;\mathcal{S}) \to \mathfrak{H}^k_{\overline{p}}(U;\mathcal{S}) \to \mathfrak{H}^{k+1}_{\overline{p}}(X,U;\mathcal{S}) \to \cdots$$

(cf. [8, Theorem 7.1.11])[3].

e. **Universal Coefficients Theorem.** There are two natural exact sequences

$$0 \to \mathrm{Ext}(H^{\overline{p}}_{k-1}(X;\mathcal{S}), R) \to H^{k}_{\overline{p}}(X;\mathcal{S}) \to \mathrm{Hom}(H^{\overline{p}}_{k}(X;\mathcal{S}), R) \to 0,$$

$$0 \to \mathrm{Ext}(\mathfrak{H}^{\overline{p}}_{k-1}(X;\mathcal{S}), R) \to \mathfrak{H}^{k}_{\overline{p}}(X;\mathcal{S}) \to \mathrm{Hom}(\mathfrak{H}^{\overline{p}}_{k}(X;\mathcal{S}), R) \to 0,$$

for every $k \in \mathbb{N}$. We find the proof of the second assertion in [8, Proposition 7.1.4]. But the proof is the same for the first sequence.

1.6. **Intersection homology from Borel-Moore point of view (cf.** [10, 23]**).** The *Borel-Moore* $\overline{p}$*-intersection homology* $H^{BM,\overline{p}}_{*}(X;\mathcal{S})$ and the *Borel-Moore* $\overline{p}$*-tame intersection homology* $\mathfrak{H}^{BM,\overline{p}}_{*}(X;\mathcal{S})$ are defined in the same way as the homologies defined in 1.4 have been defined but considering locally finite chains instead of finite chains.

When X is compact, we have $H^{BM,\overline{p}}_{*}(X;\mathcal{S}) = H^{\overline{p}}_{*}(X;\mathcal{S})$ and $\mathfrak{H}^{BM,\overline{p}}_{*}(X;\mathcal{S}) = \mathfrak{H}^{\overline{p}}_{*}(X;\mathcal{S})$.

1.7. **Main properties for Borel-Moore (tame) intersection homology.** We suppose that X is a *hemicompact* space, that is, there exists an increasing sequence of compact subsets $K_0 \subset K_1 \subset \cdots K_n \subset \cdots$ such that, each compact $K \subset X$ is included on some K_n. We have proved in [7, Proposition 2.2] that the Borel-Moore intersection homology can be computed in terms of the intersection homology in the following way[4]:

$$H^{BM,\overline{p}}_{*}(X) = \varprojlim_{n\in\mathbb{N}} H^{\overline{p}}_{*}(X, X\backslash K_n) \quad \text{and} \quad \mathfrak{H}^{BM,\overline{p}}_{*}(X) = \varprojlim_{n\in\mathbb{N}} \mathfrak{H}^{\overline{p}}_{*}(X, X\backslash K_n). \tag{5}$$

1.8. **Blown-up intersection cohomologies (cf.** [3]**).** Let $N_*(\Delta)$ and $N^*(\Delta)$ be the simplicial chain and cochain complexes of an euclidean simplex Δ, with coefficients in R. For each simplex $F \in N_*(\Delta)$, we write $\mathbf{1}_F$ the element of $N^*(\Delta)$ taking the value 1 on F and 0 otherwise. Given a face F of Δ, we denote by $(F,0)$ the same face viewed as face of the cone $\mathtt{c}\Delta = \Delta * [\mathtt{w}]$ and by $(F,1)$ the face $\mathtt{c}F$ of $\mathtt{c}\Delta$. Here, $[\mathtt{w}] = (\emptyset, 1) = \mathtt{c}\emptyset$ is the apex o f the cone $\mathtt{c}\Delta$. Cochains on the cone are denoted $\mathbf{1}_{(F,\varepsilon)}$ for $\varepsilon = 0$ or 1. For defining the blown-up intersection complex, we first set

$$\widetilde{N}^*(\Delta) = N^*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N^*(\mathtt{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

A basis of $\widetilde{N}^*(\Delta)$ is composed of the elements $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n}$, where $\varepsilon_i \in \{0,1\}$ and F_i is a face of Δ_i for $i \in \{0,\ldots,n\}$ or the empty set with $\varepsilon_i = 1$ if $i < n$. We set $|\mathbf{1}_{(F,\varepsilon)}|_{>s} = \sum_{i>s}(\dim F_i + \varepsilon_i)$, with the convention $\dim \emptyset = -1$.

Let $\ell \in \{1,\ldots,n\}$ and $\mathbf{1}_{(F,\varepsilon)} \in \widetilde{N}^*(\Delta)$. The *$\ell$-perverse degree* of $\mathbf{1}_{(F,\varepsilon)} \in N^*(\Delta)$ is

$$\|\mathbf{1}_{(F,\varepsilon)}\|_\ell = \begin{cases} -\infty & \text{if } \varepsilon_{n-\ell} = 1, \\ |\mathbf{1}_{(F,\varepsilon)}|_{>n-\ell} & \text{if } \varepsilon_{n-\ell} = 0. \end{cases}$$

Given $\omega = \sum_b \lambda_b \, \mathbf{1}_{(F_b,\varepsilon_b)} \in \widetilde{N}^*(\Delta)$ with $0 \neq \lambda_b \in R$ for all b, the *ℓ-perverse degree* is

$$\|\omega\|_\ell = \max_b \|\mathbf{1}_{(F_b,\varepsilon_b)}\|_\ell.$$

[3]Only the tame case is considered in this reference but the non-tame case can be treated in the same way.

[4]In the op.cit. the result is proved for the Borel-Moore tame intersection homology. The same proof works for the Borel-Moore intersection homology.

By convention, we set $\|0\|_\ell = -\infty$.

Let $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ be a filtered simplex. We set $\widetilde{N}^*_\sigma = \widetilde{N}^*(\Delta)$. If $\delta_\ell\colon \Delta' \to \Delta$ is an inclusion of a face of codimension 1, we denote by $\partial_\ell\sigma$ the filtered simplex defined by $\partial_\ell\sigma = \sigma \circ \delta_\ell\colon \Delta' \to X$. If $\Delta = \Delta_0 * \cdots * \Delta_n$ is filtered, the induced filtration on Δ' is denoted $\Delta' = \Delta'_0 * \cdots * \Delta'_n$. The *blown-up intersection complex* of X is the cochain complex $\widetilde{N}^*(X)$ composed of the elements ω associating to each regular filtered simplex $\sigma\colon \Delta_0 * \cdots * \Delta_n \to X$ an element $\omega_\sigma \in \widetilde{N}^*_\sigma$ such that $\delta^*_\ell(\omega_\sigma) = \omega_{\partial_\ell\sigma}$, for any face operator $\delta_\ell\colon \Delta' \to \Delta$ with $\Delta'_n \neq \emptyset$. The differential $d\omega$ is defined by $(d\omega)_\sigma = d(\omega_\sigma)$. The *perverse degree of* ω *along a singular stratum* S equals

$$\|\omega\|_S = \sup\left\{\|\omega_\sigma\|_{\operatorname{codim} S} \mid \sigma\colon \Delta \to X \text{ regular simplex such that } \operatorname{Im}\sigma \cap S \neq \emptyset\right\}.$$

We denote $\|\omega\|$ the map $S \mapsto ||\omega||_S$, where $||\omega||_S = 0$ if S is a regular stratum. A *cochain* $\omega \in \widetilde{N}^*(X;\mathcal{S})$ *is* $\overline{p}$*-allowable* if $\|\omega\| \leqslant \overline{p}$ and of $\overline{p}$*-intersection* if ω and $d\omega$ are $\overline{p}$-allowable. We denote $\widetilde{N}^*_{\overline{p}}(X;\mathcal{S})$ the complex of $\overline{p}$-intersection cochains and $\mathscr{H}^{\overline{p}}_*(X;\mathcal{S})$ its homology called *blown-up intersection cohomology* of X for the perversity $\overline{p}$.

A subset $K \subset X$ is a *support* of the cochain $\omega \in \widetilde{N}^*_{\overline{p}}(X;\mathcal{S})$ if $\omega_\sigma = 0$, for any regular simplex $\sigma\colon \Delta \to X$ such that $\operatorname{Im}\sigma \cap K = \emptyset$. We also say that $\omega \equiv 0$ on $X\backslash K$.

We denote $\widetilde{N}^*_{\overline{p},c}(X;\mathcal{S})$ the complex of $\overline{p}$-intersection cochains with compact supports and $\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S})$ its cohomology.

1.9. **Main properties for blown up intersection cohomologies.** We group here the main properties of blown-up intersection cohomology. We fix a perverse space $(X,\mathcal{S},\overline{p})$.

a. **Mayer-Vietoris.** Suppose X paracompact. Given an open cover $\{U,V\}$ of X we have the long exact sequence (cf. [3, Corollary 10.1])

$$\cdots \to \mathscr{H}^k_{\overline{p}}(X;\mathcal{S}) \to \mathscr{H}^k_{\overline{p}}(U;\mathcal{S}) \oplus \mathscr{H}^*_{\overline{p}}(V;\mathcal{S}) \to \mathscr{H}^k_{\overline{p}}(U \cap V;\mathcal{S}) \to \mathscr{H}^{k+1}_{\overline{p}}(X;\mathcal{S}) \to \cdots.$$

b. **Local calculations.** We have the isomorphism

$$\mathscr{H}^k_{\overline{p}}(\mathbb{R}^m \times X, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{S}) = \mathscr{H}^k_{\overline{p}}(X,\mathcal{I}),$$

coming from the inclusion $\iota\colon X \to \mathbb{R}^m \times X$ defined by $\iota(x) = (0,x)$ (cf. [3, Theorem D]). If L is compact, we have the isomorphism

$$\mathscr{H}^k_{\overline{p}}(\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathcal{S}) = \begin{cases} \mathscr{H}^k_{\overline{p}}(L,\mathcal{S}) & \text{if } k \leqslant \overline{p}(\mathtt{v}), \\ 0 & \text{if } k > \overline{p}(\mathtt{v}), \end{cases}$$

where the isomorphism (first line) comes from the inclusion $\iota\colon L \to \mathbb{R}^m \times \mathring{\mathtt{c}}L$ defined by $\iota(x) = (0,[x,1/2])$ (cf. [3, Theorem E]).

c. **Join.** Using the above calculations, one gets the isomorphism:

$$\mathscr{H}^k_{\overline{p}}(S^m * X, \mathcal{S}_{\star m+1}) = \begin{cases} \mathscr{H}^k_{\overline{p}}(X,\mathcal{S}) & \text{if } k \leqslant \overline{p}(S^m), \\ 0 & \text{if } \overline{p}(S^m) < k \leqslant \overline{p}(S^m) + m + 1, \\ \mathscr{H}^{k-m-1}_{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant \overline{p}(S^m) + m + 2, \end{cases}$$

where the first isomorphism comes from the inclusion $\iota\colon X \to S^m * X$ defined by $\iota(x) = [0,x]$.

d. **Relative cohomology.** We consider an open subset $U \subset X$. The *complex of relative* $\overline{p}$*-intersection cochains* is $\widetilde{N}^*_{\overline{p}}(X,U;\mathcal{S}) = \widetilde{N}^*_{\overline{p}}(X;\mathcal{S}) \oplus \widetilde{N}^{*-1}_{\overline{p}}(U;\mathcal{S})$, endowed with the differential

$D(\alpha,\beta) = (d\alpha, \alpha - d\beta)$. Its homology is the *relative blown-up $\overline{p}$-intersection cohomology of the perverse pair* $(X, U, \overline{p})$, denoted by $\mathscr{H}^*_{\overline{p}}(X, U)$.

By definition, we have a long exact sequence associated to the perverse pair $(X, U, \overline{p})$:

$$\ldots \to \mathscr{H}^i_{\overline{p}}(X;\mathcal{S}) \xrightarrow{\imath^\star} \mathscr{H}^i_{\overline{p}}(U;\mathcal{S}) \to \mathscr{H}^{i+1}_{\overline{p}}(X,U;\mathcal{S}) \xrightarrow{\mathrm{pr}^*} \mathscr{H}^{i+1}_{\overline{p}}(X;\mathcal{S}) \to \ldots,$$

where $\mathrm{pr}\colon \widetilde{N}^*_{\overline{p}}(X,U) \to \widetilde{N}^*_{\overline{p}}(X)$ is defined by $\mathrm{pr}(\alpha,\beta) = \alpha$ and $\imath\colon \widetilde{N}^*_{\overline{p}}(X) \to \widetilde{N}^*_{\overline{p}}(U)$ is the restriction map (cf. [3, Sec. 12.2]).

e. **Injective limit.** Analogously to the Borel-Moore intersection homology, the blown-up intersection cohomology with compact supports can be computed through the relative blown-up intersection cohomology by using an injective limit. Let us see that.

Proposition 1.1. *Let $(X, \mathcal{S}, \overline{p})$ be a normal and hemicompact perverse space. Then, there exists an isomorphism*

$$\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \varinjlim_{K\subset X} \mathscr{H}^*_{\overline{p}}(X, X\backslash K;\mathcal{S}),$$

where K runs over the family of compact subsets of X.

Proof. By hemicompactness there exists an increasing sequence of compact subsets $\{K_n\}$ with

$$K_0 \subset \mathrm{int}(K_1) \subset K_1 \subset \mathrm{int}(K_2) \subset K_2 \subset \cdots K_n \subset \cdots,$$

and $X = \bigcup_{n\geqslant 0} K_n$. In particular, the family $\{K_n\}$ is cofinal in the family of compact subsets of X. So, it suffices to prove that the chain map

$$B\colon \widetilde{N}^*_{\overline{p},c}(X;\mathcal{S}) \to \varinjlim_{n\in\mathbb{N}} \widetilde{N}^*_{\overline{p}}(X, X\backslash K_n;\mathcal{S}),$$

defined by $B(\omega) = \langle(\omega, 0), m\rangle$, where K_m is a compact support of ω, is a quasi-isomorphism.

An element $\langle(\alpha,\beta), m\rangle \in \varinjlim_{n\in\mathbb{N}} \widetilde{N}^*_{\overline{p}}(X, X\backslash K_n;\mathcal{S})$ is characterized by these two properties:

- $(\alpha,\beta) \in \mathscr{H}^*_{\overline{p}}(X, X\backslash K_m;\mathcal{S})$, and
- $\langle(\alpha,\beta), m\rangle = \langle(\alpha',\beta'), m'\rangle$ if $(\alpha,\beta) = (\alpha',\beta')$ on $\widetilde{N}^*_{\overline{p}}(X, X\backslash K_{m'};\mathcal{S})$ if $m \leqslant m'$.

We proceed in several steps.

• *Step 1. Bump functions.*

Since X is normal then, for each each $n \in \mathbb{N}$, we can find a continuous map $f_n\colon X \to [0,1]$ with $f_n \equiv 0$ on K_{n+1} and $f_n \equiv 1$ on $X\backslash\mathrm{int}(K_{n+2})$. Associated to f_n we have constructed a cochain $\tilde{f}_n \in \widetilde{N}^0_{\overline{0}}(X;\mathcal{S})$ verifying $\tilde{f}_n \equiv 0$ on K_{n+1} and $\tilde{f}_n \equiv 1$ on $X\backslash\mathrm{int}(K_{n+2})$ (cf. [3, Lemma 10.2]). Consider the open covering $\mathcal{U}_n = \{X\backslash K_n, \mathrm{int}(K_{n+1})\}$ of X. Notice that[5],

$$\gamma \in \widetilde{N}^*_{\overline{p}}(X\backslash K_n;\mathcal{S}) \Longrightarrow \tilde{f}_n \smile \gamma \in \widetilde{N}^{*,\mathcal{U}_n}_{\overline{p}}(X;\mathcal{S}) \text{ and } \tilde{f}_n \smile \gamma = \gamma \text{ on } X\backslash K_{n+3}. \tag{6}$$

• *Step 2. The operator B^* is a monomorphism.*

Let $[\omega] \in \mathrm{Ker}\, B^*$. So, there exists $m \in \mathbb{N}$ and $\langle(\gamma,\eta), m\rangle \in \widetilde{N}^*_{\overline{p}}(X, X\backslash K_m;\mathcal{S})$ with K_m compact support of $\omega \in \widetilde{N}^*_{\overline{p},c}(X;\mathcal{S})$ and $\langle(\omega,0), m\rangle = \langle D(\gamma,\eta), m\rangle = \langle(d\gamma, \gamma - d\eta), m\rangle$. In particular, we get $\omega = d\gamma$ on X and $\gamma - d\eta = 0$ on $K\backslash K_m$.

We get the claim if we prove that $\rho_{\mathcal{U}_m,c}(\omega) = d\theta$ for some $\theta \in \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p},c}(X;\mathcal{S})$ (cf. [3, Theorem B]). Here, $\rho_{\mathcal{U}_m,c}\colon \widetilde{N}^*_{\overline{p},c}(X;\mathcal{S}) \to \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p},c}(X;\mathcal{S})$ is the canonical restriction. It suffices to consider $\theta = \rho_{\mathcal{U}_m}(\gamma) - d(\tilde{f}_m \smile \eta)$, where $\rho_{\mathcal{U}_m}\colon \widetilde{N}^*_{\overline{p},c}(X;\mathcal{S}) \to \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p}}(X;\mathcal{S})$ is the canonical restriction, since

[5] We refer the reader to (cf. [3, Section 4]) for the definition of the $\smile$-product.

(i) $\tilde{f}_m \smile \eta \in \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p}}(X;\mathcal{S})$ (cf. (6)).

(ii) $\gamma - d(\tilde{f}_m \smile \eta) \overset{(6)}{=} \gamma - d\eta = 0$ on $X\backslash K_{m+3}$, giving that K_{m+3} is a compact support of θ,

(iii) $d\theta = d\rho_{\mathcal{U}_m}(\gamma) = \rho_{\mathcal{U}_m}(\omega) = \rho_{\mathcal{U}_m,c}(\omega)$.

• *Step 3. The operator $B^{\mathcal{U}_m}$.*

For each $n, m \in \mathbb{N}$ we define $\widetilde{N}^{*,\mathcal{U}_m}_{\overline{p}}(X, X\backslash K_n;\mathcal{S}) = \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p}}(X;\mathcal{S}) \oplus \widetilde{N}^{*-1,\mathcal{U}_m}_{\overline{p}}(X\backslash K_n;\mathcal{S})$ (cf. [3, Definition 9.6]). We consider the chain map

$$B^{\mathcal{U}_m} : \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p},c}(X;\mathcal{S}) \to \varinjlim_{n\in\mathbb{N}} \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p}}(X, X\backslash K_n;\mathcal{S}),$$

defined by $B^{\mathcal{U}_m}(\omega) = \langle(\omega, 0), p\rangle_{\mathcal{U}_m}$, where K_p is a compact support of ω (cf. [5, Definition 2.6]). We have the commutative diagram

$$\begin{array}{ccc} \widetilde{N}^{*}_{\overline{p},c}(X;\mathcal{S}) & \xrightarrow{B} & \varinjlim_{n\in\mathbb{N}} \widetilde{N}^{*}_{\overline{p}}(X, X\backslash K_n;\mathcal{S}) \\ \downarrow{\rho_{\mathcal{U}_m,c}} & & \downarrow{\rho'_{\mathcal{U}_m}} \\ \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p},c}(X;\mathcal{S}) & \xrightarrow{B^{\mathcal{U}_m}} & \varinjlim_{n\in\mathbb{N}} \widetilde{N}^{*,\mathcal{U}_m}_{\overline{p}}(X, X\backslash K_n;\mathcal{S}) \end{array} \tag{7}$$

where the vertical maps are defined by restriction. Both are quasi-isomorphisms. It suffices to apply [5, Proposition 2.6] (for the left one) and the fact that inductive limits commute with cohomology and [3, Theorem B] (for the right one).

• *Step 4. The operator B^* is an epimorphism.*

Let $\Xi = \langle(\gamma, \eta), m\rangle \in \varinjlim_{\mathbb{N}} \widetilde{N}^{*}_{\overline{p}}(X, X\backslash K;\mathcal{S})$ be a cycle. Then $\langle D(\gamma, \eta), m\rangle = \langle(d\gamma, \gamma - d\eta), m\rangle = 0$. The cochain $\theta = \rho_{\mathcal{U}_m}(\gamma) - d(\tilde{f}_m \smile \eta)$ is a cycle of $\widetilde{N}^{*,\mathcal{U}_m}_{\overline{p},c}(X;\mathcal{S})$ since (i), (ii) and $d\theta = d\rho_{\mathcal{U}_m}(\gamma) = 0$. In fact,

$$\begin{aligned} B^{\mathcal{U}_m,*}[\theta] &= [\langle(\theta, 0), m+3\rangle_{\mathcal{U}_m}] = \left[\left\langle(\rho_{\mathcal{U}_m}((\gamma) - d(\tilde{f}_m \smile \eta), 0), m+3\right\rangle_{\mathcal{U}_m}\right] \\ &=_{(1)} \left[\left\langle(\rho_{\mathcal{U}_m}(\gamma), \tilde{f}_m \smile \eta), m+3\right\rangle_{\mathcal{U}_m}\right] =_{(2)} [\langle(\rho_{\mathcal{U}_m}(\gamma), \eta), m+3\rangle_{\mathcal{U}_m}] \\ &= [\rho'_{\mathcal{U}_m}(\langle(\gamma, \eta), m+3\rangle)] =_{(3)} [\rho'_{\mathcal{U}_m}(\langle(\gamma, \eta), m\rangle)] = \rho'^*_{\mathcal{U}_m}[\Xi]. \end{aligned}$$

where $=_{(1)}$ comes from $D(\tilde{f}_m \smile \eta, 0) = (d(\tilde{f}_m \smile \eta), -\tilde{f}_m \smile \eta)$, $=_{(2)}$ from $\tilde{f}_m = 1$ on $X\backslash K_{m+2}$ and $=_{(3)}$ from the fact that $\eta \in \widetilde{N}^{*}_{\overline{p}}(X\backslash K_m)$.

The properties of the previous diagram (7) give the existence of $[\omega] \in \mathscr{H}^{*}_{\overline{p},c}(X;\mathcal{S})$ with $\rho^*_{\mathcal{U}_m,c}[\omega] = [\theta]$ verifying $\rho'^*_{\mathcal{U}_m}(B^*[\omega]) = B^{\mathcal{U}_m,*}(\rho^*_{\mathcal{U}_m,c}[\omega]) = B^{\mathcal{U}_m,*}[\theta] = \rho'^*_{\mathcal{U}_m}[\Xi]$, which gives $[\Xi] = B^*[\omega]$. ♣

2. Stratified sets and refinements

A refinement of a stratified space $(X, \mathcal{S})$ is another stratified space $(X, \mathcal{T})$ whose strata are formed using the strata of the original stratification. We prove that it is possible to go from $\mathcal{S}$ to $\mathcal{T}$ by modifying just a discrete family of strata: the simple refinement.

2.1. **Stratified spaces.** A *stratified space*[6] is a Hausdorff topological space X endowed with a partition $\mathcal{S}$, whose elements are called *strata*, verifying the following conditions (S1)-(S6).

(S1) The family $\mathcal{S}$ is locally finite.
(S2) An element of $\mathcal{S}$ is a connected manifold.
(S3) *Frontier Condition.* Given two strata $S, S' \in \mathcal{S}$, we have[7]: $S \cap \overline{S'} \neq \emptyset \Longrightarrow S \subset \overline{S'}$.
(S4) Given two strata $S, S' \in \mathcal{S}$, we have: $S \cap \overline{S'} \neq \emptyset$ and $S \neq S' \Longrightarrow \dim S < \dim S'$.
(S5) The family $\{\dim S \in \mathcal{S}\}$ is bounded.

Stratified and filtered spaces are related as follows.

Lemma 2.1. *Let $(X, \mathcal{S})$ be a stratified space. Then the filtration $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subsetneq X_n = X$, given by*

$$X_k = \sqcup\{S \in \mathcal{S} \mid \dim S \leqslant k\},$$

(cd. (S5)) defines a filtered space on X whose associated stratification is $\mathcal{S}$.

Proof. For the first statement it suffices to prove that each X_k is a closed subset of X. Let us consider $x \in \overline{X_k}$. Property (S1) gives $x \in \overline{X_k} = \overline{\{S \in \mathcal{S} \mid \dim S \leqslant k\}} = \{\overline{S} \in \mathcal{S} \mid \dim S \leqslant k\}$. So, there exists $S \in \mathcal{S}$ with $x \in \overline{S}$ and $\dim S \leqslant k$. If S' is the stratum of $\mathcal{S}$ containing x then condition (S4) give $\dim S' \leqslant \dim S$ and therefore $x \in X_k$.

We have $X_k \backslash X_{k-1} = \sqcup\{S \in \mathcal{S} \mid \dim S = k\}$. Again, conditions (S1) and (S4) imply that the elements of the RHS of the equality are closed subsets of $X_k \backslash X_{k-1}$. So, the stratification of the filtered space is $\mathcal{S}$. ♣

These are not equivalent notions since, for example, the strata of a filtered space are not necessarily manifolds.

The relation $S \preceq S'$ defined by $S \subset \overline{S'}$, is an order relation on $\mathcal{S}$ (see [4, Proposition A.22]). The notation $S \prec S'$ means $S \preceq S'$ and $S \neq S'$. So, condition (S4) is equivalent to

(S4) $S \prec S' \Longrightarrow \dim S < \dim S'$.

The *depth* of a family of strata $\mathcal{S}' \subset \mathcal{S}$ is $\operatorname{depth} \mathcal{S}' = \sup\{i \in \mathbb{N} \mid \exists S_0 \prec S_1 \prec \cdots \prec S_i \text{ with } S_0, \ldots, S_i \in \mathcal{S}'\}$. Conditions (S4) and (S5) give

Lemma 2.2. *Let $(X, \mathcal{S})$ be a stratified space. Then* $\operatorname{depth} \mathcal{S}' < \infty$.

In this work we shall use the formula

$$\overline{S} = S \sqcup \bigsqcup_{Q \prec S} Q. \tag{8}$$

Let us see that. The inclusion $\supset$ is clear. Let $x \in \overline{S}$. Consider $Q \in \mathcal{S}$ containing x. Since $Q \cap \overline{S} \neq \emptyset$ then we get $\subset$ from (S3).

The examples of 1.2: induced, product, cone and join stratification, are also stratified spaces, if we begin with a stratified space $(X, \mathcal{S})$. Any point $x \in X$, with $\{x\} \notin \mathcal{S}$, can be added as a new stratum. This gives the stratification $\mathcal{S}_x = \{x\} \sqcup \{(S \backslash \{x\})_{cc} \mid S \in \mathcal{S}\}$.

The following result will be important for the understanding of the local structure of the stratified spaces we are interested in.

Proposition 2.3. *Let $(X, \mathcal{S})$ be a stratified space and let $m \in \mathbb{N}^*$. Then, there exists a stratified homeomorphism*

$$h\colon (\mathring{\mathtt{c}}(S^m * X), \mathring{\mathtt{c}}\mathcal{S}_{\star m}) \to (B^{m+1} \times \mathring{\mathtt{c}}X, (\mathcal{I} \times \mathring{\mathtt{c}}\mathcal{S})_{(0,\mathrm{v})}). \tag{9}$$

[6]This definition is not a standard one in all sources. For example, it is more restrictive than that of [4, 8].

[7]This condition is equivalent to say that the closure of a stratum is the union of strata.

Proof. We find in [2, 5.7.4] the homeomorphism $h\colon \mathring{\mathtt{c}}(S^m * X) \to B^{m+1} \times \mathring{\mathtt{c}}X$: defined by

$$h([[z,y],r]) = \begin{cases} (2rz, [y,r]) & \text{si } ||z|| \leqslant 1/2 \\ (rz/||z||, [y, 2r(1-||z||)]) & \text{si } ||z|| \geqslant 1/2. \end{cases} \tag{10}$$

Let us verify that h preserves the stratifications. We write $\mathtt{u}$ the apex of the cone $\mathring{\mathtt{c}}(S^m * X)$. We distinguish three cases.

+ $h(\mathtt{u}) = (0, \mathtt{v})$.
+ $h(S^m \times]0,1[) = (B^{m+1} \times \{\mathtt{v}\}) \backslash \{(0,\mathtt{v})\}$ since $h([[z,y],r] = (rz, \mathtt{v})$ if $||z|| = 1$.
+ The restriction $h\colon B^{m+1} \times X \times]0,1[\to B^{m+1} \times X \times]0,1[$ is given by

$$(z,y,r) \mapsto \begin{cases} (2rz, y, r) & \text{si } ||z|| \leqslant 1/2 \\ (rz/||z||, y, 2r(1-||z||)) & \text{si } ||z|| \geqslant 1/2. \end{cases}$$

It is clearly a stratified homeomorphism. ♣

2.2. **Refinements.** We say that the stratified space $(X, \mathcal{S})$ is a *refinement* of the stratified space $(X, \mathcal{T})$[8], written $(X,\mathcal{S}) \lhd (X,\mathcal{T})$, if $\mathcal{S} \neq \mathcal{T}$ and

(S6) $\forall S \in \mathcal{S} \quad \exists T \in \mathcal{T}$ such that S is embedded submanifold of T.

The stratum T is also denoted by S^I. We have

$$\dim S \leqslant \dim S^I \text{ and } \operatorname{codim} S^I \leqslant \operatorname{codim} S, \quad \text{for each } S \in \mathcal{S}. \tag{11}$$

Notice that

$$S, Q \in \mathcal{S} \text{ with } S \preceq Q \Longrightarrow S^I \preceq Q^I \tag{12}$$

(cf. (S3)). In this work, we shall distinguish several types of strata.

Definition 2.4. *Let $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ be a refinement. A stratum $S \in \mathcal{S}$ is a* source stratum *if $\dim S = \dim S^I$. In this case, we also say that S is a* source stratum of $T \in \mathcal{T}$, *if $T = S^I$. We also use the following types of strata:*

- $\mathcal{A} = \{S \in \mathcal{S} \ / \ \dim S = \dim S^I\}$: source strata.
- $\mathcal{V} = \{S \in \mathcal{S} \ / \ \dim S < \dim S^I\}$: virtual strata.
- $\mathcal{M} = \{$*maximal strata of* $\mathcal{V}$ *with* $\dim M^I$ *maximal*$\}$: v-maximal strata.
- $\mathcal{O} = \{S \in \mathcal{A} \ / \ \exists M \in \mathcal{M}$ *with* $M \preceq S, M^I = S^I\}$: stable strata.

The refinement $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ is simple *when* $\operatorname{depth} \mathcal{V} = 0$. *We always have* $\operatorname{depth} \mathcal{M} = 0$.

Definition 2.5. *Let $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ be a refinement. A stratum $S \in \mathcal{S}^{sing}$ is* exceptional *if $S^I \in \mathcal{T}^{reg}$. Moreover, if $\operatorname{codim} S = 1$ we say that S is an* 1-exceptional stratum.

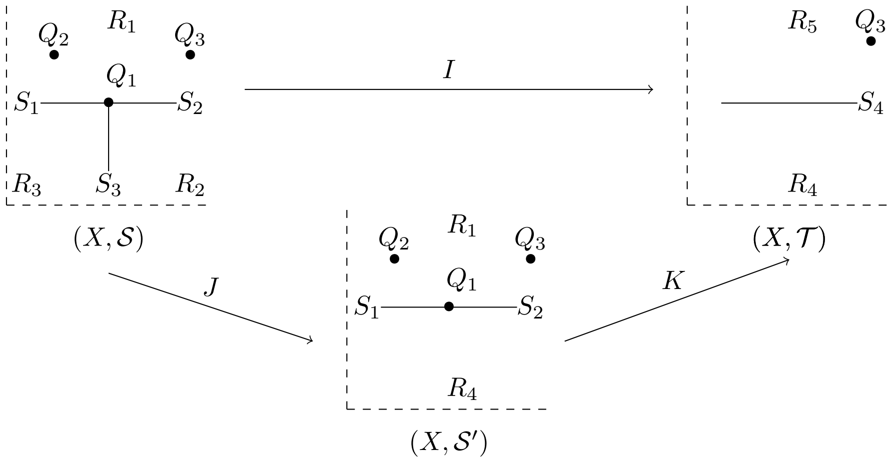
Any exceptional stratum is a virtual stratum.

Definition 2.6. *A* simple decomposition *of a refinement $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ is a finite sequence of simple refinements: $(X,\mathcal{S}) = (X,\mathcal{R}_0) \lhd \cdots \lhd (X,\mathcal{R}_m) = (X,\mathcal{T})$.*

Example 2.7. A key result of this work is the Proposition 2.10 giving the existence of simple refinements. The relevance of these kind of refinements is given by Proposition 3.4, where we get a nice local description of the a simple refinement in the framework of CS-sets.

Before proving these results, we give an example of a refinement $(X,\mathcal{S}) \lhd_I (X,\mathcal{T})$ described as composition of two simple refinements $(X,\mathcal{S}) \lhd_J (X,\mathcal{S}')$ and $(X,\mathcal{S}') \lhd_K (X,\mathcal{T})$ through a stratified space $(X,\mathcal{S}')$.

[8] We also say that $(X,\mathcal{T})$ is a *coarsening* of $(X,\mathcal{S})$.



Refinement J	Refinement I	Refinement K
$\mathcal{A}\backslash\mathcal{O} = \{Q_1, Q_2, Q_3, R_1, S_1, S_2\}$	$\mathcal{A}\backslash\mathcal{O} = \{Q_3, R_1, S_1, S_2\}$	$\mathcal{A}\backslash\mathcal{O} = \{Q_3, R_4\}$
$\mathcal{V} = \mathcal{M} = \{S_3\}$	$\mathcal{V}\backslash\mathcal{M} = \{Q_1, Q_2\}$	$\mathcal{V} = \mathcal{M} = \{Q_1, Q_2\}$
$\mathcal{O} = \{R_2, R_3\}$	$\mathcal{M} = \{S_3\}$	$\mathcal{O} = \{R_1, S_1, S_2\}$
	$\mathcal{O} = \{R_2, R_3\}$	

In the simple refinement J (resp. K) a stratum of $\mathcal{M}$ melts into a stratum of $\mathcal{S}'$ (resp. $\mathcal{T}$) and, for each of them, 1 or 2 strata of $\mathcal{O}$ also disappear into a bigger stratum with same dimension: $S_3, R_2, R_3 \rightsquigarrow R_4$ (resp. $Q_1, S_1, S_2 \rightsquigarrow S$ or $Q_2, R_1 \rightsquigarrow R_5$). Among the strata of $\mathcal{V}$ those of $\mathcal{M}$ are the first to disappear.

The objective of the following Lemmas is to prove Proposition 2.10: a refinement can be decomposed as a sequence of simple refinements.

Lemma 2.8. *Let $(X, \mathcal{S}) \lhd_I (X, \mathcal{T})$ be a refinement with $\mathcal{S} \neq \mathcal{T}$. Then*

(a) $\mathcal{M} \neq \varnothing$.

(b) For each $S \in \mathcal{S}$ there exists a source stratum $P \in \mathcal{S}$ of S^{I} with $S \preceq P$.

(c) Given two strata $R, Q \in \mathcal{T}$ with $R \preceq Q$ there exist two source strata $R', Q' \in \mathcal{S}$ of R and Q respectively with $R' \preceq Q'$.

Proof. (a) Since $\mathcal{S} \neq \mathcal{T}$ then there exists a stratum $S \in \mathcal{S}$ with $S \neq S^{I}$. If $\mathcal{M} = \varnothing$ then $\mathcal{V} = \varnothing$ and then $S^{I} = \sqcup\{P \in \mathcal{A} \mid P^{I} = S^{I}\}$, open connected subsets of S^{I} (cf. (S6)). By connectedness of S^{I} we conclude that $\{P \in \mathcal{A} \mid P^{I} = S^{I}\}$ contains just one element, necessarily S. The contradiction $S^{I} = S$ implies that $\mathcal{M} \neq \varnothing$.

(b) By definition we have

$$S^{I} = \sqcup\{P \in \mathcal{A} \mid P^{I} = S^{I}\} \sqcup \left(\sqcup\{P \in \mathcal{V} \mid P^{I} = S^{I}\}\right), \tag{13}$$

where the elements of the first term are open subsets of S^{I}. This decomposition is locally finite (cf. (S1)). By dimension reasons, $O = \sqcup\{P \in \mathcal{A} \mid P^{I} = S^{I}\} = \sqcup\{\text{source strata of } S^{I}\}$ is an open dense subset of S^{I} (cf. (S4), (S6)). Condition $S \subset \overline{O}$ implies the existence of a source stratum P of S^{I} with $S \cap \overline{P} \neq \varnothing$. Property (S3) gives (b).

(c) Item (b) gives a source stratum $R' \in \mathcal{S}$ of R. Since $\sqcup$ $\{\text{source strata of } Q\}$ $= \sqcup_{i \in I} Q_i$ is an open dense subset of Q then $R' \subset R \subset \overline{Q} = \overline{\sqcup_{i \in I} Q_i} =_{(S1)} \cup_{i \in I} \overline{Q_i}$. So, there exists $Q_i \in \mathcal{S}$, source stratum of Q, with $R' \cap \overline{Q_i} \neq \varnothing$. Since $R' \preceq Q_i$ (cf. (S3)) we end the proof taking $Q' = Q_i$. ♣

The subsets $M_{_I}$ we study now play an important rôle in the construction of the simple decomposition of a refinement. They are the new strata on the first step of this decomposition.

Lemma 2.9. *Let $(X,\mathcal{S}) \lhd_I (X,\mathcal{T})$ be a refinement. Consider a stratum $M \in \mathcal{M}$. We define*

$$M_{_I} = \sqcup\{Q \mid Q \in \mathcal{O} \sqcup \mathcal{M} \text{ and } Q^{^I} = M^{^I}\}.$$

Then

(a) $M_{_I}$ is a connected open subset of $M^{^I}$.

(b) Q is an embedded sub-manifold of $M_{_I}$, if $Q \in \mathcal{M}$ and $Q^{^I} = M^{^I}$, and

(c) Q is an open subset of $M_{_I}$, if $Q \in \mathcal{O}$ and $Q^{^I} = M^{^I}$.

Proof. Without loss of generality we can suppose $X = M^{^I}$. We have $S^{^I} = M^{^I}$ for each $S \in \mathcal{S}$.

(a) The subset $F = \sqcup\{S \in \mathcal{V}\backslash\mathcal{M}\}$ is a closed subset of X (cf. (8) and (S4)) not meeting $M_{_I}$. Given $S \in \mathcal{V}\backslash\mathcal{M}$ we have $\dim S \neq \dim X$ (since $S \in \mathcal{V}$) and $\dim S \neq \dim X - 1$ (since $S \notin \mathcal{M}$). Then, F it is a locally finite union of sub-manifolds of X whose codimension is at least 2 (cf. (S1), (S4)). So, $Y = X\backslash F$ is a $\mathcal{S}$-saturated connected open subset of $M^{^I}$ containing $M_{_I}$.

By construction, we have $\mathcal{V}_Y = \mathcal{M}$, that is, $\mathcal{S}_Y = \mathcal{M} \sqcup \mathcal{A}$. Let us suppose $\mathcal{A} \neq \mathcal{O}$. By dimension reasons, if $S \in \mathcal{A}\backslash\mathcal{O}$ then $\overline{S} = S$ (cf. (8) and (S4)). Property (S2) gives $S = Y$ and then $S = M$ which is impossible. So, $\mathcal{S}_Y = \mathcal{M} \sqcup \mathcal{O}$. Then $M_{_I} = Y$ which is a connected open subset of $M^{^I}$.

(b) Condition (S6) implies that Q is an embedded sub-manifold of $M^{^I}$. Since $Q \subset M_{_I}$ then (a) gives the result.

(c) Finally, $\dim Q = \dim Q^{^I} = \dim M^{^I}$ implies that Q is an open subset of $M^{^I}$ (cf. (S6)). Since $Q \subset M_{_I}$, condition (a) gives (c). ♣

Proposition 2.10. *Any refinement $(X,\mathcal{S}) \lhd (X,\mathcal{T})$, with $\mathcal{S} \neq \mathcal{T}$, has a simple decomposition.*

Proof. Let us define $d_{\mathcal{S},\mathcal{T}} = \dim M^{^I}$ where $M \in \mathcal{M}$. This number is independent of the choice of M by definition of $\mathcal{M}$. Condition $\mathcal{S} \neq \mathcal{T}$ implies $\mathcal{M} \neq \varnothing$ (cf. Lemma 2.8 (a)) and therefore $d_{\mathcal{S},\mathcal{T}} \geqslant 0$. We proceed by induction on $d_{\mathcal{S},\mathcal{T}}$. If $d_{\mathcal{S},\mathcal{T}} = 0$ then the dimension of the strata of $\mathcal{M}$ is 0. Then $\mathcal{V} = \mathcal{M}$, which gives $\operatorname{depth} \mathcal{V} = 0$. We conclude that the refinement is simple.

Now, in the inductive step, we can suppose that $d_{\mathcal{S},\mathcal{T}} > 0$. It suffices to construct a chain of refinements $(X,\mathcal{S}) \preceq (X,\mathcal{R}) \preceq (X,\mathcal{T})$, where the first one is simple and $d_{\mathcal{R},\mathcal{T}} < d_{\mathcal{S},\mathcal{T}}$.

Let $M, N \in \mathcal{M}$ be two strata with $M_{_I} \cap N_{_I} \neq \varnothing$. This implies $M^{^I} \cap N^{^I} \neq \varnothing$ and therefore $M^{^I} = N^{^I}$. So, $M_{_I} = \sqcup\{Q \in \mathcal{O} \sqcup \mathcal{M} \mid Q \subset M^{^I}\} = \sqcup\{Q \in \mathcal{O} \sqcup \mathcal{M} \mid Q \subset N^{^I}\} = N_{_I}$. We get the dichotomy $M_{_I} = N_{_I}$ or $M_{_I} \cap N_{_I} = \varnothing$. In order to avoid repetitions, we fix a family $\{M_i \subset \mathcal{M} \mid i \in \nabla\}$ such that $\cup\{M_{_I} \mid M \in \mathcal{M}\} = \sqcup\{M_{_{i,I}} \mid i \in \nabla\}$, We define

$$\mathcal{R} = \mathcal{S}\backslash(\mathcal{O} \sqcup \mathcal{M}) \sqcup \{M_{_{i,I}} \mid i \in \nabla\}. \tag{14}$$

Let us verify all the properties.

• $(\boldsymbol{X},\boldsymbol{\mathcal{R}})$ **is a stratified space**. By definition of stable strata we have $\sqcup\{Q \mid Q \in \mathcal{O} \sqcup \mathcal{M}\} = \sqcup\{M_{_{i,I}} \mid i \in \nabla\}$. Then $\mathcal{R}$ is a partition of X. Condition $(\text{S1})_{\mathcal{S}}$ gives condition $(\text{S1})_{\mathcal{R}}$. Condition $(\text{S2})_{\mathcal{R}}$ comes from $(\text{S2})_{\mathcal{S}}$ and Lemma 2.9 (a). For the proof of $(\text{S3})_{\mathcal{R}}$ and $(\text{S4})_{\mathcal{R}}$, it suffices to prove:

(a) $S \cap \overline{P} \neq \varnothing \quad \Rightarrow \quad S \subset \overline{P}$ and $\dim S < \dim P$.

(b) $S \cap \overline{M_{_I}} \neq \varnothing \quad \Rightarrow \quad S \subset \overline{M_{_I}}$ and $\dim S < \dim M_{_I}$,

(c) $\overline{S} \cap M_{_I} \neq \varnothing \quad \Rightarrow \quad \overline{S} \supset M_{_I}$ and $\dim M_{_I} < \dim S$,

(d) $\overline{N_{_I}} \cap M_{_I} \neq \varnothing \quad \Rightarrow \quad M_{_I} = N_{_I}$.

where $S, P \in \mathcal{S}\backslash(\mathcal{O} \sqcup \mathcal{M})$ and $M, N \in \mathcal{M}$. Let us see that.

(a) It follows directly from $(S3)_{\mathcal{S}}$ and $(S4)_{\mathcal{S}}$.

(b) Locally finiteness of $\mathcal{S}$ (cf. $(S3)_{\mathcal{S}}$) gives $\overline{M_I} = \cup\{\overline{Q} \mid Q \in \mathcal{S} \text{ and } Q \subset M_I\}$. So, there exists $Q \in \mathcal{O} \sqcup \mathcal{M}$ with $Q^I = M^I$ and $S \cap \overline{Q} \neq \emptyset$. So, $S \subset \overline{Q} \subset \overline{M_I}$ (cf. $(S3)_{\mathcal{S}}$). Since $S \notin \mathcal{O} \sqcup \mathcal{M}$ we get $S \neq Q$ and then $\dim S \overset{(S4)_{\mathcal{S}}}{<} \dim Q \overset{(11)}{\leqslant} \dim Q^I = \dim M^I \overset{Lemma\ 2.9(a)}{=} \dim M_I$.

(c) Condition $\overline{S} \cap M_I \neq \emptyset$ implies the existence of $Q \in \mathcal{M} \sqcup \mathcal{O}$ with $Q^I = M^I$ and $Q \preceq S$ (cf. $(S3)_{\mathcal{S}}$). By definition of stable strata we can suppose that $Q \in \mathcal{M}$, which implies $S \in \mathcal{A}$ since $S \notin \mathcal{M}$. If $M^I = S^I$ then $S \in \mathcal{O}$, which is impossible. So, $M^I \neq S^I$. Since $M^I = Q^I \overset{(12)}{\preceq} S^I$ then $M^I \prec S^I$ and we get $\dim M^I \overset{(S4)_{\mathcal{T}}}{<} \dim S^I$.

Let us consider a virtual stratum $V \in \mathcal{V}$ included in S^I. There exists a maximal stratum $W \in \mathcal{V}$ with $V \preceq W$ (cf. Lemma 2.2). Since $V^I \preceq W^I$ (cf. (12)) then we have

$$\dim M^I < \dim S^I = \dim V^I \overset{(S4)_{\mathcal{S}}}{\leqslant} \dim W^I,$$

which is impossible by definition of $\mathcal{M}$. So, the subset S^I does not contain any virtual stratum.

By connectedness of S^I the formula (13) implies that S^I contains just one stratum of $\mathcal{S}$, that is, $S^I = S$. We get $M_I \subset M^I = Q^I \subset \overline{S^I} = \overline{S}$ and $\dim M_I = \dim M^I = \dim Q^I \overset{(S4)_{\mathcal{S}}}{\leqslant} \dim S^I = \dim S$ (cf. Lemma 2.9 (a)).

(d) If $\overline{N_I} \cap M_I \neq \emptyset$ then $\overline{N^I} \cap M^I \neq \emptyset$ and therefore $M^I \preceq N^I$ (cf. $(S3)_{\mathcal{T}}$). Lemma 2.9 (a), $(S4)_{\mathcal{T}}$ and (12) give $\dim M_I = \dim M^I \leqslant \dim N^I = \dim N_I$. By definition of $\mathcal{M}$ we get that previous $\leqslant$ becomes $=$. Finally, condition $(S4)_{\mathcal{T}}$ gives $M^I = N^I$ and therefore $M_I = N_I$.

• $\boldsymbol{(X, \mathcal{S}) \lhd (X, \mathcal{R})}$ **is a simple refinement**. The strata of $\mathcal{S} \backslash (\mathcal{M} \sqcup \mathcal{O})$ remain equal. The other strata verify condition $(S6)_{\mathcal{S},\mathcal{R}}$ following Lemma 2.9. So, $(X, \mathcal{S}) \lhd (X, \mathcal{R})$ is a refinement. The only strata whose dimension increases when passing from $\mathcal{S}$ to $\mathcal{R}$ are the strata of $\mathcal{M}$: $\dim M < \dim M_I$. So

$$\mathcal{V}_{\mathcal{S},\mathcal{R}} = \mathcal{M}_{\mathcal{S},\mathcal{R}} = \mathcal{M} = \mathcal{M}_{\mathcal{S},\mathcal{T}} \tag{15}$$

which gives $\operatorname{depth} \mathcal{V}_{\mathcal{S},\mathcal{R}} = \operatorname{depth} \mathcal{M}_{\mathcal{S},\mathcal{T}} = 0$.

• $\boldsymbol{(X, \mathcal{R}) \lhd (X, \mathcal{T})}$ **is a refinement with** $\boldsymbol{d_{\mathcal{R},\mathcal{T}} < d_{\mathcal{S},\mathcal{T}}}$. A stratum $Q \in \mathcal{S} \backslash (\mathcal{O} \sqcup \mathcal{M})$ goes to Q^I, where it is an embedded sub-manifold from $(S6)_{\mathcal{S},\mathcal{R}}$. The strata M_I, $M \in \mathcal{M}$, are open subsets of M^I. So, $(X, \mathcal{R}) \lhd (X, \mathcal{T})$ is a refinement. Since $\dim M_I = \dim M^I$, for each $M \in \mathcal{M}$, then $M_I \in \mathcal{R}$ is a source stratum. The same is true for the strata of $\mathcal{A} \backslash \mathcal{O}$. This gives $\mathcal{V}_{\mathcal{R},\mathcal{T}} = \mathcal{V} \backslash \mathcal{M} = \mathcal{V}_{\mathcal{S},\mathcal{T}} \backslash \mathcal{M}_{\mathcal{S},\mathcal{T}}$ and therefore $d_{\mathcal{R},\mathcal{T}} < d_{\mathcal{S},\mathcal{T}}$. ♣

3. CS-sets

The invariance result we study in this work applies to CS-sets, a weaker notion than that of stratified pseudomanifold. Here, a link of a stratum is not necessarily a CS-set but a filtered space [8, example 2.3.6]. We also describe the local structure of a simple refinement between two CS-sets.

3.1. **CS-sets.** A filtered space $(X, \mathcal{S})$ is a *n-dimensional CS-set* if any regular stratum is an n-dimensional manifold, and for any singular stratum $S \in \mathcal{S}$ and for any $x \in S$ there exists a stratum preserving homeomorphism[9]

$$\varphi \colon (\mathbb{R}^i \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathtt{c}\mathcal{L}) \to (V, \mathcal{S}),$$

[9] A homeomorphism which is also a stratified map. The involved stratifications are described in Example 1.2 .

where

(a) $V \subset X$ is a open subset containing x,
(b) $(L;\mathcal{L})$ is a compact filtered space,
(c) $\varphi(0,\mathtt{v}) = x$ and $\varphi(\mathbb{R}^i \times \{\mathtt{v}\}) = V \cap S$.

The pair (V,φ) is a *$\mathcal{S}$-conical chart*, or simply *conical chart*, of x. The *link* of φ is $(L,\mathcal{L})$. Since the links are always non-empty sets then the open subset $X\backslash\Sigma$ is dense. Closed strata of $\mathcal{S}$ are exactly the minimal strata of $\mathcal{S}$. On the other hand, the open strata of $\mathcal{S}$ are the maximal strata of X, they coincide with the n-dimensional strata of X.

A *perverse CS-set* is a triple $(X,\mathcal{S},\overline{p})$ where $(X,\mathcal{S})$ is a CS-set and $\overline{p}$ is a perversity on $(X,\mathcal{S})$.

We find in [20] a comparison between different notions of stratification. In this work we need the following property.

Proposition 3.1. *Any CS-set is a stratified space.*

Proof. Conditions (S2) and (S5) come from definition. Property (S1) is proved in [8, Lemma 2.3.8]. Let us verify (S3) and (S4). Since it is a local question, we set $X = \mathbb{R}^i \times \mathring{c}L$ with $S = \mathbb{R}^i \times \{\mathtt{v}\}$. We can suppose $S \neq S'$ and therefore $S' = \mathbb{R}^i \times Q \times]0,1[$ for some $Q \in \mathcal{L}$. Since $\overline{S'} = \mathbb{R}^i \times \mathtt{c}\overline{Q}$ we get $S \subset \overline{S'}$. We also have $\dim S < \dim S'$. ♣

Consider a refinement $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ between two CS-sets, which makes sense following previous Proposition. The identity $I\colon (X,\mathcal{S}) \to (X,\mathcal{T})$ is in fact a stratified map (cf. (11)). We write $(X,\mathcal{S}) \lhd_I (X,\mathcal{T})$.

Simple decompositions and CS-sets are compatible.

Proposition 3.2. *A refinement $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ between two different CS-sets possesses a simple decomposition made up of CS-sets.*

Proof. It suffices to prove that the first element $(X,\mathcal{R})$ of the simple decomposition constructed in the proof of Proposition 2.10 is a CS-set.

We use the following notation: $(X,\mathcal{S}) \lhd_I (X,\mathcal{R}) \lhd_J (X,\mathcal{T})$ and $(X,\mathcal{S}) \lhd_E (X,\mathcal{T})$ the original refinement. We know that the manifolds $X\backslash\Sigma_{\mathcal{S}}$ and $X\backslash\Sigma_{\mathcal{T}}$ are dense open subsets of X. So, $\dim(X,\mathcal{S}) = \dim(X,\mathcal{T})$.

It remains to construct a $\mathcal{R}$-conical chart of any point $x \in \Sigma_{\mathcal{S}}$. We consider the strata $S \in \mathcal{S}$ and $S^I \in \mathcal{R}$ containing x. We distinguish two cases.

+ $\boldsymbol{S \in \mathcal{A}_{\mathcal{S},\mathcal{R}}}$. Let $\varphi\colon (\mathbb{R}^m \times \mathring{c}L, \mathcal{I} \times \mathring{c}\mathcal{L}) \to (V,\mathcal{S})$ be a $\mathcal{S}$-conical chart of x with link $(L,\mathcal{L})$. Since $\dim S = \dim S^I$ then $S \cap V = S^I \cap V = \varphi(\mathbb{R}^m \times \{\mathtt{v}\})$. A stratum of $\mathcal{R}_{V\backslash S^I}$ is a union of strata of $\mathcal{S}_{V\backslash S}$, then it is of the form $\varphi(\mathbb{R}^m \times]0,1[\times \bullet)$. So, there exists a filtration $\mathcal{L}'$ on L such that $\varphi\colon (\mathbb{R}^m \times]0,1[\times L, \mathcal{I} \times \mathcal{I} \times \mathcal{L}') \to (V\backslash S^I, \mathcal{R})$ is a stratified homeomorphism. This is also the case for $\varphi\colon (\mathbb{R}^m \times \mathring{c}L, \mathcal{I} \times \mathring{c}\mathcal{L}') \to (V,\mathcal{R})$. We get that (φ, V) is a $\mathcal{R}$-conical neighborhood of x with link $(L,\mathcal{L}')$.

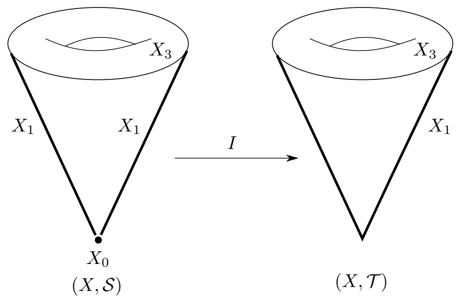
+ $\boldsymbol{S \in \mathcal{V}_{\mathcal{S},\mathcal{R}}}$. Notice first that $\mathcal{V}_{\mathcal{S},\mathcal{R}} = \mathcal{M}_{\mathcal{S},\mathcal{T}}$ (cf. (15)). By construction of $\mathcal{R}$, the stratum of $\mathcal{R}$ containing the point x is S_I (cf. (14)). Let $\varphi\colon (\mathbb{R}^m \times \mathring{c}L, \mathcal{I} \times \mathring{c}\mathcal{L}) \to (V,\mathcal{T})$ be a $\mathcal{T}$-conical chart of x with link $(L,\mathcal{L})$. It suffices to prove that $(V,\mathcal{T}) = (V,\mathcal{R})$.

Since S_I is an open subset of S^I (cf. Lemma 2.9 (a)) we can suppose

$$S_I \cap V = S^I \cap V = \varphi(\mathbb{R}^m \times \{\mathtt{v}\}). \tag{16}$$

By definition of $\mathcal{M}_{\mathcal{S},\mathcal{T}}$ we have that the only virtual $\mathcal{S}$-stratum on V is $V \cap S$. So, there are no virtual $(\mathcal{S},\mathcal{T})$-strata on $V\backslash S^I$. We conclude from Lemma 2.8 (a) that $(V\backslash S^I, \mathcal{S}) = (V\backslash S^I, \mathcal{T})$ and therefore $(V\backslash S^I, \mathcal{S}) = (V\backslash S^I, \mathcal{R})$. Using (16) we get the claim $(V,\mathcal{T}) = (V,\mathcal{R})$. ♣

Remark 3.3. Notice that the coarsening of a CS-set is not necessarily a CS-set. Let us give an example.



On the CS-set $(X, \mathcal{S})$ the link of the strata of X_1 (resp. of the stratum X_0) is S^1 (resp. T^2). This lack of uniformity implies that the coarsening $(X, \mathcal{T})$ is not a CS-set.

We construct a CS-set Y by adding a cone on each boundary of M. This CS-set has two singular points: P_1 and P_2.

The following result describes the construction of compatible conical charts associated to a simple refinement.

Proposition 3.4. *Let $(X, \mathcal{S}) \lhd_I (X, \mathcal{T})$ be a simple refinement between two CS-sets. We consider a point $x \in \Sigma_{\mathcal{S}}$ and we write $S \in \mathcal{S}$ and $S^{I} \in \mathcal{T}$ the strata containing x. We distinguish three cases.*

(a) S is a source stratum. *Then there exists*
- *a $\mathcal{S}$-conical chart (φ, V) of x, whose link is $(L, \mathcal{L})$, and*
- *a $\mathcal{T}$-conical chart (φ, V) of x, whose link is $(L, \mathcal{L}')$ for some filtration $\mathcal{L}'$ on L.*

(b) S is an exceptional stratum. *Let $b = \dim S^{I} - \dim S \geqslant 1$. Then there exists*
- *a $\mathcal{S}$-conical chart (ϕ, W) of x, whose link is $(S^{b-1}, \mathcal{I})$.*

(c) S is a virtual stratum and S^{I} is a singular stratum. *Let $b = \dim S^{I} - \dim S \geqslant 1$. Then there exists*
- *a $\mathcal{T}$-conical chart (ψ, W) of x, whose link is $(E; \mathcal{E})$, and*
- *a $\mathcal{S}$-conical chart (ϕ, W) of x, whose link is $(S^{b-1} * E, \mathcal{E}_{\star b-1})$.*

Proof. The case (a) has been studied in the proof of Proposition 3.2, since $S \in \mathcal{A}$.

We treat the cases (b) and (c), where $\dim S < \dim S^{I}$. We have $S \in \mathcal{V} = \mathcal{M}$, since the decomposition is simple. Notice that depth $\mathcal{M} = 0$. Then we can suppose that $\mathcal{M} = \{S\}$, since (b) and (c) are local questions. In other words, $\mathcal{S} = \{S\} \sqcup \mathcal{A}$. This implies $\mathcal{S} = \mathcal{T}$ on $X \backslash S$ and therefore on $S^{I} \backslash S$. The stratification $\mathcal{S}$ induces on S^{I} the stratification

$$\{S, (S^{I} \backslash S)_{cc}\} \quad \text{with } S \preceq (S^{I} \backslash S)_{cc} \tag{17}$$

(b) Since S is an embedded sub-manifold of S^{I} (cf. (S6)) then there exists a homeomorphism $\phi \colon \mathbb{R}^a \times \mathbb{R}^b = \mathbb{R}^a \times \mathring{c}S^{b-1} \to W$, where $W \subset S^{I}$ is an open neighborhood of x and $\phi(\mathbb{R}^a \times \{\mathtt{v}\}) = S \cap W$, with $a = \dim S$. Do not forget that S^{I} is a regular stratum of $\mathcal{T}$, which implies that W is an open subset of X. From (17), we conclude that $\phi \colon (\mathbb{R}^a \times \mathring{c}S^{b-1}, \mathcal{I} \times \mathring{c}\mathcal{I}) \to (W, \mathcal{T})$ is a stratified homeomorphism an therefore (ϕ, W) is a $\mathcal{T}$-chart of x whose link is $(S^{b-1}, \mathcal{I})$.

(c) Without loss of generality we can suppose that: $(W, \mathcal{T}) = (\mathbb{R}^{a+b} \times \mathring{c}E, \mathcal{I} \times \mathring{c}\mathcal{E})$, $\psi = \mathrm{Id}$, $S^{I} = \mathbb{R}^{a+b} \times \{\mathtt{v}\}$ and $x = (0, \mathtt{v})$. Since S is an embedded sub-manifold of S^{I} (cf. (S6)) then we can suppose $S = \mathbb{R}^a \times \{0\} \times \{\mathtt{v}\}$. From (17) we get that the stratification $\mathcal{S}$ induces the stratification $\{\mathbb{R}^a \times \{0\} \times \{\mathtt{v}\}, \mathbb{R}^a \times (\mathbb{R}^b \backslash \{0\})_{cc} \times \{\mathtt{v}\}\}$ on $S^{I} = \mathbb{R}^{a+b} \times \{\mathtt{v}\}$.

Since all the strata of $(\mathcal{S}, W\backslash S)$ are source strata then $\mathcal{S} = \mathcal{T}$ on $W\backslash S^{^I} = \mathbb{R}^{a+b} \times (\mathring{\mathtt{c}}E\backslash\{\mathtt{v}\})$ (cf. Lemma 2.8 (a)). This gives that $\psi\colon (\mathbb{R}^a \times (\mathbb{R}^b \times \mathring{\mathtt{c}}E), \mathcal{I} \times (\mathcal{I} \times \mathring{\mathtt{c}}\mathcal{E})_{(0,\mathtt{v})}) \to (W, \mathcal{S})$ is a stratified homeomorphism. We consider the homeomorphism $g\colon \mathbb{R}^b \to B^b$ given by $g(x) = 2\arctan(||x||) \cdot x/\pi$. Finally, we define

$$\phi = \psi \circ (\mathrm{Id} \times g^{-1} \times \mathrm{Id}) \circ (\mathrm{Id} \times h)\colon (\mathbb{R}^a \times \mathring{\mathtt{c}}(S^{b-1} * E), \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{E}_{\star b-1}) \to (W, \mathcal{S}),$$

which is a stratified homeomorphism (cf. Proposition 2.3). We get the $\mathcal{S}$-conical chart (ϕ, W) of x whose link is $(S^{b-1} * E, \mathcal{E}_{\star b-1})$. ♣

3.2. **Charts and perversities.** Consider a CS-set $(X, \mathcal{S})$ and a conical chart

$$\varphi\colon (\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}L) \to (V, \mathcal{S})$$

of a point $x \in S$, where $S \in \mathcal{S}^{sing}$. A perversity $\overline{p}$ on $(X, \mathcal{S})$ induces a perversity on the LHS which is described as follows. By restriction, $\overline{p}$ determines a perversity on $(V, \mathcal{S})$ still denoted by $\overline{p}$. We call again $\overline{p}$ the perversity induced on $(\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}L)$ by the stratified homeomorphism φ. A such perversity is determined by a perversity on the link $(L, \mathcal{L})$, also denoted by $\overline{p}$, and by the number $\overline{p}(S)$ following these formulæ:

$$\overline{p}(\underbrace{\mathbb{R}^m \times Q\times]0,1[}_{=_\varphi V\cap R}) = \overline{p}(Q) = \overline{p}(R), \quad \text{and} \quad \overline{p}(\mathbb{R}^m \times \{\mathtt{v}\}) = \overline{p}(\mathtt{v}) = \overline{p}(S), \tag{18}$$

where $\mathtt{v}$ is the apex of $\mathring{\mathtt{c}}L$.

We study the behavior of the perversities concerning the charts of Proposition 3.4. More precisely, if $I\colon (X, \mathcal{S}) \to (X, \mathcal{T})$ is the stratified map induced by the refinement $(X, \mathcal{S}) \lhd (X, \mathcal{T})$ and $\overline{p}$ is a perversity on $(X, \mathcal{S})$ we study the relation between $\overline{p}$ and $I_\star\overline{p}$ under the previous conventions (18) following the three cases presented in Proposition 3.4.

(a) The map $I\colon (V, \mathcal{S}) \to (V, \mathcal{T})$ becomes $(\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathcal{L}) \to (\mathbb{R}^m \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathcal{L}')$, given by $x \mapsto x$, which is a stratified map. Recall that $\varphi(V \cap S) = \mathbb{R}^m \times \{\mathtt{v}\} = \varphi(V \cap S^{^I})$ (cf. (16)). Previous conventions give the equalities

$$\overline{p}(S) = \overline{p}(\mathbb{R}^m \times \{\mathtt{v}\}) = \overline{p}(\mathtt{v}) \ \text{ and } \ I_\star\overline{p}(S^{^I}) = I_\star\overline{p}(\mathbb{R}^m \times \{\mathtt{v}\}) = I_\star\overline{p}(\mathtt{v}) \tag{19}$$

(b) The map $I\colon (W, \mathcal{S}) \to (W, \mathcal{T})$ becomes the stratified map $(\mathbb{R}^a \times \mathring{\mathtt{c}}S^{b-1}, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{I}) \to (\mathbb{R}^{a+b}, \mathcal{I})$ given by $(x, [z, r]) \mapsto (x, g^{-1}(rz))$ where $g(x) = \frac{2\arctan(||x||)}{\pi||x||} \cdot x$. The strata $S, (S^{^I}\backslash S)_{cc} \in \mathcal{S}$ and $S^{^I} \in \mathcal{T}$ become respectively $\mathbb{R}^a \times \{\mathtt{u}\}, \mathbb{R}^a \times \left(S^{b-1}\right)_{cc} \times]0,1[$ and $\mathbb{R}^{a+b}$, where $\mathtt{u}$ is the apex of the cone $\mathring{\mathtt{c}}S^{b-1}$. We have $I_\star\overline{p} = 0$ and previous convention (18) gives the formula

$$\overline{p}(S) = \overline{p}(\mathbb{R}^a \times \{\mathtt{u}\}) = \overline{p}(\mathtt{u}) \tag{20}$$

(c) The map $I\colon (W, \mathcal{S}) \to (W, \mathcal{T})$ becomes the stratified map

$$\psi^{-1} \circ \phi\colon (\mathbb{R}^a \times \mathring{\mathtt{c}}(S^{b-1} * E), \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{E}_{\star b-1}) \to (\mathbb{R}^a \times \mathbb{R}^b \times \mathring{\mathtt{c}}E, \mathcal{I} \times \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{E})$$

given by

$$(x, [[z, y], r]) \mapsto \begin{cases} (x, g^{-1}(2rz), [y, r]) & \text{si } ||z|| \leqslant 1/2 \\ (x, g^{-1}(rz/||z||), [y, 2r(1 - ||z||)]) & \text{si } ||z|| \geqslant 1/2. \end{cases}$$

The strata $S, (S^{^I}\backslash S)_{cc} \in \mathcal{S}$ and $S^{^I} \in \mathcal{T}$ become respectively $\mathbb{R}^a \times \{\mathtt{u}\}, \mathbb{R}^a \times \left(S^{b-1}\right)_{cc} \times]0,1[$ and $\mathbb{R}^a \times \mathbb{R}^b \times \{\mathtt{v}\}$, where $\mathtt{u}$ is the apex of the cone $\mathring{\mathtt{c}}(S^{b-1} * E)$ and $\mathtt{v}$ is the apex of the cone $\mathring{\mathtt{c}}E$.

The other strata oh the LHS are source strata. Previous convention (18) gives the formulæ

$$\begin{array}{rcccl} \overline{p}(R) & = & \overline{p}(\underbrace{\mathbb{R}^a \times D^b \times Q \times]0,1[}_{=_\phi V \cap R}) & = & \overline{p}(Q) \\ I_\star\overline{p}(R^I) & = & I_\star\overline{p}(\underbrace{\mathbb{R}^a \times \mathbb{R}^b \times Q \times]0,1[}_{=_\psi V \cap R^I}) & = & I_\star\overline{p}(Q), \\ \overline{p}((S^I \backslash S)_{cc}) & = & \overline{p}(\mathbb{R}^a \times \left(S^{b-1}\right)_{cc} \times]0,1[) & = & \overline{p}(\left(S^{b-1}\right)_{cc}) \\ \overline{p}(S) & = & \overline{p}(\mathbb{R}^a \times \{\mathtt{u}\}) & = & \overline{p}(\mathtt{u}) \\ I_\star\overline{p}(S^I) & = & I_\star\overline{p}(\mathbb{R}^a \times \mathbb{R}^b \times \{\mathtt{v}\}) & = & I_\star\overline{p}(\mathtt{v}), \end{array} \tag{21}$$

where $R \in \mathcal{R}$ with $R^I \neq S^I$.

3.3. **Comparison tools.** The invariance results we prove in two final sections follow the same pattern: a stratified map induces an isomorphism in (co)homology. To achieve this objective we use these two results. The first one is used with compact supports (cf. [6, Theorem 5.1]) and the second one is used with closed supports (cf. [3, Proposition 13.2]).

Proposition 3.5. *Let $\mathcal{F}_X$ be the category whose objects are (stratified homeomorphic to) open subsets of a given CS set $(X, \mathcal{S})$ and whose morphisms are stratified homeomorphisms and inclusions. Let $\mathcal{A}b_*$ be the category of graded abelian groups. Let F_*, $G_*\colon \mathcal{F}_X \to \mathcal{A}b$ be two functors and $\Phi\colon F_* \to G_*$ a natural transformation satisfying the conditions listed below.*

(a) *F_* and G_* admit exact Mayer-Vietoris sequences and the natural transformation Φ induces a commutative diagram between these sequences,*
(b) *If $\{U_\alpha\}$ is a increasing collection of open subsets of X and $\Phi\colon F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each α, then $\Phi\colon F^*(\cup_\alpha U_\alpha) \to G^*(\cup_\alpha U_\alpha)$ is an isomorphism.*
(c) *Consider (φ, V) a conical chart of a singular point $x \in S$ with $S \in \mathcal{S}$. If $\Phi\colon F^*(V\backslash S) \to G^*(V\backslash S)$ is an isomorphism, then so is $\Phi\colon F^*(V) \to G^*(V)$.*
(d) *If U is an open subset of X contained within a single stratum and homeomorphic to an Euclidean space, then $\Phi\colon F^*(U) \to G^*(U)$ is an isomorphism.*

Then $\Phi\colon F^*(X) \to G^*(X)$ is an isomorphism.

Proposition 3.6. *Let $\mathcal{F}_X$ be the category whose objects are (stratified homeomorphic to) open subsets of a given paracompact second countable[10] CS-set X and whose morphisms are stratified homeomorphisms and inclusions. Let $\mathcal{A}b_*$ be the category of graded abelian groups. Let F^*, $G^*\colon \mathcal{F}_X \to \mathcal{A}b$ be two functors and $\Phi\colon F^* \to G^*$ a natural transformation satisfying the conditions listed below.*

(a) F^* and G^* admit exact Mayer-Vietoris sequences and the natural transformation Φ induces a commutative diagram between these sequences,
(b) If $\{U_\alpha\}$ is a disjoint collection of open subsets of X and $\Phi\colon F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each α, then $\Phi\colon F^*(\bigsqcup_\alpha U_\alpha) \to G^*(\bigsqcup_\alpha U_\alpha)$ is an isomorphism.
(c) Consider (φ, V) a conical chart of a singular point $x \in S$ with $S \in \mathcal{S}$. If $\Phi\colon F^*(V\backslash S) \to G^*(V, \backslash S)$ is an isomorphism, then so is $\Phi\colon F^*(V) \to G^*(V)$.
(d) If U is an open subset of X contained within a single stratum and homeomorphic to an Euclidean space, then $\Phi\colon F^*(U) \to G^*(U)$ is an isomorphism.

Then $\Phi\colon F^*(X) \to G^*(X)$ is an isomorphism.

[10]In the original reference [3, Proposition 13.2] the pseudomanifold X needs to be separable. A second countable space is separable (see for example [25, Theorem 16.9]) so we can change this last hypothesis in the statement of the Proposition.

Remark 3.7. A priori, in order to apply Proposition 3.5 and Proposition 3.6 one needs to verify condition (c) for any conical chart of X. Reading carefully the proof of these Propositions one notices that it is enough to verify (c) for a neighborhood basis of each point x of X.

Associated to a conical chart $\varphi\colon \mathbb{R}^i \times \mathring{c}L \to V$ of the point x, we can construct a neighborhood basis $B_x = \left\{\varphi_\varepsilon \colon]-\varepsilon,\varepsilon[^i \times \mathring{c}_\varepsilon L \to V_\varepsilon \mid \varepsilon > 0\right\}$ of x, where $\mathring{c}_\varepsilon = L \times [0,\varepsilon[/L \times \{0\}$ and $V_\varepsilon = \varphi(]-\varepsilon,\varepsilon[^i \times \mathring{c}_\varepsilon L)$. Notice that all this open subsets are stratified homeomorphic after homotethy.

So, in order to apply Proposition 3.5 and Proposition 3.6 it suffices to verify conditions (c) for a conical chart of each point of X.

Notice that the family $F_x = \left\{\varphi_\varepsilon \colon [-\varepsilon,\varepsilon] \times \mathtt{c}_\varepsilon L \to V_\varepsilon \mid \varepsilon > 0\right\}$ is a 1neighborhood basis for the point x made up of closed subsets. In other words, the space X is locally compact.

3.4. **Morphisms.** Consider a stratified map $f\colon (X,\mathcal{S},\overline{p}) \to (Y,\mathcal{T},\overline{q})$ between two perverse CS-sets. If the perversities verify $f^*D\overline{q} \leqslant D\overline{p}$ then we have the following induced morphisms.

(a) $f_*\colon H_*^{\overline{p}}(X;\mathcal{S}) \to H_*^{\overline{q}}(Y;\mathcal{T})$ and $f^*\colon H^*_{\overline{p}}(X;\mathcal{S}) \to H^*_{\overline{q}}(Y;\mathcal{T})$ (cf. [6, Proposition 3.11]).

(b) $f^*\colon H^*_{\overline{p},c}(X;\mathcal{S}) \to H^*_{\overline{q},c}(Y;\mathcal{T})$ if the map f is a proper map. This comes from 1.5.d and from the fact that the family $\{f^{-1}(K) \mid K \subset Y \text{ compact}\}$ is cofinal in the family of compact subsets of X if f is proper.

(c) $f_*\colon \mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}) \to \mathfrak{H}_*^{\overline{q}}(Y;\mathcal{T})$, if $f(X_{\overline{p}}) \subset \Sigma_{(Y,\mathcal{T})}$ where $X_{\overline{p}} = \sqcup\{\overline{S} \mid S \in \mathcal{S}^{sing} \text{ and } \overline{p}(S) > \overline{t}(S)\}$ (cf. [6, Proposition 3.11]). An adapted version of this result is needed in this work (see Lemma 4.5).

(d) $f_*\colon H_*^{BM,\overline{p}}(X;\mathcal{S}) \to H_*^{BM,\overline{q}}(X,\mathcal{T})$ and $f_*\colon \mathfrak{H}_*^{BM,\overline{p}}(X;\mathcal{S}) \to \mathfrak{H}_*^{BM,\overline{q}}(X,\mathcal{T})$ (cf. 1.5.d).

If the perversities verify $f^*\overline{q} \leqslant \overline{p}$ then we have the following induced morphisms:

(e) $f^*\colon \mathscr{H}^*_{\overline{q}}(Y;\mathcal{T}) \to \mathscr{H}^*_{\overline{p}}(X;\mathcal{S})$ if $f^\star\overline{q} \leqslant \overline{p}$ (cf. [3, Theorem A]).

(f) $f^*\colon \mathscr{H}^*_{\overline{q},c}(Y;\mathcal{T}) \to \mathscr{H}^*_{\overline{p},c}(X;\mathcal{S})$ if the map f is a proper map.This comes from 1.9.d and from the fact that the family $\{f^{-1}(K) \mid K \subset Y \text{ compact}\}$ is cofinal in the family of compact subsets of X if f is proper.

4. Refinement invariance for CS-sets

We prove the main result of this work: the refinement invariance of all the homologies and cohomologies of Section 1 : Theorem A for coarsenings and Theorem B for refinements. In the first case, we need to work with a particular type of perversities, the K-perversities.

4.1. K**-perversities.** These are the perversities for which refinement invariance holds. Roughly speaking, they are M-perversities defined on the LHS of a refinement $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ whose restriction to the strata of the RHS is a classical perversity verifying the growing condition of a Goresky-MacPherson perversity.

Definition 4.1. *Let* $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ *be a refinement. A perversity* $\overline{p}$ *on* $(X,\mathcal{S})$ *is a* K-perversity *if it verifies conditions (K1) and (K2).*

(K1) We have, for any strata $S,Q \in \mathcal{S}$ *with* $S \preceq Q$ *and* $S^I = Q^I$,

$$\overline{p}(Q) \leqslant \overline{p}(S) \leqslant \overline{p}(Q) + \overline{t}(S) - \overline{t}(Q), \tag{22}$$

(K2) We have, for any strata $S,Q \in \mathcal{S}$ *with* $\dim S = \dim Q$ *and* $S^I = Q^I$,

$$\overline{p}(Q) = \overline{p}(S), \tag{23}$$

Remark 4.2. Notice these two conditions are equivalent to conditions

$$D\overline{p}(Q) \leqslant D\overline{p}(S) \leqslant D\overline{p}(Q) + \overline{t}(S) - \overline{t}(Q) \quad \text{and } D\overline{p}(Q) = D\overline{p}(S). \tag{24}$$

Also, condition (22) is always verified when both strata S and Q are regular strata. If the stratum Q is regular and the stratum S is singular (id est, S is an exceptional stratum), then condition (22) becomes

$$0 \leqslant \overline{p}(S) \leqslant \overline{t}(S). \tag{25}$$

In particular, the existence of a K-perversity implies the non-existence of 1-exceptional strata since $0 \leqslant \overline{t}(S) = -1$ is not possible.

Before proving the main results of this work, we need some technical Lemmas.

Lemma 4.3. *Let $(X,\mathcal{S}) \lhd_I (X,\mathcal{T})$ be a refinement. Any K-perversity $\overline{p}$ verifies $I_\star\overline{p}(T) = \overline{p}(S)$ for each $T \in \mathcal{T}$ where $S \in \mathcal{S}$ is a source stratum of T.*

Proof. We know from Section 1.3 that $I_\star\overline{p}(T) = \min\{\overline{p}(Q) \mid Q \in \mathcal{S} \text{ and } Q^I = T\}$. For any $Q \in \mathcal{S}$ with $Q^I = T$ there exists a source stratum $S \in \mathcal{S}$ with $Q \preceq S$ and $S^I = T$ (cf. Lemma 2.8 (b)). So, $I_\star\overline{p}(T) =_{(K1)} \min\{\overline{p}(S) \mid S \in \mathcal{S} \text{ source stratum of } T\}$. Condition (23) ends the proof. ♣

Lemma 4.4. *Let $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ be a refinement. For any K-perversity $\overline{p}$ we have $I^\star DI_\star\overline{p} \leqslant D\overline{p}$.*

Proof. Given a stratum $S \in \mathcal{S}$, there exists a source stratum $Q \in \mathcal{S}$ of S^I verifying $S \preceq Q$ (cf. Lemma 2.8 (b)). We have

$$\begin{aligned} I^\star DI_\star\overline{p}(S) &= DI_\star\overline{p}(S^I) = \overline{t}(S^I) - I_\star\overline{p}(S^I) \overset{source}{=} \overline{t}(Q^I) - \overline{p}(Q) \leqslant_{(1)} \overline{t}(Q) - \overline{p}(Q) \\ &= D\overline{p}(Q) \overset{(K1)}{\leqslant} D\overline{p}(S), \end{aligned}$$

where ${}_{(1)}$ comes from (11) except when Q is an exceptional stratum. In this case $\operatorname{codim} Q \geqslant 2$ and therefore $\overline{t}(Q^I) = 0 \leqslant \overline{t}(Q)$ (cf. Remark 4.2). ♣

Lemma 4.5. *Let $(X,\mathcal{S}) \lhd_I (X,\mathcal{T})$ be a refinement. between two CS-sets. For any K-perversity $\overline{p}$ we have the induced morphisms $I_\star \colon \mathfrak{H}^{\overline{p}}_*(X;\mathcal{S}) \to \mathfrak{H}^{I_\star\overline{p}}_*(X;\mathcal{T})$, $I^\star \colon \mathfrak{H}^*_{I_\star\overline{p}}(X;\mathcal{T}) \to \mathfrak{H}^*_{\overline{p}}(X;\mathcal{S})$, $I^\star \colon \mathfrak{H}^*_{I_\star\overline{p},c}(X;\mathcal{T}) \to \mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S})$.*

Proof. If we prove that the operator $I_\star \colon \mathfrak{C}^{\overline{p}}_*(X;\mathcal{S}) \to \mathfrak{C}^{I_\star\overline{p}}_*(X;\mathcal{T})$ is well defined then, by duality, the operator $I^\star \colon \mathfrak{C}^*_{I_\star\overline{p}}(X;\mathcal{T}) \to \mathfrak{C}^*_{\overline{p}}(X;\mathcal{S})$ is also well defined. Following [6, Proposition 3.11] and Lemma 4.4 it suffices to prove $I(X_{\overline{p}}) \subset \Sigma_{(X,\mathcal{T})}$. If this is not true, then there exist $Q \in \mathcal{S}$ and $S \in \mathcal{S}^{sing}$ with $Q \preceq S$, $\overline{p}(S) > \overline{t}(S)$ and $Q^I \in \mathcal{T}^{reg}$. Since $Q^I \preceq_{(12)} S^I$ then $S^I \in \mathcal{S}^{reg}$. Then S is an exceptional stratum. This is impossible (cf. (25)). Last point comes from 1.5.d. ♣

Lemma 4.6. *Let $(X,\mathcal{S}) \lhd_I (X,\mathcal{R}) \lhd_J (X,\mathcal{T})$ be two refinements. If $\overline{p}$ is a K-perversity on $(X,\mathcal{S})$, relatively to the refinement $E = J \circ I$, then*

(a) *$\overline{p}$ is a K-perversity, relatively to the refinement I, and*
(b) *$I_\star\overline{p}$ is a K-perversity, relatively to the refinement J.*

Proof. Property (a) comes directly from the fact that $S^I = S^J$ implies $S^E = S^{I^J} = Q^{I^J} = Q^E$, if $S, Q \in \mathcal{S}$. Let us prove (b) in two steps.

$(\mathrm{K1})_{I_\star\overline{p}}$ Consider $S, Q \in \mathcal{R}$ with $S \preceq Q$ and $S^J = Q^J$. Lemma 2.8 (c) gives two I-source strata $S', Q' \in \mathcal{S}$, of S and Q respectively, with $S' \preceq Q'$. We have $I_\star\overline{p}(S) = \overline{p}(S')$ and $I_\star\overline{p}(Q) = \overline{p}(Q')$ (cf. Lemma 4.3) and

$$S'^E = S'^{I^J} = S^J = Q^J = Q'^{I^J} = Q'^E \tag{26}$$

then

$$I_\star\overline{p}(Q) \quad = \quad \overline{p}(Q') \overset{(K1)_{\overline{p}}}{\leqslant} \overline{p}(S') = I_\star\overline{p}(S) \overset{(K1)_{\overline{p}}}{\leqslant} \overline{p}(Q') + \overline{t}(S') - \overline{t}(Q')$$
$$\underset{source\ strata}{=} \quad I_\star\overline{p}(Q) + \overline{t}(S'^{I}) - \overline{t}(Q'^{I}) = I_\star\overline{p}(Q) + \overline{t}(S) - \overline{t}(Q).$$

$(K2)_{I_\star\overline{p}}$ Consider two strata $S, Q \in \mathcal{R}$ with $\dim S = \dim Q$ and $S^{J} = Q^{J}$. Lemma 2.8 (b) gives two source strata $S', Q' \in \mathcal{S}$ of S and Q respectively, relatively to the refinement I. Then $S'^{E} = Q'^{E}$ (cf. (26)) and $\dim S' = \dim S'^{I} = \dim S = \dim Q = \dim Q'^{I} = \dim Q'$. Applying $(K2)_{\overline{p}}$ we get $\overline{p}(S') = \overline{p}(Q')$. On the other hand, Lemma 4.3 gives $I_\star\overline{p}(S) = \overline{p}(S')$, $I_\star\overline{p}(Q) = \overline{p}(Q')$ and therefore we get the claim $I_\star\overline{p}(S) = I_\star\overline{p}(Q)$. ♣

4.2. **Main results.** We give the two invariance results of the various intersection (co)homologies: by coarsening and by refinement.

Theorem A (Invariance by coarsening). *Let $(X,\mathcal{S}) \lhd (X,\mathcal{T})$ be a refinement between two CS-sets. For any K-perversity $\overline{p}$ on $(X,\mathcal{S})$ the identity $I\colon X \to X$ induces the isomorphisms*

(R1) $H_*^{\overline{p}}(X;\mathcal{S}) \cong H_*^{I_\star\overline{p}}(X;\mathcal{T})$, (R2) $H^*_{\overline{p}}(X;\mathcal{S}) \cong H^*_{I_\star\overline{p}}(X;\mathcal{T})$,

(R3) $H^*_{\overline{p},c}(X;\mathcal{S}) \cong H^*_{I_\star\overline{p},c}(X;\mathcal{T})$, (R4) $\mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(X;\mathcal{T})$,

(R5) $\mathfrak{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}^*_{I_\star\overline{p}}(X;\mathcal{T})$, (R6) $\mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathfrak{H}^*_{I_\star\overline{p},c}(X;\mathcal{T})$,

If in addition, X is second countable then

(R7) $H_*^{BM,\overline{p}}(X;\mathcal{S}) \cong H_*^{BM,I_\star\overline{p}}(X;\mathcal{T})$, (R8) $\mathfrak{H}_*^{BM,\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}_*^{BM,I_\star\overline{p}}(X;\mathcal{T})$,

(R9) $\mathscr{H}^*_{I_\star\overline{p}}(X;\mathcal{T}) \cong \mathscr{H}^*_{\overline{p}}(X;\mathcal{S})$, (R10) $\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathcal{H}^*_{I_\star\overline{p},c}(X;\mathcal{T})$

Proof. Notice first that the identity I induces the morphisms (R1), . . . , (R10). This comes from Lemma 4.4, 1.3, Paragraph 3.4 and Lemma 4.5. We proceed in several steps.

(R2) and (R5). Apply the Universal coefficient Theorem of 1.5.e to (R1) and (R4).

(R3) and (R6). Considering (4) it suffices to prove that I induces the isomorphisms $H^*_{\overline{p}}(X, X\backslash K;\mathcal{S}) \cong H^*_{I_\star\overline{p}}(X, X\backslash K;\mathcal{T})$ and $\mathfrak{H}^*_{\overline{p}}(X, X\backslash K;\mathcal{S}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(X, X\backslash K;\mathcal{T})$, for each compact subset $K \subset X$. Properties (R2), (R5) and the long exact sequences of 1.5.d give the result.

(R7) and (R8). Since X second countable then it is hemicompact (see [23, Remark 1.3]). Considering (5) it suffices to prove $H_*^{\overline{p}}(X, X\backslash K\mathcal{S}) \cong H_*^{I_\star\overline{p}}(X, X\backslash K;\mathcal{T})$ and $\mathfrak{H}_*^{\overline{p}}(X, X\backslash K;\mathcal{S}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(X, X\backslash K;\mathcal{T})$, where K is a compact subset of X. Properties (R1), (R4) and the long exact sequences of 1.5.d give the result.

(R10). Since X second countable then it is hemicompact, paracompact and therefore normal (see [23, Remark 1.3], [25, Theorem 20.10]). Considering Proposition 1.1 it suffices to prove $\mathscr{H}^*_{\overline{p}}(X, X\backslash K;\mathcal{S}) \cong \mathscr{H}^*_{I_\star\overline{p}}(X, X\backslash K;\mathcal{T})$, where $K \subset X$ is compact. Property (R9) and the long exact sequence of 1.9.d give the result.

(R1), (R4) and (R9). Without loss of generality we can suppose that the refinement is simple (cf. Proposition 3.2 and Lemma 4.6). We verify the conditions of Proposition 3.5, for (R1) and (R4), and Proposition 3.6, for (R9). Th functor Φ comes from $I\colon X \to X$.

(a) It suffices to consider the Mayer-Vietoris sequences of 1.5.a, 1.7.a and 1.9.a[11].

[11]Notice that X is second countable, Hausdorff and locally compact (Remark 3.7). Then, the pseudomanifold X is paracompact (cf. [1, II.12.12]).

(b) The chains have compact support, so we get (R1) and (R4). The case (R9) is immediate.

(d) Since $\mathcal{S}_U = \mathcal{I}$ implies $\mathcal{T}_U = \mathcal{I}$ then property (D) becomes a tautology.

(c) Consider a singular point $x \in X$. Following Remark 3.7 we distinguish three cases.

(C-a) $\boldsymbol{x} \in \boldsymbol{S}$, **source stratum of** $\boldsymbol{\mathcal{S}}$. Considering Proposition 3.4 (a) and using the local calculations 1.5.b and 1.9.b, we need to prove

$$\begin{array}{lll} (R1) & H^{\overline{p}}_{*}(L,\mathcal{L})\cong H^{I_\star\overline{p}}_{*}(L,\mathcal{L}') & \Longrightarrow\ H^{\overline{p}}_{*}(\mathring{c}L,\mathring{c}\mathcal{L})\cong H^{I_\star\overline{p}}_{*}(\mathring{c}L,\mathring{c}\mathcal{L}') \\ (R4) & \mathfrak{H}^{\overline{p}}_{*}(L,\mathcal{L})\cong \mathfrak{H}^{I_\star\overline{p}}_{*}(L,\mathcal{L}') & \Longrightarrow\ \mathfrak{H}^{\overline{p}}_{*}(\mathring{c}L,\mathring{c}\mathcal{L})\cong \mathfrak{H}^{I_\star\overline{p}}_{*}(\mathring{c}L,\mathring{c}\mathcal{L}') \\ (R9) & \mathscr{H}^{\overline{p}}_{*}(L,\mathcal{L})\cong \mathscr{H}^{I_\star\overline{p}}_{*}(L,\mathcal{L}') & \Longrightarrow\ \mathscr{H}^{\overline{p}}_{*}(\mathring{c}L,\mathring{c}\mathcal{L})\cong \mathscr{H}^{I_\star\overline{p}}_{*}(\mathring{c}L,\mathring{c}\mathcal{L}'). \end{array}$$

Since the perversity $\overline{p}$ verifies $\overline{p}(S) = I_\star\overline{p}(S^I)$ (cf. Lemma 4.3) then we have $\overline{p}(\mathtt{v}) = I_\star\overline{p}(\mathtt{v})$ (cf. (19)). The result comes now directly from the local calculations 1.5.b and 1.9.b.

(C-b) $\boldsymbol{x} \in \boldsymbol{S}$, **exceptional stratum of** $\boldsymbol{\mathcal{S}}$. Considering Proposition 3.4 (b) and using the local calculations 1.5.b and 1.9.b, we need to prove

$$(R1)\ H^{\overline{p}}_{*}\big(\mathring{c}S^{b-1},\mathring{c}\mathcal{I}\big)\cong G, \qquad (R4)\ \mathfrak{H}^{\overline{p}}_{*}\big(\mathring{c}S^{b-1},\mathring{c}\mathcal{I}\big)\cong G, \qquad (R9)\ \mathscr{H}^{*}_{\overline{p}}\big(\mathring{c}S^{b-1},\mathring{c}\mathcal{I}\big)\cong R.$$

where $b = \operatorname{codim} S \geqslant 1$. Since $0 \leqslant \overline{p}(S) \leqslant \overline{t}(S) = b-2$ (cf. (25)) then we have $0 \leqslant \overline{p}(\mathtt{u}) \leqslant b-2$ (cf. (20)). The result comes now directly from the local calculations 1.5.b and 1.9.b.

(C-c) $\boldsymbol{x} \in \boldsymbol{S}$, **virtual stratum, with** $\boldsymbol{S^I}$ **singular stratum of** $\boldsymbol{\mathcal{S}}$. Considering Proposition 3.4 (c) and using the local calculations 1.5.b and 1.9.b, we need to prove

$$\begin{array}{lll} (R1) & H^{\overline{p}}_{*}\big(\mathring{c}(S^{b-1} * E),\mathring{c}\mathcal{E}_{\star b-1}\big) & \cong\ H^{I_\star\overline{p}}_{*}(\mathring{c}E,\mathring{c}\mathcal{E}) \\ (R4) & \mathfrak{H}^{\overline{p}}_{*}\big(\mathring{c}(S^{b-1} * E),\mathring{c}\mathcal{E}_{\star b-1}\big) & \cong\ \mathfrak{H}^{I_\star\overline{p}}_{*}(\mathring{c}E,\mathring{c}\mathcal{E}) \\ (R9) & \mathscr{H}^{*}_{\overline{p}}\big(\mathring{c}(S^{b-1} * E),\mathring{c}\mathcal{E}_{\star b-1}\big) & \cong\ \mathscr{H}^{*}_{I_\star\overline{p}}(\mathring{c}E,\mathring{c}\mathcal{E}), \end{array}$$

where $b = \dim S^I - \dim S \geqslant 1$.

Since $S \preceq (S^I\backslash S)_{cc}$ (cf. (17)) and $\overline{p}((S^I\backslash S)_{cc}) \leqslant \overline{p}(S) \leqslant \overline{p}((S^I\backslash S)_{cc}) + b$ (cf. (22)) then we have $\overline{p}((S^{b-1})_{cc}) \leqslant \overline{p}(\mathtt{u}) \leqslant \overline{p}((S^{b-1})_{cc}) + b$ (cf. (21)). Similarly, we get $D\overline{p}((S^{b-1})_{cc}) \leqslant D\overline{p}(\mathtt{u}) \leqslant D\overline{p}((S^{b-1})_{cc}) + b$ (cf. (24)).

Since $\overline{p}(S) = I_\star\overline{p}(S^I)$ (cf. Lemma 4.3) then we have $\overline{p}(\mathtt{v}) = I_\star\overline{p}(\mathtt{v})$ (cf. (21)).

Applying the local calculations 1.5.b,c and 1.9.b,c the question becomes

$$(R1)\ H^{\overline{p}}_{*}(E,\mathcal{E})\cong H^{I_\star\overline{p}}_{*}(E,\mathcal{E}), \qquad (R4)\ \mathfrak{H}^{\overline{p}}_{*}(E,\mathcal{E})\cong \mathfrak{H}^{I_\star\overline{p}}_{*}(E,\mathcal{E}) \qquad (R9)\ \mathscr{H}^{*}_{\overline{p}}(E,\mathcal{E})\cong \mathscr{H}^{*}_{I_\star\overline{p}}(E,\mathcal{E}),$$

The stratum S belongs to $\mathcal{V} = \mathcal{M}$ (cf. (15)). Since any other $R \in \mathcal{S}$ meeting the conical chart W verifies $S \prec R$ then R is a source stratum and then $\overline{p}(R) = I_\star\overline{p}(R)$ (cf. Lemma 4.3). From (21) we get $\overline{p} = I_\star\overline{p}$ on E. The claim is proved. ♣

Remark 4.7. The existence of 1-exceptional strata may impeach the above isomorphisms. This is the case for (R4), … (R10). For example $\mathfrak{H}^{\overline{0}}_{*}\big(\mathring{c}S^0,\mathring{c}\mathcal{I}\big) = 0 \neq G = \mathfrak{H}^{\overline{0}}_{*}(]-1,1[,\mathcal{I})$. But we have $H^{\overline{0}}_{*}\big(\mathring{c}S^0\big) = G = H^{\overline{0}}_{*}(]-1,1[,\mathcal{I},\mathring{c}\mathcal{I})$. In fact, the local calculations $H^{\overline{p}}_{0}\big(\mathring{c}S^0,\mathcal{S}\big)$ and $\mathfrak{H}^{\overline{p}}_{0}\big(\mathring{c}S^0;\mathcal{S}\big)$ are different:

$$H^{\overline{p}}_{0}\big(\mathring{c}S^0,\mathring{c}\mathcal{I}\big) = \begin{cases} H_0(S^0) & \text{if } D\overline{p}(\mathtt{v}) \geqslant 0 \\ G & \text{if } D\overline{p}(\mathtt{v}) < 0 \end{cases} \qquad \mathfrak{H}^{\overline{p}}_{0}\big(\mathring{c}S^0,\mathring{c}\mathcal{I}\big) = \begin{cases} H_0(S^0) & \text{if } D\overline{p}(\mathtt{v}) \geqslant 0 \\ 0 & \text{if } D\overline{p}(\mathtt{v}) < 0. \end{cases}$$

We observe that condition $\mathfrak{H}^{\overline{p}}_{*}\big(\mathring{c}S^0,\mathring{c}\mathcal{I}\big)\cong G$ of (C-c) is never fulfilled, while we just need $D\overline{p}(\mathtt{v}) < 0$ to have $H^{\overline{p}}_{*}\big(\mathring{c}S^0,\mathring{c}\mathcal{I}\big)\cong G$ of (C-c).

Condition (22) can be weakened in cases (R1), (R2) and (R3) as follows: dealing with 1-exceptional strata S, it suffices to ask $D\overline{p}(S) < 0$, that is, $\overline{p}(S) \geqslant 0$ and not $0 \leqslant \overline{p}(S) \leqslant \overline{t}(S)$. So, these strata are allowed for (R1), (R2) and (R3).

Theorem B (Invariance by refinement). *Let $(X, \mathcal{S}) \lhd (X, \mathcal{T})$ be a refinement between two CS-sets. We suppose that there are no 1-exceptional strata. For any perversity $\overline{q}$ on $(X, \mathcal{T})$ the identity $I \colon X \to X$ induces the isomorphisms*

$$\text{(R1)} \quad H_*^{I^\star\overline{q}}(X; \mathcal{S}) \cong H_*^{\overline{q}}(X; \mathcal{T}), \qquad \text{(R2)} \quad H^*_{I^\star\overline{q}}(X; \mathcal{S}) \cong H^*_{\overline{q}}(X; \mathcal{T}),$$

$$\text{(R3)} \quad H^*_{I^\star\overline{q},c}(X; \mathcal{S}) \cong H^*_{\overline{q},c}(X; \mathcal{T}), \qquad \text{(R4)} \quad \mathfrak{H}_*^{I^\star\overline{q}}(X; \mathcal{S}) \cong \mathfrak{H}_*^{\overline{q}}(X; \mathcal{T}),$$

$$\text{(R5)} \quad \mathfrak{H}^*_{I^\star\overline{q}}(X; \mathcal{S}) \cong \mathfrak{H}^*_{\overline{q}}(X; \mathcal{T}), \qquad \text{(R6)} \quad \mathfrak{H}^*_{I^\star\overline{q},c}(X; \mathcal{S}) \cong \mathfrak{H}^*_{\overline{q},c}(X; \mathcal{T}).$$

If in addition, X is second countable then

$$\text{(R7)} \quad H_*^{BM,I^\star\overline{q}}(X; \mathcal{S}) \cong H_*^{BM,\overline{q}}(X; \mathcal{T}), \qquad \text{(R8)} \quad \mathfrak{H}_*^{BM,I^\star\overline{q}}(X; \mathcal{S}) \cong \mathfrak{H}_*^{BM,\overline{q}}(X; \mathcal{T}),$$

$$\text{(R9)} \quad \mathscr{H}^*_{I^\star\overline{q}}(X; \mathcal{T}) \cong \mathscr{H}^*_{\overline{q}}(X; \mathcal{S}), \qquad \text{(R10)} \quad \mathscr{H}^*_{I^\star\overline{q},c}(X; \mathcal{S}) \cong \mathcal{H}^*_{\overline{q},c}(X; \mathcal{T})$$

Proof. It suffices to apply apply Theorem A to the perversity $I^\star\overline{p}$ (cf. Paragraph 1.3), if this perversity is a K-perversity. This is the case when 1-codimensional exceptional strata do not appear. Let us verify properties (K1) and (K2).

(K1) We have $I^\star\overline{q}(Q) = \overline{q}(Q^I) = \overline{q}(S^I) = I^\star\overline{q}(S) \leqslant I^\star\overline{q}(Q) + \overline{t}(S) - \overline{t}(Q)$, if we prove $\overline{t}(Q) \leqslant \overline{t}(S)$. This is clear if S and Q are regular strata or singular strata at the same time (cf. (S4)). It remains the case where S is an exceptional stratum and Q is a regular stratum. The inequality becomes $\overline{t}(S) \geqslant 0$, that is, $\operatorname{codim} S \geqslant 2$. This comes from the non-existence of 1-exceptional strata.

(K2) We have $I^\star\overline{q}(Q) = \overline{q}(Q^I) = \overline{q}(S^I) = I^\star\overline{q}(S)$. ♣

In cases (R1), (R2) and (R3), 1-exceptional strata S may appear if $\overline{p}(S) \geqslant 0$ (cf. Remark 4.7).

4.3. **Topological invariance.** One of the two more important properties of the intersection homology is the topological invariance [15]. Next Corollaries show that the refinement invariance implies topological invariance in some cases. We find the well known topological invariance of the intersection homology [15] (see also [19, 13]) and those of tame intersection homology [9] (closed supports) and [11] (compact supports). We also get the topological invariance of the blown-up intersection cohomology [3, Theorem G] (closed supports) and [5, Theorem A] (compact supports).

Before giving the result, there are two important tools to highlight.

• *Intrinsic stratification* (cf. [18, 19]). Any stratified space $(X, \mathcal{S})$ has a smallest refinement: the *intrinsic stratified space* $(X, \mathcal{S}^*)$. It is a canonical object: we have $\mathcal{S}^* = \mathcal{T}^*$ for any stratification $\mathcal{T}$ defined on X. If $(X, \mathcal{S})$ is a CS-set then $(X, \mathcal{S}^*)$ is also a CS-set.

• *Classical perversities versus M-perversities.* The former depend on the codimension of the strata while the latter are defined on the strata themselves.

A *King perversity* is a map $\overline{p} \colon \mathbb{N} \to \mathbb{Z}$ verifying $\overline{p}(0) = 0$ and $\overline{p}(k) \leqslant \overline{p}(k+1) \leqslant \overline{p}(k) + 1$ for each $k \in \mathbb{N}^*$ (cf. [19]). It verifies

$$\overline{p}(k) \leqslant \overline{p}(\ell) \leqslant \overline{p}(k) + \ell - k, \tag{27}$$

if $1 \leqslant k \leqslant \ell$. A King perversity $\overline{p}$ induces a perversity, still denoted by $\overline{p}$: $\overline{p}(S) = \overline{p}(\operatorname{codim} S)$.

A *Goresky-MacPherson perversity* is a King perversity $\overline{p}$ with $\overline{p}(0) = \overline{p}(1) = \overline{p}(2) = 0$ (cf. [15]). It verifies, for each $k \geqslant 2$,

$$\overline{0} \leqslant \overline{p}(k) \leqslant k-2 = \overline{t}(k) \tag{28}$$

Corollary 4.8. *Let $(X,\mathcal{S})$ be a CS-set endowed with a positive King perversity $\overline{p}$. Consider the intrinsic refinement $(X,\mathcal{S}) \lhd_I (X,\mathcal{S}^*)$. The identity map $I\colon X \to X$ induces the isomorphisms*

$$H_*^{\overline{p}}(X;\mathcal{S}) \cong H_*^{\overline{p}}(X;\mathcal{S}^*) \qquad H^*_{\overline{p}}(X;\mathcal{S}) \cong H^*_{\overline{p}}(X;\mathcal{S}^*) \qquad H^*_{\overline{p},c}(X;\mathcal{S}) \cong H^*_{\overline{p},c}(X;\mathcal{S}^*),$$

if $\overline{p}(\ell) \geqslant 0$ when ℓ is the codimension of an exceptional stratum. We also have

$$\mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}^*) \qquad \mathfrak{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}^*_{\overline{p}}(X;\mathcal{S}^*) \qquad \mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}^*),$$

if $0 \leqslant \overline{p}(\ell) \leqslant \overline{t}(\ell)$. If in addition, X is second countable the we have

$$\mathscr{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathscr{H}^*_{\overline{p}}(X;\mathcal{S}^*) \qquad \mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}^*) \qquad H_*^{BM,\overline{p}}(X;\mathcal{S}) \cong H_*^{BM,\overline{p}}(X;\mathcal{S}^*)$$

Proof. Let us verify that $\overline{p}$ is a K-perversity.

(K1) By definition of the perversity $\overline{p}$, we need to prove

$$\overline{p}(\operatorname{codim} Q) \leqslant \overline{p}(\operatorname{codim} S) \leqslant \overline{p}(\operatorname{codim} Q) + \overline{t}(\operatorname{codim} S) - \overline{t}(\operatorname{codim} Q).$$

This is clear if S, Q are regular strata or singular strata (cf. (S4) and (27)). It remains the case where S is an exceptional stratum and Q is a regular stratum. The inequality becomes $0 \leqslant \overline{p}(\operatorname{codim} S) \leqslant \overline{t}(\operatorname{codim} S)$ which is true from hypothesis and Remark 4.7.

(K2) We have $\overline{p}(S) = \overline{p}(\operatorname{codim} S) = \overline{p}(\operatorname{codim} Q) = \overline{p}(Q)$.

The classical perversity $\overline{p}$ induces the perversity $\overline{p}$ on $(X,\mathcal{S})$ by formula $\overline{p}(S) = \overline{p}(\operatorname{codim} S)$. In fact, the perversity $I_\star\overline{p}$ of $(X,\mathcal{T})$ also comes from the classical perversity $\overline{p}$: $I_\star\overline{p}(T) \overset{Lemma\ 4.3}{=} \overline{p}(S) = \overline{p}(\operatorname{codim} S) \overset{source}{=} \overline{p}(\operatorname{codim} T) = \overline{p}(T)$, where $T \in \mathcal{T}$ and $S \in \mathcal{S}$ is source stratum of T. Now, it suffices to apply Theorem A. ♣

Remark 4.9. (1) - Let $(X,\mathcal{S})$ be a CS-set endowed with a Goresky-MacPherson perversity $\overline{p}$. Since $\overline{p} \geqslant \overline{0}$ (cf. (28)), then the previous Corollary implies that the cohomologies $H_*^{\overline{p}}(X;\mathcal{S})$, $H^*_{\overline{p}}(X;\mathcal{S})$ and $H^*_{\overline{p},c}(X;\mathcal{S})$ are independent of the stratification $\mathcal{S}$. We do not have a similar result for tame intersection homologies since condition $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ (cf. (28)) implies that tame intersection homology coincides with the usual intersection homology.

Let us suppose that X is second countable. When 1-exceptional strata do not exist then we can apply the above Corollary and conclude that the cohomologies $\mathscr{H}^*_{\overline{p}}(X;\mathcal{S})$, $\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S})$ and $H_*^{BM,\overline{p}}(X;\mathcal{S})$ are independent of the stratification $\mathcal{S}$. (cf. (28)).

(2) - Consider $\overline{p}$ a K-perversity. Condition (K2) means that the restriction of $\overline{p}$ to the $\mathcal{S}$-stratification lying on each stratum $T \in \mathcal{T}$ is in fact a classical perversity (excepted the condition $\overline{p}(0) = 0$). On the other hand, property (K1) is in fact a growing condition of the type (27), even weaker. Although it is not completely exact, we can think a K-perversity as a perversity whose restriction to any stratum $T \in \mathcal{T}$ is a King perversity.

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