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Abstract. We give estimates of the entropy numbers of composition operators on the Hardy space of the disk and of the polydisk.

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1 Introduction

This short paper was motivated by a question of J. Wengenroth ([19]) about entropy numbers of composition operators on Hardy spaces H^2, which stand a little apart in the jungle of "s-numbers", even though they seem the most natural for the study of compactness, since their membership in c_0 characterizes compactness, even in the general framework of arbitrary Banach spaces. Indeed, in various papers (see [1, 10, 11, 12, 13]), we studied in detail the approximation numbers of composition operators, and here we will essentially transfer those results to entropy numbers thanks to the polar (Schmidt) decomposition and a general result on entropy numbers of diagonal operators on ℓ^2.

So, the proofs are easy, but the statements feature a very different behavior of those entropy numbers. In particular, we will investigate a few properties related with a so-called "spectral radius type formula" which we obtained, in dimension one through a result of Widom ([12]), and, partially in dimension N ([13, 14]), through a result of Nivoche ([16]) and Zakharyuta ([22]).

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2 Entropy numbers

We begin by recalling some facts on s-numbers.

Given an operator $T\colon X \to Y$ between Banach spaces, recall ([4]) that we can attach to this operator five non-increasing sequences (a_n), (b_n), (c_n), (d_n), (e_n) of non-negative numbers (depending on T), respectively the sequences of *approximation*, *Bernstein*, *Gelfand*, *Kolmogorov*, and *entropy* numbers of T. The latter are defined ([4, Chapter 1], or [17, Chapter 5]), for $n \geq 1$ by:

$$e_n(T) = \inf\{\varepsilon > 0\,;\ N\big(T(B_X), \varepsilon B_Y\big) \leq 2^{n-1}\}\,, \tag{2.1}$$

where B_X and B_Y are the respective closed unit balls of X and Y, and where, for $A, B \subseteq Y$, $N(A, B)$ denotes the smallest number of translates of B needed to cover A.

All those sequences (a_n), (b_n), (c_n), (d_n), (e_n), say (u_n), share the ideal property:

$$u_n(ATB) \leq \|A\|\, u_n(T)\, \|B\|\,.$$

For Hilbert spaces, it turns out that $a_n = b_n = c_n = d_n = s_n$, where (s_n) designates the sequence of singular numbers, but entropy numbers stay a little apart.

For general Banach spaces X and Y and $T\colon X \to Y$, we have, in general ([3, Theorem 1], see also [17, Theorem 5.2]), for $\alpha > 0$:

$$\sup_{1 \leq k \leq n} k^\alpha e_k(T) \leq C_\alpha \sup_{1 \leq k \leq n} k^\alpha a_k(T)\,,$$

and, if X and Y^* are of type 2:

$$a_n(T) \leq K\, e_n(T)\,, \qquad \text{for all } n \geq 1$$

([7, Corollary 1.6]), where $K = \kappa\, [T_2(X) T_2(Y^*)]^2$; in particular, if T acts between Hilbert spaces (see [17, Theorem 5.3]):

$$a_n(T) \leq 4\, e_n(T)\,, \qquad \text{for all } n \geq 1\,.$$

Those inequalities indicate that entropy numbers are always bigger than singular numbers, up to a constant, and that, as far as the scale of powers n^α

is implied, they are dominated by approximation numbers in a weak sense. But it turns out that, individually, they can be much bigger than the latter for composition operators, as we shall see.

We will rely on the following estimate ([4, p. 17]), in which ℓ^2 denotes the space of square-summable sequences $x = (x_k)_{k\geq 1}$ of complex numbers. This estimate is given for the sequence (ε_n) of covering numbers and with the scale of powers of 2, but $e_n = \varepsilon_{2^{n-1}}$, by definition, and the change of 2 to e only affects constants.

Theorem 2.1. (see [4, p. 17]) *There exist absolute constants* $0 < a < b$ *such that, for any diagonal compact operator* $\Delta\colon \ell^2 \to \ell^2$ *with positive and non-increasing eigenvalues* $(\sigma_k)_{k\geq 1}$*, namely* $\Delta\big((x_k)_k\big) = (\sigma_k x_k)_k$*, we have, for all* $n \geq 1$*:*

$$a \sup_{k\geq 1} \left[\mathrm{e}^{-n/k} \bigg(\prod_{j=1}^{k} \sigma_j \bigg)^{1/k} \right] \leq e_n(\Delta) \leq b \sup_{k\geq 1} \left[\mathrm{e}^{-n/k} \bigg(\prod_{j=1}^{k} \sigma_j \bigg)^{1/k} \right]. \tag{2.2}$$

A useful corollary of Theorem 2.1 is the following.

Theorem 2.2. *Let* $T\colon H_1 \to H_2$ *be a compact operator between the Hilbert spaces* H_1 *and* H_2*, and let* $(a_n)_{n\geq 1}$ *be its sequence of approximation numbers. Then, for all* $n \geq 1$*:*

$$\alpha \sup_{k\geq 1} \left[\mathrm{e}^{-n/k} \bigg(\prod_{j=1}^{k} a_j \bigg)^{1/k} \right] \leq e_n(T) \leq \beta \sup_{k\geq 1} \left[\mathrm{e}^{-n/k} \bigg(\prod_{j=1}^{k} a_j \bigg)^{1/k} \right], \tag{2.3}$$

where α *and* β *are positive numerical constants.*

Proof. Let $Tx = \sum_{n=1}^{\infty} s_n (x \mid u_n) v_n$ the Schmidt decomposition of T, where $(u_n)_n$ and $(v_n)_n$ are orthonormal sequences of H_1 and H_2, respectively, and $(s_n)_n$ is the sequence of singular numbers of T. Let $\Delta\colon \ell_2 \to \ell_2$ the diagonal operator with diagonal values s_n, $n \geq 1$. Then $T = V_1 \Delta U_1$ and $\Delta = V_2 T U_2$, with $U_1 x = \big((x \mid u_n)\big)_n$, $V_1\big((t_n)_n\big) = \sum_n t_n v_n$, $U_2\big((t_n)_n\big) = \sum_n t_n u_n$ and $V_2 x = \big((x \mid v_n)\big)_n$. We have $\|U_1\|, \|V_1\|, \|U_2\|, \|V_2\| \leq 1$; hence the result follows from Theorem 2.1 and the ideal property. □

This theorem might be thought useless, because we don't know better the a_n's than the e_n's! In our situation, this is not the case, since we made a more or less systematic study of the a_n's for composition operators in [1, 11, 10, 12] for example.

We now pass to applications to composition operators C_φ, defined as $C_\varphi(f) = f \circ \varphi$ when they act on the Hardy space $H^2(\mathbb{D}^N)$ (which is always the case if $N = 1$). Here, φ denotes an analytic and non-degenerate self-map of $\mathbb{D}^N$. For clarity, we separate the cases of dimension $N = 1$ and of dimension $N \geq 2$.

3 Applications in dimension 1

3.1 General results

In [12], we had coined the parameter:

$$\beta_1(T) = \lim_{n\to\infty} \big[a_n(T)\big]^{1/n} \tag{3.1}$$

and its versions $\beta_1^+(T), \beta_1^-(T)$ with a upper limit and a lower limit respectively. The following result ([12]) shows in particular that no lower or upper limit is needed for $\beta = \beta_1$, and provides a simpler proof of the second item in Theorem 3.1 than in our initial proof of [10].

For the definition of the Green capacity $\mathrm{Cap}\,(A)$ of a Borel subset A of $\mathbb{D}$, $0 \leq \mathrm{Cap}\,(A) \leq \infty$, we refer to [12].

Theorem 3.1. *Let $\Omega = \varphi(\mathbb{D})$, with $\varphi\colon \mathbb{D} \to \mathbb{D}$ a non-constant analytic map. Then:*

1) *One always has $\beta_1^-(C_\varphi) = \beta_1^+(C_\varphi) =: \beta_1(C_\varphi)$ and:*

$$\beta_1(C_\varphi) = \exp[-1/\mathrm{Cap}\,(\Omega)] > 0\,. \tag{3.2}$$

2) *In particular, one has the equivalence:*

$$\beta_1(C_\varphi) = 1 \quad \Longleftrightarrow \quad \|\varphi\|_\infty = 1\,. \tag{3.3}$$

Here, another parameter emerges.

$$\gamma_1(T) = \lim_{n\to\infty} \big[e_n(T)\big]^{1/\sqrt{n}} \tag{3.4}$$

and its $\gamma_1^+(T), \gamma_1^-(T)$ versions.

Theorem 3.2. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a symbol and $\Omega = \varphi(\mathbb{D})$. Then:*

1) *$\gamma_1^-(C_\varphi) = \gamma_1^+(C_\varphi) =: \gamma_1(C_\varphi)$ and:*

$$\gamma_1(C_\varphi) = \exp\big[-\sqrt{2/\mathrm{Cap}\,(\Omega)}\big] > 0\,. \tag{3.5}$$

2) *In particular, one has the equivalence:*

$$\gamma_1(C_\varphi) = 1 \quad \Longleftrightarrow \quad \|\varphi\|_\infty = 1\,. \tag{3.6}$$

Proof. Set $\rho = 1/\mathrm{Cap}\,(\Omega)$ for simplicity of notations. Let $\varepsilon > 0$, and C_ε a positive constant which depends only on ε and can vary from a formula to another. Theorem 3.1 implies $a_k \leq C_\varepsilon\,\mathrm{e}^{\varepsilon k}\mathrm{e}^{-k\rho}$, whence:

$$(a_1 \cdots a_k)^{1/k} \leq C_\varepsilon\,\mathrm{e}^{\varepsilon k/2}\mathrm{e}^{-\rho k/2}\,.$$

Theorem 2.2 now gives:

$$e_n(C_\varphi) \leq C_\varepsilon \sup_{k\geq 1} \big[\mathrm{e}^{\varepsilon k/2}\,\mathrm{e}^{-(n/k+\rho k/2)}\big]\,.$$

This supremum is essentially attained for $k = \big[\sqrt{2n/\rho}\big]$ where $[\,.\,]$ stands for the integer part, and gives:

$$e_n(C_\varphi) \leq C_\varepsilon \mathrm{e}^{\varepsilon\sqrt{n/(2\rho)}}\mathrm{e}^{-\sqrt{2n\rho}}\,.$$

This implies $\gamma_1^+(C_\varphi) \leq \mathrm{e}^{\varepsilon\sqrt{1/(2\rho)}}\mathrm{e}^{-\sqrt{2\rho}}$, and finally:

$$\gamma_1^+(C_\varphi) \leq \mathrm{e}^{-\sqrt{2\rho}}\,.$$

The lower bound $\gamma_1^-(C_\varphi) \geq \mathrm{e}^{-\sqrt{2\rho}}$ is proved similarly.

This clearly ends the proof, since we know from [12] that $\mathrm{Cap}\,(\Omega) = \infty$ if ond only if $\|\varphi\|_\infty = 1$. □

3.2 Specific results

For $0 < \theta < 1$, the lens map λ_θ of parameter θ is defined by:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta} \tag{3.7}$$

(see [18] or [10]).

Theorem 3.3. *Let λ_θ be the lens map with parameter θ. Then, with positive constants a, b, a', b' depending only on θ:*

$$a'\,\mathrm{e}^{-b' n^{1/3}} \leq e_n(C_{\lambda_\theta}) \leq a\,\mathrm{e}^{-b n^{1/3}}\,. \tag{3.8}$$

Proof. We proved in [8, Theorem 2.1] (see also [10, Proposition 6.3] that $a_k = a_k(C_{\lambda_\theta}) \leq a\,\mathrm{e}^{-b\sqrt{k}}$. It follows, using Theorem 2.2, that $(a_1 \cdots a_k)^{1/k} \leq a\,\mathrm{e}^{-b\sqrt{k}}$ and that, for some positive constant C:

$$e_n(C_{\lambda_\theta}) \leq C\,\exp\big[-\big((n/k) + b k^{1/2}\big)\big]\,.$$

Taking $k = [n^{2/3}]$ gives the claimed upper bound. The lower bound is proved similarly, using the left inequality in Theorem 2.2, since we know ([12]) that $a_k \geq a'\,\mathrm{e}^{-b'\sqrt{k}}$. □

We refer to [11, Section 4.1] for the definition of the cusp map χ. We have:

Theorem 3.4. *Let χ be the cusp map. Then, with positive constants a, b, a', b':*

$$a' \, \mathrm{e}^{-b'\sqrt{n/\log n}} \leq e_n(C_\chi) \leq a \, \mathrm{e}^{-b\sqrt{n/\log n}} . \tag{3.9}$$

Proof. We proved in [11] that:

$$a' \, \mathrm{e}^{-b' k/\log k} \leq a_k(C_\chi) \leq a \, \mathrm{e}^{-bk/\log k} . \tag{3.10}$$

The proof then follows the same lines as in Theorem 3.3, with the choice $k = [\sqrt{n \log n}]$. □

4 The multidimensional case

4.1 General results

Let $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$ be an analytic map. We will say that φ is non-degenerate if $\varphi(\mathbb{D}^N)$ has non-empty interior, equivalently if $\det \varphi'(z) \neq 0$ for at least one point $z \in \mathbb{D}^N$.

Let now $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$ be a non-degenerate analytic map inducing a bounded composition operator $C_\varphi \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$ (this is not always the case as soon as $N > 1$, even if φ is injective and hence non-degenerate, see for example [5, p. 246], when the polydisk is replaced by the ball; but similar examples exist for the polydisk). Assume moreover that C_φ is a compact operator.

Theorem 4.1. *Let $C_\varphi \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$ be a compact composition operator, with φ non-degenerate. We have:*

1) $e_n(C_\varphi) \geq c \exp\big(-C\, n^{\frac{1}{N+1}}\big)$*, for some constants $C > c > 0$, depending on φ;*

2) *if $\|\varphi\|_\infty < 1$, then $e_n(C_\varphi) \leq C \exp\big(-c\, n^{\frac{1}{N+1}}\big)$, with $C > c > 0$ depending on φ.*

Proof. 1) It is proved in [1, Theorem 3.1] that, for a non-degenerate map φ, it holds:

$$a_k(C_\varphi) \geq a' \, \mathrm{e}^{-b' k^{1/N}} .$$

As in the previous section, it follows from Theorem 2.2, that $(a_1 \cdots a_k)^{1/k} \geq \mathrm{e}^{-b'' k^{1/N}}$, and then, taking $k = [n^{N/(N+1)}]$, that:

$$e_n(C_\varphi) \geq c\, \mathrm{e}^{-C n^{1/(N+1)}} .$$

2) Similarly, for $\|\varphi\|_\infty < 1$, it is proved in [1, Theorem 5.2] that:

$$a_k(C_\varphi) \leq C\, \mathrm{e}^{-c k^{1/N}} ;$$

and we get the result from Theorem 2.2. □

Those estimates motivate the introduction of the parameter:

$$\gamma_N(C_\varphi) = \lim_{n\to\infty} \big[e_n(C_\varphi)\big]^{\frac{1}{n^{1/(N+1)}}} . \tag{4.1}$$

We define similarly $\gamma_N^{\pm}(C_\varphi)$, and will say more on it in next section.

4.2 Specific results

4.2.1 Multi-lens maps

Let λ_θ be lens maps with parameter θ. We define the multi-lens map Λ_θ of parameter θ on the polydisk $\mathbb{D}^N$ as:

$$\Lambda_\theta(z_1, \ldots, z_N) = \big(\lambda_\theta(z_1), \lambda_\theta(z_2), \ldots, \lambda_\theta(z_N)\big) , \tag{4.2}$$

for $(z_1, \ldots, z_N) \in \mathbb{D}^N$.

The following result is proved in [1, Theorem 6.1].

Theorem 4.2. *Let Λ_θ be the multi-lens map with parameter θ. Then, for positive constants a, b, a', b' depending only on θ and N, one has:*

$$a'\, \mathrm{e}^{-b' n^{1/(2N)}} \leq a_n(C_{\Lambda_\theta}) \leq a\, \mathrm{e}^{-b\, n^{1/(2N)}} . \tag{4.3}$$

The version of Theorem 4.2 for entropy numbers, stated without proof, is:

Theorem 4.3. *Let Λ_θ be the multi-lens map with parameter θ. Then:*

$$a' \exp\left(-b' n^{1/(2N+1)}\right) \leq e_n(C_{\Lambda_\theta}) \leq a \exp\left(-b\, n^{1/(2N+1)}\right) . \tag{4.4}$$

4.2.2 Multi-cusp maps

Let $\chi\colon \mathbb{D} \to \mathbb{D}$ be the cusp map and $\varphi\colon \mathbb{D}^N \to \mathbb{D}^N$ be the multi-cusp map defined by:

$$\Xi\,(z_1, \ldots, z_N) = \big(\chi(z_1), \chi(z_2), \ldots, \chi(z_N)\big)\,. \tag{4.5}$$

It is proved in [1, Theorem 6.2]:

Theorem 4.4. *Let $\chi\colon \mathbb{D} \to \mathbb{D}$ be the cusp map and $\Xi\colon \mathbb{D}^N \to \mathbb{D}^N$ be the multi-cusp map. Then:*

$$a'\,\mathrm{e}^{-b'\,n^{1/N}/\log n} \leq a_n(C_\Xi) \leq a\,\mathrm{e}^{-b\,n^{1/N}/\log n}\,, \tag{4.6}$$

where a, b, a', b' are positive constants depending only on N.

The version of Theorem 4.4 for entropy numbers, stated without proof, is:

Theorem 4.5. *let $\chi\colon \mathbb{D} \to \mathbb{D}$ be the cusp map and $\Xi\colon \mathbb{D}^N \to \mathbb{D}^N$ be the multi-cusp map. Then:*

$$\begin{aligned} &a'\,\exp\big[-b'\,n^{1/(N+1)}\,(\log n)^{-N/(N+1)}\big] \\ &\qquad\qquad \leq e_n(C_\Xi) \leq a\,\exp\big[-b\,n^{1/(N+1)}\,(\log n)^{-N/(N+1)}\big]\,. \end{aligned} \tag{4.7}$$

5 Connections with pluricapacity and Zakharyuta's results

Here, in dimension $N \geq 2$, the situation is satisfactory for upper bounds (see [13]); for lower bounds, see [14]. The notion involved is now that of pluricapacity, or Monge-Ampère capacity, coined by Bedford and Taylor in [2]. More precisely, if A is a Borel subset of $\mathbb{D}^N$, we refer to [13] or [14] for the definition of its pluricapacity $\mathrm{Cap}_N\,(A)$, belonging to $[0, +\infty]$, and set:

$$\tau_N(A) = \frac{1}{(2\pi)^N}\,\mathrm{Cap}_N\,(A) \tag{5.1}$$

$$\Gamma_N(A) = \exp\bigg[-\Big(\frac{N!}{\tau_N(A)}\Big)^{1/N}\bigg] \tag{5.2}$$

$$\beta_N^+(T) = \limsup_{n\to\infty}\,\big[a_n(T)\big]^{1/n^{1/N}}\,. \tag{5.3}$$

We temporarily assume that $\|\varphi\|_\infty < 1$ so that $K = \overline{\varphi(\mathbb{D}^N)}$ is a compact subset of $\mathbb{D}^N$. We proved in [13, Theorem 6.4], relying on positive results of Nivoche ([16]) and Zaharyuta ([22, Proposition 6.1]) on the so-called Kolmogorov conjecture, that:

Theorem 5.1. *It holds:*

$$\beta_N^+(C_\varphi) \leq \Gamma_N(K)\,. \tag{5.4}$$

We have the following result, which extends the previous result in dimension 1.

Theorem 5.2. *The following upper bound holds:*

$$\gamma_N^+(C_\varphi) \leq \exp\big(-\beta_N \rho^{N/(N+1)}\big)\,, \tag{5.5}$$

where:

$$\rho = \bigg(\frac{N!}{\tau_N(K)}\bigg)^{1/N} = 2\pi\bigg(\frac{N!}{\mathrm{Cap}_N\,(K)}\bigg)^{1/N}, \tag{5.6}$$

and

$$\begin{aligned}\beta_N = \bigg(\frac{N}{N+1}\bigg)^{N/(N+1)} & \big(N^{-N/(N+1)} + N^{1/(N+1)}\big) \\ & \geq \mathrm{e}^{-1/(N+1)} N^{1/(N+1)}\,.\end{aligned} \tag{5.7}$$

Proof. Abbreviate $a_n(C_\varphi)$ and $e_n(C_\varphi)$ to a_n and e_n, and set $\alpha = N/(N+1)$. Let $\varepsilon > 0$. Theorem 5.1 implies:

$$a_k \leq C_\varepsilon\, \mathrm{e}^{\varepsilon k^{1/N}}\, \mathrm{e}^{-\rho k^{1/N}}\,,$$

so:

$$(a_1 \cdots a_k)^{1/k} \leq C_\varepsilon\, \mathrm{e}^{\varepsilon k^{1/N}} \mathrm{e}^{-\rho\alpha k^{1/N}}\,.$$

Apply once more Theorem 2.2 to obtain:

$$e_n \leq C_\varepsilon \sup_{k \geq 1} \mathrm{e}^{\varepsilon k^{1/N}} \exp\big[-(n/k + \rho\alpha k^{1/N})\big]\,.$$

The supremum is essentially attained for k the integral part of $(N/\rho\alpha)^\alpha n^\alpha$ and then, in view of (5.7) and $\alpha/N = 1 - \alpha$, up to a negligible term:

$$\begin{aligned}\frac{n}{k} + \rho\,\alpha\, k^{1/N} &= n^{1-\alpha}\bigg(\frac{\rho\,\alpha}{N}\bigg)^{\alpha} + \rho\,\alpha\, n^{1-\alpha}\bigg(\frac{N}{\rho\alpha}\bigg)^{1-\alpha} \\ &= n^{1-\alpha}(\rho\,\alpha)^\alpha (N^{-\alpha} + N^{1-\alpha})\,.\end{aligned}$$

Finally,

$$e_n \leq C_\varepsilon\, \mathrm{e}^{\varepsilon n^{1-\alpha}} \exp\big(-\beta_N \rho^\alpha n^{1-\alpha}\big) = C_\varepsilon\, \mathrm{e}^{\varepsilon n^{1/(N+1)}} \exp\big(-\beta_N \rho^\alpha n^{1/(N+1)}\big)\,.$$

This clearly ends the proof of Theorem 5.2. □

Remark. We have so far no sharp lower bound for entropy numbers, at least when $\|\varphi\|_\infty = 1$, since we already fail to have one in general for approximation numbers (see however [14]).

Besides, let $J\colon H^\infty(\mathbb{D}^N) \to \mathcal{C}(K)$ be the canonical embedding, when $K \subseteq \mathbb{D}^N$ is a "condenser", namely a compact subset of $\mathbb{D}^N$ such that any bounded analytic function on $\mathbb{D}^N$ which vanishes on K vanishes identically, which is moreover "regular". The positive solution to the Kolmogorov conjecture can be expressed in terms of the Kolmogorov numbers $d_n(J)$ of J or equivalently, in terms of the entropy numbers $e_n(J)$ of J ([21, Theorem 5], generalizing Erokhin's result in dimension 1 appearing in his posthumous paper [6] and methods due to Mityagin [15] and Levin and Tikhomirov [9]; see also [22, Lemma 2.2]). The result is that, taking $K = \overline{\varphi(\mathbb{D}^N)}$, one has, with sharp constants c_K, c'_K depending on the pluricapacity of K in $\mathbb{D}^N$:

$$d_n(J) \approx \mathrm{e}^{-c_K\, n^{1/N}} \quad \text{and} \quad e_n(J) \approx \mathrm{e}^{-c'_K\, n^{1/(N+1)}} . \tag{5.8}$$

This jump from the exponent $1/N$ to the exponent $1/(N+1)$ is reflected in our Theorem 5.2, through the new parameter γ_N^+.

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Daniel Li
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec
Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille.fr

Luis Rodríguez-Piazza
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n
41 012 SEVILLA, SPAIN
piazza@us.es