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Abstract. *We construct an analytic self-map* Φ *of the bidisk* $\mathbb{D}^2$ *whose image touches the distinguished boundary, but whose approximation numbers of the associated composition operator on* $H^2(\mathbb{D}^2)$ *are small in the sense that* $\limsup_{n\to\infty}[a_{n^2}(C_\Phi)]^{1/n} < 1$.

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1 Introduction

For composition operators $C_\Phi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ on the Hardy space of the unit disk, the decay of their approximation numbers $a_n(C_\Phi)$ cannot be arbitrarily fast, and actually cannot supersede a geometric speed ([16]; see also [10, Theorem 3.1]): there exists a positive constant c such that:

$$a_n(C_\Phi) \gtrsim \mathrm{e}^{-cn}\,, \qquad n = 1, 2, \ldots$$

It is easy to see that this speed occurs when $\|\Phi\|_\infty < 1$, and we proved in [10, Theorem 3.4] that a geometrical speed only takes place in this case; in other words:

$$\|\Phi\|_\infty = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} [a_n(C_\Phi)]^{1/n} = 1\,. \tag{1.1}$$

This leads to the introduction, for an operator T between Banach spaces, of the parameters:

$$\beta^-(T) = \liminf_{n\to\infty}\, [a_n(T)]^{1/n} \quad \text{and} \quad \beta^+(T) = \limsup_{n\to\infty}\, [a_n(T)]^{1/n}\,, \tag{1.2}$$

where $a_n(T)$ is the n-th approximation number of T. When $[a_n(T)]^{1/n}$ actually has a limit, i.e. when $\beta^-(T) = \beta^+(T)$, we write it $\beta(T)$.

What is proved in [10, Theorem 3.4] is that $\beta(C_\Phi) = 1$ if and only if $\|\Phi\|_\infty = 1$. Later, in [12], we gave, when $\|\Phi\|_\infty < 1$, a formula for this parameter in terms of the Green capacity of $\Phi(\mathbb{D})$, which allowed us to recover (1.1).

More generally, for $N \geq 1$, we introduce:

$$\beta_N^-(T) = \liminf_{n\to\infty} [a_{n^N}(T)]^{1/n} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n\to\infty} [a_{n^N}(T)]^{1/n}\,, \tag{1.3}$$

and:

$$\beta_N(T) = \lim_{n\to\infty} [a_{n^N}(T)]^{1/n} \tag{1.4}$$

when the limit exists. It is clear that $0 \leq \beta_N^\pm(T) \leq 1$, and it is interesting to know when the extreme cases $\beta_N^\pm(T) = 0$ or $\beta_N^\pm(T) = 1$ occur. For example:

$$\begin{aligned} \beta_N^-(T) > 0 \quad &\Longleftrightarrow \quad a_{n^N}(T) \gtrsim \mathrm{e}^{-\tau n}\,, \quad \text{with } \tau > 0 \\ \beta_N^-(T) = 1 \quad &\Longleftrightarrow \quad a_{n^N}(T) \gtrsim \mathrm{e}^{-n\varepsilon_n}\,, \quad \text{with } \varepsilon_n \to 0\,. \end{aligned}$$

It is coined in [1] (see also [13] and [14]) that $\beta_N^\pm(C_\Phi)$ are the suitable parameters for the composition operators on $H^2(\mathbb{D}^N)$, and it is proved, for any $N \geq 1$, that $\beta_N^-(C_\Phi) > 0$, as soon as Φ is non degenerate (i.e. the Jacobian J_Φ is not identically 0) and the operator C_Φ is bounded on $H^2(\mathbb{D}^N)$. As for an expression of $\beta_N^\pm(C_\Phi)$ in terms of "capacity", only partial results are known so far ([13] and [14]) and the application to a result like (1.1) fails in general. We gave an example of such a phenomenon in [13, Theorem 5.12]. In the present paper we give a shaper result.

2 Background and notation

Let $\mathbb{D}$ be the open unit disk, $H^2(\mathbb{D}^N)$ the Hardy space of the polydisk $\mathbb{D}^N$, and $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ an analytic map. When $N = 1$, it is well-known (see [4] or [17]) that Φ induces a composition operator $C_\Phi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ by the formula:

$$C_\Phi(f) = f \circ \Phi\,,$$

and the connection between the "symbol" Φ and the properties of the operator C_Φ, in particular its compactness, can be further studied (see [4] or [17]). When $N > 1$, C_Φ is not bounded in general (see [4]).

Let $\mathbb{T}$ be the unit circle, and m the normalized Haar measure on $\mathbb{T}^N$. A positive Borel measure μ on $\mathbb{D}^N$ is called a Carleson measure (for the space $H^2(\mathbb{D}^N)$) if the canonical injection $J\colon H^2(\mathbb{D}^N) \to L^2(\mu)$ is bounded. When $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ is analytic and induces a bounded composition operator on $H^2(\mathbb{D}^N)$, the pullback measure $m_\Phi = \Phi^*(m)$, defined, for any test function u, by:

$$\int_{\mathbb{D}^N} u(w)\, dm_\Phi(w) = \int_{\mathbb{T}^N} u[\Phi^*(\xi)]\, dm(\xi)\,,$$

is a Carleson measure. Here Φ^* is the radial limit function, defined for m-almost every $\xi \in \mathbb{T}^N$, by $\Phi^*(\xi) = \lim_{r\to 1^-} \Phi(r\xi)$.

For $\xi \in \mathbb{T} = \partial\mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ is defined as:

$$S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| \leq h\}\,. \tag{2.1}$$

If $f \in \mathrm{Hol}\,(\mathbb{D}^2)$, $D_j^k f$ denotes the k-th derivative of f with respect to the j-th variable ($j = 1, 2$).

We denote by $A(\mathbb{D})$ the disk algebra, i.e. the space of functions holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We similarly define the bidisk algebra $A(\mathbb{D}^2)$.

Let H_1 and H_2 be Hilbert spaces, and $T\colon H_1 \to H_2$ an operator. The n-th approximation number $a_n(T)$ of T, $n = 1, 2, \ldots$, is defined (see [2]) as the distance (for the operator-norm) of T to operators of rank $< n$:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\|\,. \tag{2.2}$$

The approximation numbers have the ideal property:

$$a_n(ATB) \leq \|A\|\, a_n(T)\, \|B\|\,.$$

The n-th Gelfand number $c_n(T)$ of T is defined by:

$$c_n(T) = \inf_{\operatorname{codim} E < n} \|T_{|E}\|\,. \tag{2.3}$$

As an easy consequence of the Schmidt decomposition, we have for any compact operator between Hilbert spaces:

$$c_n(T) = a_n(T)\,. \tag{2.4}$$

If $T, T_1, T_2\colon H \to H'$ are operators between Hilbert spaces H and H', we write $T = T_1 \oplus T_2$ if $T = T_1 + T_2$ and:

$$\|Tx\|^2 = \|T_1 x\|^2 + \|T_2 x\|^2\,, \quad \text{for all } x \in H\,.$$

The subaddivity of approximation numbers is then expressed by:

$$a_{j+k}(T_1 \oplus T_2) \leq a_j(T_1) + a_k(T_2)\,. \tag{2.5}$$

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers, and by $[x]$ the integral part of the real number x.

We write $X \lesssim Y$ to indicate that $X \leq c\,Y$ for some constant $c > 0$, and $X \approx Y$ to indicate that $X \lesssim Y$ and $Y \lesssim X$.

3 Purpose of the paper

Let us recall that the Hardy space of the polydisk is the space:

$$H^2(\mathbb{D}^N) = \Big\{ f\colon \mathbb{D}^N \to \mathbb{C}\,;\ f(z) = \sum_{\alpha \in \mathbb{N}^N} a_\alpha z^\alpha \text{ and } \|f\|_2^2 := \sum |a_\alpha|^2 < \infty \Big\}\,.$$

If $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ is an analytic map, the associated composition operator C_Φ (which is not always bounded on $H^2(\mathbb{D}^N)$) is defined by:

$$C_\Phi(f) = f \circ \Phi\,.$$

We will mainly here be interested in the case $N = 2$.

The reproducing kernel K_a of $H^2(\mathbb{D}^2)$ is, with $a = (a_1, a_2)$ and $z = (z_1, z_2)$:

$$K_a(z) = \frac{1}{(1 - \overline{a_1} z_1)(1 - \overline{a_2} z_2)} \,\cdot \tag{3.1}$$

As a consequence:

$$|f(a)| = |\langle f, K_a \rangle| \leq \frac{\|f\|_2}{\sqrt{(1 - |a_1|^2)(1 - |a_2|^2)}} \,\cdot \tag{3.2}$$

In particular, the functions in the unit ball of $H^2(\mathbb{D}^2)$ are uniformly bounded on compact subsets of $\mathbb{D}^2$.

In [13, Theorem 5.12], we gave an example of a holomorphic self-map $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$, continuous on the closure $\overline{\mathbb{D}^2}$, such that $\|\Phi\|_\infty = 1$, that is:

$$\Phi(\mathbb{T}^2) \cap \partial\mathbb{D}^2 \neq \emptyset\,, \tag{3.3}$$

and yet:

$$\beta_2^+(C_\Phi) < 1\,, \tag{3.4}$$

in contrast with the one-dimensional case ([10, Theorem 3.4]).

Understanding where the difference really lies when we pass to the multidimensional case is a big challenge: it does not seem to be a matter of regularity of the boundary, and a similar example probably holds for the Hardy space of the ball. It might be a matter of boundary: the Shilov boundary of the ball is its usual boundary, but that of the polydisk is its distinguished boundary:

$$\partial_e \mathbb{D}^N = \{z = (z_j)\,;\ |z_j| = 1 \text{ for all } j = 1, \dots, N\} = \mathbb{T}^N$$

(indeed, the distinguished maximum principle tells that, for f analytic in $\mathbb{D}^N$ and continuous on $\overline{\mathbb{D}^N}$, it holds $\max_{z\in\mathbb{D}^N} |f(z)| = \max_{z\in\partial_e\mathbb{D}^N} |f(z)|$). The aim of this paper is to show that this is not the case and, improving on ([13, Theorem 5.12]) and (3.3), to build an analytic self-map $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$, continuous on $\overline{\mathbb{D}^2}$, non-degenerate and such that:

$$\Phi(\mathbb{T}^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset\,, \quad \text{but} \quad \beta_2^+(C_\Phi) < 1\,. \tag{3.5}$$

The paper is organized as follows. In Section 4, we recall with some detail the definition and main properties of a so-called *cusp map* $\chi \in A(\mathbb{D})$, to be of essential use in our counterexample. In Section 5, we prove several lemmas which constitute the core or the proof. In Section 6, we state and prove our main theorem.

4 The cusp map

The *cusp map* $\chi\colon \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and extends continuously on $\overline{\mathbb{D}}$. The boundary of its image is formed by three circular arcs of respective centers $\frac{1}{2}$, $1 + \frac{i}{2}$, $1 - \frac{i}{2}$, and of radius $\frac{1}{2}$ (see Figure 1). However, the parametrization $t \mapsto \chi(\mathrm{e}^{it})$ involves logarithms.

It was often used by the authors ([11], [8]) as an extremal example.

We first recall the definition of χ.

Let $\mathbb{D}^+ = \{z \in \mathbb{D}\,;\ \mathfrak{Re}\, z > 0\}$ be the right half-disk. Let now $\mathbb{H}$ be the upper half-plane, and $T\colon \mathbb{D} \to \mathbb{H}$ defined by:

$$T(u) = i\,\frac{1+u}{1-u}\,, \quad \text{with} \quad T^{-1}(s) = \frac{s-i}{s+i}\cdot$$

Taking the square root of T, we map $\mathbb{D}$ onto the first quadrant defined by $Q_1 = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}$; we go back to the half-disk $\{z \in \mathbb{D}\,;\ \mathfrak{Im}\, z < 0\}$ by T^{-1}. Finally, make a rotation by i to go onto $\mathbb{D}^+$. We get:

$$\chi_0(z) = \frac{\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} - i}{-i\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} + 1}\cdot$$

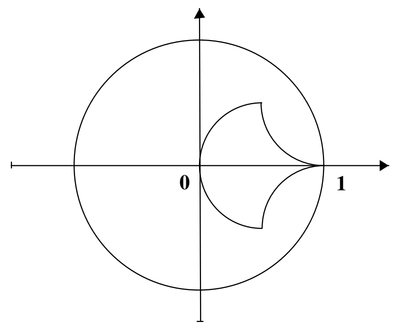

Figure 1: *Cusp map domain*

One has $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, and $\chi_0(-i) = i$. The half-circle $\{z \in \mathbb{T}\,;\ \mathfrak{Re}\, z \geq 0\}$ is mapped by χ_0 onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively:

$$\chi_1(z) = \log \chi_0(z)\,, \quad \chi_2(z) = -\frac{2}{\pi}\,\chi_1(z) + 1\,, \quad \chi_3(z) = \frac{1}{\chi_2(z)} \tag{4.1}$$

and finally:

$$\chi(z) = 1 - \chi_3(z)\,. \tag{4.2}$$

We now summarize the properties of the cusp map χ in the following proposition.

Proposition 4.1. *The cusp map satisfies:*

1) $1 - |\chi(z)| \lesssim \dfrac{1}{\log(2/|1-z|)}$*;*
2) $|1-\chi(z)| \leq K(1-|\chi(z)|)$ *for all* $z \in \mathbb{D}$*, where* K *is a positive constant;*
3) $\chi(\mathbb{D})$ *is the intersection of the open disk* $D\big(\frac{1}{2}, \frac{1}{2}\big)$ *with the exterior of the two open disks* $D\big(1 + \frac{i}{2}, \frac{1}{2}\big)$ *and* $D\big(1 - \frac{i}{2}, \frac{1}{2}\big)$*;*
4) $\chi(1) = 1$*,* $\chi(\overline{z}) = \overline{\chi(z)}$ *and* $|\chi(z) - 1| \leq 1$ *for all* $z \in \mathbb{D}$*;*
5) *for* $0 < |t| \leq \pi/4$*, we have* $1 - \mathfrak{Re}\,\chi(\mathrm{e}^{it}) \approx 1/(\log 1/|t|)$*;*
6) $\chi(\overline{\mathbb{D}}) \subseteq \{z = x + iy\,;\ 0 \leq x \leq 1 \text{ and } |y| \leq 2(1-x)^2\}$.

Proof. Items 1) to 5) are proved in [11, Lemma 4.2]. To prove 6), write $\chi(z) = (1-h) + iy$. Since $\chi(\overline{z}) = \overline{\chi(z)}$, we can assume $y \geq 0$. Since $\chi(\mathbb{D}) \cap D\big(1+\frac{i}{2}, \frac{1}{2}\big) = \emptyset$, we have $\big|\chi(z) - \big(1+\frac{i}{2}\big)\big| \geq \frac{1}{2}$; hence:

$$h^2 + \Big(y - \frac{1}{2}\Big)^2 = \Big|\chi(z) - \Big(1 + \frac{i}{2}\Big)\Big|^2 \geq \frac{1}{4}\,,$$

so that $y \leq y^2 + h^2$. But $y \leq 1/2$, since $\chi(z) \in D\big(\frac{1}{2}, \frac{1}{2}\big)$; therefore $y^2 \leq y/2$, so we get $y \leq 2h^2$. □

5 Preliminary lemmas

In this section, we collect some lemmas, which will reveal essential in the proof of our counterexample.

We consider the map $\varphi = \varphi_\theta$, $0 < \theta < 1$, defined, for $z \in \overline{\mathbb{D}} \setminus \{1\}$, by:

$$\varphi\,(z) = \exp\big(-(1-z)^{-\theta}\big)\,. \tag{5.1}$$

We observe, since $\mathfrak{Re}\,(1-z) \geq 0$ for $z \in \overline{\mathbb{D}}$, that:

$$|\varphi\,(z)| \leq \exp\big(-\delta\,|1-z|^{-\theta}\big)\,, \tag{5.2}$$

where $\delta = \cos \pi\theta/2 > 0$. Moreover, (5.2) shows that $\varphi \in A(\mathbb{D})$, since:

$$\lim_{z \to 1, z \in \overline{\mathbb{D}}} \varphi\,(z) = 0 =: \varphi\,(1)\,.$$

Our first lemma will allow us to define our symbol Φ.

Lemma 5.1. *One can adjust $0 < c < 1$ so as to get:*

$$|\chi(z)| + 2\,c\,|\varphi \circ \chi(z)| < 1 \qquad \textit{for all } z \in \mathbb{D}\,. \tag{5.3}$$

Hence, if we set, for any $g \in A(\mathbb{D})$ with $\|g\|_\infty \leq 1$:

$$\Phi(z_1, z_2) = \big(\chi(z_1), \chi(z_1) + c\,(\varphi \circ \chi)(z_1)\,g(z_2)\big)\,, \tag{5.4}$$

we have $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$.

Remark. The factor 2 in (5.3) is needed in order to get the following inequalities, to be used later, for $z \in \mathbb{D}$ and $w = \chi(z) + c\,(\varphi \circ \chi)(z)\,u$, with $|u| \leq 1$:

$$|w| \leq \frac{1 + |\chi(z)|}{2}\,, \tag{5.5}$$

or, equivalently:

$$1 - |w| \geq \frac{1 - |\chi(z)|}{2} \cdot \tag{5.6}$$

Indeed:

$$|w| \leq |\chi(z)| + c\,|\varphi \circ \chi(z)| \leq |\chi(z)| + \frac{1 - |\chi(z)|}{2} = \frac{1 + |\chi(z)|}{2} \cdot$$

Proof of Lemma 5.1. Set $X = |1 - \chi(z)|$, so that, with K the constant of Proposition $4.1, 2)$:

$$|\chi(z)| \leq 1 - \frac{|1 - \chi(z)|}{K} = 1 - \frac{X}{K} \cdot \tag{5.7}$$

For $z \in \mathbb{D}$ and X close enough to zero, say $X < \eta$, we have $2\exp(-\delta X^{-\theta}) < \frac{X}{K}$. If we adjust $0 < c < 1$ so as to have $c < \frac{\eta}{2K}$, it follows from (5.2) and (5.7) that, for $X < \eta$:

$$|\chi(z)| + 2\,c\,|\varphi \circ \chi(z)| \leq 1 - \frac{X}{K} + 2\exp(-\delta X^{-\theta}) < 1\,.$$

However, for $X \geq \eta$, (5.7) says that $|\chi(z)| \leq 1 - \frac{\eta}{K}$, so:

$$|\chi(z)| + 2\,c\,|\varphi \circ \chi(z)| \leq 1 - \frac{\eta}{K} + 2\,c < 1\,,$$

as well and this ends the proof of Lemma 5.1. □

Our second lemma estimates some integrals and ensures that Φ induces a compact composition operator on $H^2(\mathbb{D}^2)$.

Lemma 5.2. *For $0 < h \leq 1$, the following estimate holds:*

$$I_0(h) := \int_{|\chi(\mathrm{e}^{it}) - 1| \leq h} \frac{1}{(1 - |\chi(\mathrm{e}^{it})|)^2}\, dt \lesssim \mathrm{e}^{-\tau/h}\,, \tag{5.8}$$

Proof. By Proposition $4.1, 5)$, there exist two constants c_1, c_2 such that:

$$\frac{c_1}{\log 1/|t|} \leq |\chi(\mathrm{e}^{it}) - 1| \leq \frac{c_2}{\log 1/|t|}\,, \qquad |t| \leq \pi\,;$$

hence:

$$I_0(h) \lesssim \int_{|t| \leq \mathrm{e}^{-c_1/h}} [\log(1/|t|)]^2\, dt = 2\int_{c_1/h}^{\infty} x^2\,\mathrm{e}^{-x}\, dx \lesssim h^{-2}\mathrm{e}^{-c_1/h}\,. \quad □$$

Corollary 5.3. *For $g \in A(\mathbb{D})$ with $0 < \|g\|_\infty \leq 1$, set:*

$$I(h) := \int_{|\chi(\mathrm{e}^{it_1})-1|\leq h} \frac{dt_1\, dt_2}{(1-|\chi(\mathrm{e}^{it_1})|)(1-|\chi(\mathrm{e}^{it_1}) + c\,(\varphi\circ\chi)(\mathrm{e}^{it_1})\, g(\mathrm{e}^{it_2})|)} \, .$$

Then:

$$I(h) \lesssim \mathrm{e}^{-\tau/h} \, .$$

Consequently, the composition operator C_Φ defined in (5.4) *is bounded from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D}^2)$ and is compact.*

Proof. Using (5.6), we have, thanks to (5.8):

$$I(h) \leq \int_{|\chi(\mathrm{e}^{it_1})-1|\leq h} \frac{2}{(1-|\chi(\mathrm{e}^{it_1})|)^2}\, dt_1\, dt_2 \lesssim \mathrm{e}^{-\tau/h} \, .$$

In particular, $I(1) < \infty$, showing that C_Φ is Hilbert-Schmidt and hence bounded. □

For the rest of the paper, we fix a number σ in $(0,1)$, that for convenience we take as:

$$\sigma = \frac{7}{8} \, , \tag{5.9}$$

a positive integer j_0 such that:

$$2\,\sigma^{j_0} \leq 1/8 \tag{5.10}$$

(i.e. $j_0 \geq 21$), and we set:

$$a_j = 1 - \sigma^j \tag{5.11}$$

and:

$$\rho_j = \frac{\sigma^j}{4} = \frac{1}{4}(1-a_j) \, . \tag{5.12}$$

We also define, for $n \geq 1$ and θ being the parameter used in (5.1):

$$N_n = \left[\frac{\log 2n}{\theta \log 1/\sigma}\right] + 1 > \frac{\log 2n}{\log 1/\sigma} \, . \tag{5.13}$$

The next lemma gives a cutting off for $\chi(\mathbb{D})$.

Lemma 5.4. *For every $n \geq 1$, the image $\chi(\mathbb{D})$ of the cusp map, deprived of the closed Euclidean disk $\overline{D}(0, 1-\sigma^{j_0}/K)$ and of $\chi(\mathbb{D}) \cap S(1, 1/n)$, can be covered by the open Euclidean disks $D(a_j, \rho_j)$, with $j_0 \leq j \leq N_n$.*

Proof. Let $z \in \mathbb{D}$ such that $|\chi(z)| > 1 - \sigma^{j_0}/K$ and $|\chi(z) - 1| > 1/n$. We write $\chi(z) = x + iy =: 1 - h + iy$.

Let j with $a_j \leq x < a_{j+1}$, i.e. $\sigma^{j+1} < h \leq \sigma^j$. We have $j \geq j_0$, since $h < \sigma^{j_0}$.

Now, since $0 \leq x - a_j < a_{j+1} - a_j = \sigma^{j+1} - \sigma^j$, that $y^2 \leq 4h^4$ (Proposition 4.1, 6)), and $h \leq \sigma^j$, we have:

$$|\chi(z) - a_j|^2 < (\sigma^j - \sigma^{j+1})^2 + y^2 \leq (1-\sigma)^2 \sigma^{2j} + 4\,\sigma^{4j}\,;$$

hence:

$$|\chi(z) - a_j| < \sigma^j(1-\sigma) + 2\,\sigma^{2j} = \sigma^j(1 - \sigma + 2\,\sigma^j)\,.$$

Subsequently, since $1 - \sigma = 1/8$, $j \geq j_0$, and $2\,\sigma^{j_0} \leq 1/8$:

$$|\chi(z) - a_j| < \sigma^j(1 - \sigma + 2\,\sigma^{j_0}) \leq \frac{\sigma^j}{4} = \rho_j\,,$$

showing that $\chi(z) \in D(a_j, \rho_j)$.

Moreover, we have $j \leq N_n$. Indeed, if $j > N_n$, we would have:

$$|\chi(z) - 1| \leq |\chi(z) - a_j| + (1 - a_j) \leq \rho_j + \sigma^j = \frac{5}{4}\,\sigma^j \leq \frac{5}{4}\,\sigma^{N_n+1} \leq 2\,\sigma^{N_n} \leq 1/n\,,$$

contradicting the fact that $\chi(z) \notin S(1, 1/n)$. □

Our next two lemmas give estimates on derivatives for the functions belonging to $H^2(\mathbb{D}^2)$.

Lemma 5.5. *Let $f \in H^2(\mathbb{D}^2)$, k a non-negative integer, $b \in \mathbb{D}$, and let $h_k(z) = (D_2^k f)(z, z)$. Then:*

$$|h_k(b)| \leq \frac{k!\,2^{k+1}}{(1 - |b|)^{k+1}}\,\|f\|_2\,.$$

Proof. The Cauchy inequalities give for $0 < s < 1 - |b|$ and $\alpha \in \mathbb{N}^2$:

$$|D^\alpha f(b, b)| \leq \frac{\alpha!}{s^{|\alpha|}} \sup_{|w_1 - b| = s, |w_2 - b| = s} |f(w_1, w_2)|\,.$$

The choice $s = \frac{1-|b|}{2}$ gives $1 - |w_j| \geq \frac{1-|b|}{2}$ for $|w_j - b| = s$, $j = 1, 2$; hence, thanks to the estimate (3.2):

$$|f(w_1, w_2)| \leq \frac{\|f\|_2}{\sqrt{(1 - |w_1|)(1 - |w_2|)}} \leq \frac{2}{1 - |b|} \, \|f\|_2 \, .$$

Specializing to $\alpha = (0, k)$ now gives the result. □

Lemma 5.6. *With the notations of Lemma 5.5, assume that $h_k^{(l)}(a) = 0$ for some $a \in \mathbb{D}$ and for $0 \leq l < n$. Then, for $0 < \rho < 1$ and $|b - a| \leq \frac{\rho}{2}\,(1 - |a|)$, it holds:*

$$|h_k(b)| \leq \rho^n \, \frac{k!\, 4^{k+1}}{(1 - |a|)^{k+1}} \, \|f\|_2 \, .$$

Proof. We may assume $\|f\|_2 \leq 1$. Consider the function defined, for $w \in \mathbb{D}$, by:

$$H_k(w) = h_k\left(a + w\, \frac{1 - |a|}{2}\right) .$$

It is a bounded and holomorphic function in $\mathbb{D}$.

For $w \in \mathbb{D}$, let $\beta = a + w\, \frac{1-|a|}{2}$, which satisfies $1 - |\beta| \geq \frac{1-|a|}{2}$. Lemma 5.5 gives:

$$|H_k(w)| = |h_k(\beta)| \leq \frac{k!\, 4^{k+1}}{(1 - |a|)^{k+1}} \, \cdot$$

Now, $H_k^{(l)}(0) = h_k^{(l)}(a) = 0$ for $0 \leq l < n$; hence the Schwarz lemma says that H_k satisfies $|H_k(w)| \leq |w|^n \, \|H_k\|_\infty$ for all $w \in \mathbb{D}$. Take $w = \frac{2(b-a)}{1-|a|}$, which satisfies $|w| \leq \rho$, to get:

$$|h_k(b)| = |H_k(w)| \leq |w|^n \, \|H_k\|_\infty \leq \rho^n \, \frac{k!\, 4^{k+1}}{(1 - |a|)^{k+1}} \, \cdot$$ □

6 The main result

Recall that χ is the cusp map and that φ is defined in (5.1). The map g appearing in the formula below plays an inert role, and is just designed to ensure that Φ is non-degenerate; we can take, for example $g(z_2) = z_2$. This seems to mean that non-degeneracy is not the only issue in the question of estimating $\beta_2^+(C_\Phi)$.

Our example appears as a perturbation of the diagonal map defined by $\Delta(z_1, z_2) = \big(\chi(z_1), \chi(z_1)\big)$ for which we already know ([15, Theorem 2.4]) that $\Delta(1,1) = (1,1)$ and $\beta_2^+(C_\Delta) < 1$. This map is degenerate, but the perturbation clearly gives a non degenerate one since its Jacobian is $J_\Phi(z_1, z_2) = c\,(\varphi \circ \chi)(z_1)\,\chi'(z_1)\,g'(z_2)$.

Theorem 6.1. *Let:*

$$\Phi(z_1, z_2) = \big(\chi(z_1), \chi(z_1) + c\,(\varphi \circ \chi)(z_1)\,g(z_2)\big)$$

be the function defined in (5.4).

Then:

1) $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$ *and* $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ *is compact;*

2) Φ *is non degenerate, and its components belong to the bidisk algebra;*

3) $\Phi(\mathbb{T}^2) \cap \mathbb{T}^2 = \Phi(\mathbb{T}^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset$*;*

4) $a_{n^2}(C_\Phi) \lesssim \exp(-\tau n)$, *for some* $\tau > 0$, *implying* $\beta_2^+(C_\Phi) < 1$.

Proof. That Φ maps $\mathbb{D}^2$ to itself is proved in Lemma 5.1 and that the composition operator $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is compact, in Corollary 5.3. Item 2) is due to the presence of g, as explained above. The fact that $\Phi(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$ is clear since $\Phi(1,1) = (1,1)$. It remains to prove 4).

Once more, the proof will be conveniently divided into several steps. We begin by a lemma which is in fact obvious, but explains well what is going on.

Lemma 6.2. *Let* $\lambda = 1 - \frac{\sigma^{j_0}}{2K}$, *where* σ, K *and* j_0 *are as in* (5.9), *Proposition 4.1, 2*), *and* (5.10). *Let* $r_n = 1 - \frac{1}{n}$, *and let* μ_1, μ_2, μ_3 *the respective restrictions of* m_Φ *to the disk* $\overline{\lambda \mathbb{D}^2}$, *the annulus* $r_n \mathbb{D}^2 \setminus \overline{\lambda \mathbb{D}^2}$, *and the annulus* $\mathbb{D}^2 \setminus r_n \mathbb{D}^2$. *We then have:*

$$C_\Phi = T_1 \oplus T_2 \oplus T_3\,,$$

where T_j *is the canonical injection of* $H^2(\mathbb{D}^2)$ *into* $L^2(\mu_j)$.

This is indeed obvious since:

$$\|C_\Phi f\|^2 = \int_{\mathbb{D}^2} |f|^2 dm_\Phi\,,$$

and by splitting the integral into three parts.

We now majorize separately the numbers $a_p(T_j)$, for $j = 1, 2, 3$. In the sequel, the positive constant τ may vary from one formula to another.

Step 1. It holds:

$$a_{n^2}(T_1) \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.1}$$

Proof. Let $V = z_1^n H^2(\mathbb{D}^2) + z_2^n H^2(\mathbb{D}^2)$; this is a subspace of $H^2(\mathbb{D}^2)$ of codimension $\leq n^2$, since:

$$V = \big\{ f \in H^2(\mathbb{D}^2)\,;\ D_1^j D_2^k f(0,0) = 0 \quad \text{for } 0 \leq j, k < n \big\}\,.$$

If $f(z) = \sum_{\max(j,k) \geq n} a_{j,k}\, z_1^j z_2^k \in V$ and $\| f\|_2 = 1$, one can write:

$$f(z) = z_1^n\, q_1(z_1, z_2) + z_2^n\, q_2(z_1, z_2)\,,$$

with:

$$q_1(z) = \sum_{j \geq n, k \geq 0} a_{j,k}\, z_1^{j-n} z_2^k \quad \text{and} \quad q_2(z) = \sum_{j < n, k \geq n} a_{j,k}\, z_1^j z_2^{k-n}\,,$$

which satisfy $\| q_j\|_2 \leq \| f\|_2 = 1$, $j = 1, 2$.

An easy estimate now gives (since $\max(|z_1|^n, |z_2^n|) \leq \lambda^n$ on $\overline{\lambda\mathbb{D}^2}$):

$$\begin{aligned} \| T_1 f\|^2 & \leq 2 \bigg(\int_{\lambda\mathbb{D}^2} \big(|z_1^n|^2 |q_1(z_1, z_2)|^2 + |z_2^n|^2 |q_2(z_1, z_2)|^2 \big)\, dm_\Phi \bigg) \\ & \lesssim \lambda^{2n} \int_{\lambda\mathbb{D}^2} \big(|q_1|^2 + |q_2|^2 \big)\, dm_\Phi \lesssim \lambda^{2n} \big(\| q_1\|_2^2 + \| q_2\|_2^2 \big) \lesssim \lambda^{2n}\,, \end{aligned}$$

since we know by Corollary 5.3 that C_Φ is bounded on $H^2(\mathbb{D}^2)$ and hence that m_Φ is a Carleson measure for $H^2(\mathbb{D}^2)$. Alternatively, we could majorize $|q_j(z_1, z_2)|$ uniformly on the support of μ_1. We hence obtain:

$$a_{n^2+1}(T_1) = c_{n^2+1}(T_1) \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.2}$$

□

Step 2. It holds:

$$a_{n^2}(T_3) \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.3}$$

Proof. In one variable, we could use the Carleson embedding theorem; but this theorem for the bidisk and the Hardy space $H^2(\mathbb{D}^2)$ notably has a more complicated statement ([3]; see also [5]), and cannot be used efficiently here. Our strategy will be to replace it by a sharp estimation of a Hilbert-Schmidt norm.

We set $h_n = 1 - r_n = 1/n$.

Clearly, denoting by S_2 the Hilbert-Schmidt class:

$$\| T_3\|^2 \leq \| T_3\|_{S_2}^2 = \int \frac{d\mu_3(w)}{(1 - |w_1|^2)(1 - |w_2|^2)} \leq \int \frac{d\mu_3(w)}{(1 - |w_1|)(1 - |w_2|)}\,.$$

Now, if $w = (w_1, w_2) = \big(\chi(z_1), \chi(z_1) + c\,(\varphi\circ\chi)(z_1)\,g(z_2)\big)$ belongs to the support of μ_3, we have $\max(|w_1|, |w_2|) \geq r_n = 1 - h_n$, and, recalling (5.5):

$$|w_1| \geq 2\,|w_2| - 1\,, \tag{6.4}$$

we have in either case $|w_1| \geq 1 - 2h_n$. By Proposition 4.1, 2), this implies that:

$$|1 - w_1| \leq 2\,K h_n\,.$$

Corollary 5.3 gives:

$$\begin{aligned}\|T_3\|^2 &\lesssim \int_{|\chi(\mathrm{e}^{it_1})-1|\leq 2Kh_n} \frac{dt_1\,dt_2}{(1 - |\chi(\mathrm{e}^{it_1})|)(1 - |\chi(\mathrm{e}^{it_1}) + c\,(\varphi\circ\chi)(\mathrm{e}^{it_1})\,g(\mathrm{e}^{it_2})|)} \\ &= I(2Kh_n) \lesssim \mathrm{e}^{-\tau/h_n}\,.\end{aligned}$$

But $h_n = 1/n$, so that:

$$a_{n^2}(T_3) \leq \|T_3\| \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.5}$$ □

Step 3. It holds:

$$a_{n^2}(T_2) \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.6}$$

This estimate follows from the following key auxiliary lemma. In fact, this lemma will give, for the Gelfand numbers, $c_{n^2}(T_2) \lesssim \mathrm{e}^{-\tau n}$, and we know that they are equal to the approximation numbers.

Let $M\colon H^2(\mathbb{D}^2) \to \mathrm{Hol}(\mathbb{D})$ be the linear map defined by:

$$Mf(z) = f(z, z)\,,$$

Recall that $a_j = 1 - \sigma^j$ and $N_n = \Big[\frac{\log 2n}{\theta \log 1/\sigma}\Big] + 1$.

Lemma 6.3. *Let E be the closed subspace of $H^2(\mathbb{D}^2)$ defined by:*

$$E = \Big\{ f \in H^2(\mathbb{D}^2)\,;\ \big[M(D_2^k f)\big]^{(l)}(a_j) = 0 \quad \text{for } 0 \leq l < n,\ 0 \leq k \leq m_j,\ 1 \leq j \leq N_n \Big\}\,.$$

Then, we can adjust the numbers m_j so as to guarantee that, for some positive constant τ:

$$\operatorname{codim} E \lesssim n^2$$

and, for all $f \in E$ with $\|f\|_2 \leq 1$:

$$\|T_2(f)\|_2 \lesssim \exp(-\tau n)\,.$$

Proof. This is the most delicate part.

Recall that:

$$h_n = 1/n\,, \quad r_n = 1 - h_n\,, \quad \lambda = 1 - \frac{\sigma^{j_0}}{2\,K}\cdot$$

We need a uniform estimate of $|f(w)|$ for $f \in E$ with $\|f\|_2 \leq 1$ and for:

$$w = (w_1, w_2) \in \operatorname{supp} m_\Phi \cap (r_n\mathbb{D}^2 \setminus \overline{\lambda\mathbb{D}^2})\,.$$

This estimate will be given by Lemma 5.4, Lemma 5.5 and Lemma 5.6. Note that we have:

$$\chi(z_1) \in \mathbb{D} \setminus [S(1, 1/n) \cup \overline{D}(0, 2\lambda - 1)]\,.$$

Indeed, if $(w_1, w_2) = \Phi(z_1, z_2) \notin \lambda\mathbb{D}^2$, we have $\max(|w_1|, |w_2|) > \lambda$; so either $|w_1| > \lambda \geq 2\lambda - 1$, or $|w_2| > \lambda$ and again $|w_1| > 2\lambda - 1$ since $|w_1| \geq 2\,|w_2| - 1$, by (5.5). Hence $w_1 \notin \overline{D}(0, 2\lambda - 1)$. Moreover, we have $|1 - w_1| \geq 1 - |w_1| > 1/n$, so $w_1 \notin S(1, 1/n)$.

Using Lemma 5.4, select $j_0 \leq j \leq N_n$ such that $|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)$. Now set:

$$A = \big(\chi(z_1), \chi(z_1)\big) \quad \text{and} \quad \Delta = \big(0, (\varphi \circ \chi)(z_1)\, g(z_2)\big)\,.$$

Our strategy will be the following. We write:

$$\begin{aligned} f[\Phi(z_1, z_2)] = f(A + \Delta) &= \sum_{k=0}^{\infty} \frac{D_2^k f(A)}{k!}\, \Delta^k \\ &= \sum_{k=0}^{\infty} \frac{M(D_2^k f)[\chi(z_1)]}{k!}\, \Delta^k = \sum_{k=0}^{\infty} \frac{h_k[\chi(z_1)]}{k!}\, \Delta^k\,, \end{aligned}$$

with $h_k = M(D_2^k f)$, and we put:

$$S_j = \sum_{k=0}^{m_j} \frac{h_k[\chi(z_1)]}{k!}\, \Delta^k$$

and

$$R_j = \sum_{k > m_j} \frac{h_k[\chi(z_1)]}{k!}\, \Delta^k\,.$$

We will estimate separately S_j and R_j.

a) *Estimation of R_j.*

Recall that j is such that $j_0 \leq j \leq N_n$ and $|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)$. We saw in the proof of this Lemma 5.4 that $1 - |\chi(z_1)| \leq |1 - \chi(z_1) \leq \frac{5}{4}\sigma^j$. Hence:

$$|\Delta| \leq |(\varphi \circ \chi)(z_1)| \leq \exp\Big(- \frac{\delta}{|1 - \chi(z_1)|^\theta} \Big) \lesssim \exp(-\tau\, \sigma^{-j\theta})\,.$$

Now, use Lemma 5.5 and (5.2) to get:

$$\begin{aligned} |R_j| &\leq \sum_{k > m_j} \frac{2^{k+1}}{(1 - |\chi(z_1)|)^{k+1}}\, |\Delta|^k \lesssim \sum_{k > m_j} 2^k \sigma^{-jk} \exp(-\tau k\, \sigma^{-j\theta}) \\ &\lesssim 2^{m_j} \sigma^{-j m_j} \exp(-\tau m_j \sigma^{-j\theta}) \lesssim \exp(-\tau m_j \sigma^{-j\theta}) \end{aligned}$$

for some absolute constant $\tau > 0$, that is:

$$|R_j| \lesssim \exp(-\tau n) \tag{6.7}$$

if we take:

$$m_j = [n\, \sigma^{j\theta}] + 1\,. \tag{6.8}$$

b) *Estimation of S_j.*

We saw in the estimation of R_j that $1 - |\chi(z_1)| \gtrsim \sigma^j$. Now, remember that $h_k^{(l)}(a_j) = 0$ for $l < n$, since $f \in E$, we then use Lemma 5.6 to get, when we take the values:

$$a = a_j\,, \quad 1 - a_j = \sigma^j\,, \quad b = \chi(z_1)\,, \quad \rho = \frac{1}{2}\,,$$

a good upper bound for $\frac{h_k[\chi(z_1)]}{k!}$ when $k \leq m_j$, namely:

$$\Big| \frac{h_k[\chi(z_1)]}{k!} \Big| \lesssim \frac{4^{k+1}}{\sigma^{j(k+1)}}\, \rho^n\,.$$

We then obtain an estimate of the form:

$$\begin{aligned} |S_j| &\lesssim \sum_{k=0}^{m_j} \rho^n \frac{4^{k+1}}{\sigma^{j(k+1)}} \lesssim \rho^n \frac{4^{m_j}}{\sigma^{j m_j}} = \exp\Big(- n\log 2 + m_j \log 4 - j m_j \log \frac{7}{8}\Big) \\ &\lesssim \exp(-4\tau n + B j m_j) \end{aligned}$$

with $\tau = \frac{1}{4}\log 2$ and $B \leq \log 4 + \log(8/7) \leq 2$; or else, using (6.8):

$$|S_j| \lesssim \exp(-4\tau n + Bjn\,\sigma^{j\theta} + Bj)\,.$$

But since $\sigma = 7/8 < 1$, the implied exponent, for $j_0 \leq j \leq N_n$:

$$-4\tau n + Bjn\,\sigma^{j\theta} + Bj = n(-4\tau + Bj\,\sigma^{j\theta}) + Bj\,,$$

is $\leq -2\tau n + B' \log n$, provided that we choose j_0 large enough, namely such that $j_0 \big(\frac{7}{8}\big)^{j_0 \theta} \leq 1/4$. This implies an inequality of the form:

$$|S_j| \lesssim \mathrm{e}^{-2\tau n} n^{B'} \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.9}$$

Putting the estimates (6.7) and (6.9) on R_j and S_j together, we obtain, for every $f \in E$ with $\|f\|_2 \leq 1$:

$$\|T_2 f\| \lesssim \mathrm{e}^{-\tau n}\,. \tag{6.10}$$

It remains to bound from above the codimension of E. Since $N_n = \big[\frac{\log 2n}{\theta \log 1/\sigma}\big] + 1$ with $\sigma = 7/8$ and $m_j = [n\,\sigma^{j\theta}] + 1$, we see that:

$$\operatorname{codim} E \leq \sum_{l=0}^{n-1} \sum_{j=1}^{N_n} m_j \leq \sum_{l=0}^{n-1} \sum_{j=1}^{N_n} (n\,\sigma^{j\theta} + 1) \lesssim n^2 \sum_{j=1}^{\infty} \sigma^{j\theta} + n \log n < q\,n^2\,.$$

Therefore (6.10) can be read as well, remembering the equality of approximation numbers and Gelfand numbers:

$$a_{q\,n^2}(T_2) = c_{q\,n^2}(T_2) \lesssim \mathrm{e}^{-\tau n}. \tag{6.11}$$

Putting the estimates (6.2), (6.5), and (6.11) together ends the proof of Lemma 6.3. □

Finally, Lemma 6.2 and (2.5) give:

$$a_{3n^2}(C_\Phi) = a_{3n^2}(T_1 \oplus T_2 \oplus T_3) \leq a_{n^2}(T_1) + a_{n^2}(T_2) + a_{n^2}(T_3) \lesssim \mathrm{e}^{-\tau n}\,,$$

thereby finishing the proof of Theorem 6.1. □

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