Pluricapacity and approximation numbers of composition operators

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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September 20, 2018

Abstract. For suitable bounded hyperconvex sets Ω in $\mathbb{C}^N$, in particular the ball or the polydisk, we give estimates for the approximation numbers of composition operators $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$ when $\varphi(\Omega)$ is relatively compact in Ω, involving the Monge-Ampère capacity of $\varphi(\Omega)$.

Key-words: approximation numbers ; composition operator; Hardy space ; hyperconvex domain ; Monge-Ampère capacity ; pluricapacity ; pluripotential theory ; Zakharyuta conjecture

MSC 2010 numbers – Primary: 47B33 – *Secondary*: 30H10 – 31B15 – 32A35 – 32U20 – 41A25 – 41A35 – 46B28 – 46E20 – 47B06

1 Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, $H^2(\mathbb{D})$ the corresponding Hardy space, φ a non-constant analytic self-map of $\mathbb{D}$ and $C_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ the associated composition operator. In [40], we proved a formula connecting the approximation numbers $a_n(C_\varphi)$ of C_φ, and the Green capacity of the image $\varphi(\mathbb{D})$ in $\mathbb{D}$, namely, when $\overline{[\varphi(\mathbb{D})]} \subset \mathbb{D}$, we have:

$$(1.1) \qquad \beta(C_\varphi) := \lim_{n\to\infty} [a_n(C_\varphi)]^{1/n} = \exp\big(-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]\big)\,,$$

where $\mathrm{Cap}\,[\varphi(\mathbb{D})]$ is the *Green capacity* of $\varphi(\mathbb{D})$.

A non-trivial consequence of that formula was the following:

$$(1.2) \qquad \|\varphi\|_\infty = 1 \quad \Longrightarrow \quad a_n(C_\varphi) \geq \delta\, \mathrm{e}^{-n\varepsilon_n} \text{ where } \varepsilon_n \to 0_+\,.$$

In other terms, as soon as $\|\varphi\|_\infty = 1$, we cannot hope better for the numbers $a_n(C_\varphi)$ than a subexponential decay, however slowly ε_n tends to 0.

In [41], we pursued that line of investigation in dimension $N \geq 2$, namely on $H^2(\mathbb{D}^N)$, and showed that in some cases the implication (1.2) still holds ([41, Theorem 3.1]):

$$(1.3) \qquad \|\varphi\|_\infty = 1 \quad \Longrightarrow \quad a_n(C_\varphi) \geq \delta\, \mathrm{e}^{-n^{1/N}\varepsilon_n} \text{ where } \varepsilon_n \to 0_+\,,$$

(the substitution of n by $n^{1/N}$ is mandatory as shown by the results of [4]).

We show in this paper that, in general, for non-degenerate symbols, we have similar formulas to (1.1) at our disposal for the parameters:

$$\text{(1.4)} \quad \beta_N^-(C_\varphi) = \liminf_{n\to\infty} [a_{n^N}(C_\varphi)]^{1/n} \quad \text{and} \quad \beta_N^+(C_\varphi) = \limsup_{n\to\infty} [a_{n^N}(C_\varphi)]^{1/n} .$$

These bounds are given in terms of the Monge-Ampère (or Bedford-Taylor) capacity of $\varphi(\mathbb{D}^N)$ in $\mathbb{D}^N$, a notion which is the natural multidimensional extension of the Green capacity when the dimension N is ≥ 2 ([41, Theorem 6.4]). We show that we have $\beta_N^-(C_\varphi) = \beta_N^+(C_\varphi)$ for well-behaved symbols.

2 Notations and background

2.1 Complex analysis

Let Ω be a domain in $\mathbb{C}^N$; a function $u\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is said *plurisubharmonic* (*psh*) if it is u.s.c. and if for every complex line $L = \{a + zw\,;\ z \in \mathbb{C}\}$ ($a \in \Omega$, $w \in \mathbb{C}^N$), the function $z \mapsto u(a + zw)$ is subharmonic in $\Omega \cap L$. We denote $\mathcal{PSH}(\Omega)$ the set of plurisubharmonic functions in Ω. If $f\colon \Omega \to \mathbb{C}$ is holomorphic, then $\log |f|$ and $|f|^\alpha$, $\alpha > 0$, are *psh*. Every real-valued convex function is *psh* (convex functions are those whose composition with all $\mathbb{R}$-linear isomorphisms are subharmonic, though plurisubharmonic functions are those whose composition with all $\mathbb{C}$-linear isomorphisms are subharmonic: see [30, Theorem 2.9.12]).

Let $dd^c = 2i\partial\bar\partial$, and $(dd^c)^N = dd^c \wedge \cdots \wedge dd^c$ (N times). When $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^2(\Omega)$, we have:

$$(dd^c u)^N = 4^N N! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) d\lambda_{2N}(z) ,$$

where $d\lambda_{2N}(z) = (i/2)^N dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_N \wedge d\bar z_N$ is the usual volume in $\mathbb{C}^N$. In general, the current $(dd^c u)^N$ can be defined for all locally bounded $u \in \mathcal{PSH}(\Omega)$ and is actually a positive measure on Ω ([5]).

Given $p_1, \ldots, p_J \in \Omega$, the pluricomplex Green function with poles $p_1, \ldots, p_J$ and weights $c_1, \ldots, c_N > 0$ is defined as:

$$g(z) = g(z, p_1, \ldots, p_J) = \sup\{v(z)\,;\ v \in \mathcal{PSH}(\Omega)\,, v \leq 0 \text{ and } v(z) \leq c_j \log|z - p_j| + \mathrm{O}\,(1)\,, \forall j = 1, \ldots, J\} .$$

In particular, for $J = 1$ and $p_1 = a$, $c_1 = 1$, $g(\,\cdot\,, a)$ is the *pluricomplex Green function* of Ω with pole $a \in \Omega$. If $0 \in \Omega$ and $a = 0$, we denote it by g_Ω and call it the *pluricomplex Green function of* Ω; hence:

$$g_a(z) = g(z, a) = \sup\{u(z)\,;\ u \in \mathcal{PSH}(\Omega)\,,\ u \leq 0 \text{ and } u(z) \leq \log|z - a| + \mathrm{O}\,(1)\} .$$

Let Ω be an open subset of $\mathbb{C}^N$. A continuous function $\rho\colon \Omega \to \mathbb{R}$ is an exhaustion function if there exists $a \in (-\infty, +\infty]$ such that $\rho(z) < a$ for all $z \in \Omega$, and the set $\Omega_c = \{z \in \Omega\,;\ \rho(z) < c\}$ is relatively compact in Ω for every $c < a$.

A domain Ω in $\mathbb{C}^N$ is said *hyperconvex* if there exists a continuous *psh* exhaustion function $\rho\colon \Omega \to (-\infty, 0)$ (see [30, p. 80]). We may of course replace the upper bound 0 by any other real number. Without this upper bound, Ω is said *pseudoconvex.*

Let Ω be a hyperconvex domain, with negative continuous *psh* exhaustion function ρ and $\mu_{\rho,r}$ the associated Demailly-Monge-Ampère measures, defined as:

$$\mu_{u,r} = (dd^c u_r)^N - \mathbb{1}_{\Omega \setminus B_{\Omega,u}(r)} (dd^c u)^N \,, \tag{2.1}$$

for $r < 0$, where $u_r = \max(u, r)$ and:

$$B_{\Omega,u}(r) = \{z \in \Omega\,;\ u(z) < r\}\,.$$

The nonnegative measure $\mu_{u,r}$ is supported by $S_{\Omega,u}(r) := \{z \in \Omega\,;\ u(z) = r\}$.

If:

$$\int_\Omega (dd^c\rho)^N < \infty\,,$$

these measures, considered as measures on $\overline{\Omega}$, weak-$*$ converge, as r goes to 0, to a positive measure $\mu = \mu_{\Omega,\rho}$ supported by $\partial\Omega$ and with total mass $\int_\Omega (dd^c\rho)^N$ ([16, Théorème 3.1], or [30, Lemma 6.5.10]).

For the pluricomplex Green function g_a with pole a, we have $(dd^c g_a)^N = (2\pi)^N \delta_a$ ([16, Théorème 4.3]) and $g_a(a) = -\infty$, so $a \in B_{\Omega,g_a}(r)$ for every $r < 0$ and $\mathbb{1}_{\Omega\setminus B_{\Omega,g_a}(r)} (dd^c g_a)^N = 0$. Hence the Demailly-Monge-Ampère mesure $\mu_{g_a,r}$ is equal to $\big(dd^c(g_a)_r\big)^N$. By [51, Lemma 1], we have $(1/|r|)\,\big(dd^c(g_a)_r\big)^N = u_{\bar{B}_{\Omega,g_a}(r),\Omega}$, the relative extremal function of $\bar{B}_{\Omega,g_a}(r) = \{z \in \Omega\,;\ g_a(z) \leq r\}$ in Ω (see (3.2) for the definition), and this measure is supported, not only by $S_{\Omega,g_a}(r)$, but merely by the Shilov boundary of $\bar{B}_{\Omega,g_a}(r)$ (see Section 2.2.1 for the definition).

Since $(dd^c g_a)^N = (2\pi)^N\delta_a$ has mass $(2\pi)^N < \infty$, these measures weak-$*$ converge, as r goes to 0, to a positive measure $\mu = \mu_{\Omega,g_a}$ supported by $\partial\Omega$ with mass $(2\pi)^N$. Demailly ([16, Définition 5.2] call the measure $\frac{1}{(2\pi)^N}\,\mu_{\Omega,g_a}$ the *pluriharmonic measure of* a. When Ω is balanced ($az \in \Omega$ for every $z \in \Omega$ and $|a| = 1$), the support of this pluriharmonic measure is the Shilov boundary of $\overline{\Omega}$ ([51, very end of the paper]).

A *bounded symmetric domain* of $\mathbb{C}^N$ is a bounded open and convex subset Ω of $\mathbb{C}^N$ which is circled ($az \in \Omega$ for $z \in \Omega$ and $|a| \leq 1$) and such that for every point $a \in \Omega$, there is an involutive bi-holomorphic map $\gamma\colon \Omega \to \Omega$ such that a is an isolated fixed point of γ (equivalently, $\gamma(a) = a$ and $\gamma'(a) = -id$: see [52, Proposition 3.1.1]). For this definition, see [13, Definition 16 and Theorem 17], or [14, Definition 5 and Theorem 4]. Note that the convexity is automatic

(Hermann Convexity Theorem; see [27, p. 503 and Corollary 4.10]). É. Cartan showed that every bounded symmetric domain of $\mathbb{C}^N$ is homogeneous, i.e. the group Γ of automorphisms of Ω acts transitively on Ω: for every $a, b \in \Omega$, there is an automorphism γ of Ω such that $\gamma(a) = b$ (see [52, p. 250]). Conversely, every homogeneous bounded convex domain is symmetric, since $\sigma(z) = -z$ is a symmetry about 0 (see [52, p. 250] or [26, Remark 2.1.2 (e)]). Moreover, each automorphism extends continuously to $\overline{\Omega}$ (see [22]).

The unit ball $\mathbb{B}_N$ and the polydisk $\mathbb{D}^N$ are examples of bounded symmetric domains. Another example is, for $N = p\,q$, bi-holomorphic to the open unit ball of $\mathcal{M}(p,q) = \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p)$ for the operator norm (see [27, Theorem 4.9]). Every product of bounded symmetric domains is still a bounded symmetric domain. In particular, every product of balls $\Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}$, $l_1 + \cdots + l_m = N$, is a bounded symmetric domain.

If Ω is a bounded symmetric domain, its gauge is a norm $\|\,.\,\|$ on $\mathbb{C}^N$ whose open unit ball is Ω. Hence every bounded symmetric domain is hyperconvex (take $\rho(z) = \|z\| - 1$).

2.2 Hardy spaces on hyperconvex domains

2.2.1 Hardy spaces on bounded symmetric domains

We begin by defining the Hardy space on a bounded symmetric domain, because this is easier.

The *Shilov boundary* (also called the Bergman-Shilov boundary or the distinguished boundary) $\partial_S\Omega$ of a bounded domain Ω is the smallest closed set $F \subseteq \partial\Omega$ such that $\sup_{z\in\overline{\Omega}} |f(z)| = \sup_{z\in F} |f(z)|$ for every function f holomorphic in some neighborhood of $\overline{\Omega}$ (see [13, § 4.1]).

When Ω is a bounded symmetric domain, it is also, since Ω is convex, the Shilov boundary of the algebra $A(\Omega)$ of the continuous functions on $\overline{\Omega}$ which are holomorphic in Ω (because every function $f \in A(\Omega)$ can be approximated by f_ε with $f_\varepsilon(z) = f\big(\varepsilon z_0 + (1-\varepsilon)z\big)$, where $z_0 \in \Omega$ is given: see [20, pp. 152–154]).

The Shilov boundary of the ball $\mathbb{B}_N$ is equal to its topological boundary, but the Shilov boundary of the bidisk is $\partial_S\mathbb{D}^2 = \{(z_1,z_2) \in \mathbb{C}^2\,;\ |z_1| = |z_2| = 1\}$, whereas, its usual boundary $\partial\mathbb{D}^2$ is $(\mathbb{T} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \mathbb{T})$; for the unit ball B_N, the Shilov boundary is equal to the usual boundary $\mathbb{S}^{N-1}$ ([13, § 4.1]). Another example of a bounded symmetric domain, in $\mathbb{C}^3$, is the set $\Omega = \{(z_1,z_2,z_3) \in \mathbb{C}^3\,;\ |z_1|^2 + |z_2|^2 < 1\,,\ |z_3| < 1\}$ and its Shilov boundary is $\partial_S\Omega = \{(z_1,z_2,z_3)\,;\ |z_1|^2 + |z_2|^2 = 1\,,\ |z_3| = 1\}$. For $p \geq q$, the matrix A is in the topological boundary of $\mathcal{M}(p,q)$ if and only if $\|A\| = 1$, but A is in the Shilov boundary if and only if $A^*A = I_q$; therefore the two boundaries coincide if and only if $q = 1$, i.e. $\Omega = \mathbb{B}_N$ (see [14, Example 2, p. 30]).

Equivalently (see [24, Corollary 9], or [13, Theorem 33], [14, Theorem 10]), $\partial_S\Omega$ is the set of the extreme points of the convex set $\overline{\Omega}$.

The Shilov boundary $\partial_S\Omega$ is invariant by the group Γ of automorphisms of Ω and the subgroup $\Gamma_0 = \{\gamma \in \Gamma\,;\ \gamma(0) = 0\}$ act transitively on $\partial_S\Omega$ (see [22]). A theorem of H. Cartan states that the elements of Γ_0 are linear transformations of $\mathbb{C}^N$ and commute with the rotations (see [24, Theorem 1] or [26, Proposition 2.1.8]). It follows that the Shilov boundary of a bounded symmetric domain Ω coincides with its topological boundary only for $\Omega = \mathbb{B}_N$ (see [35, p. 572] or [36, p. 367]); in particular the open unit ball of $\mathbb{C}^N$ for the norm $\| \, . \, \|_p$, $1 < p < \infty$, is never a bounded symmetric domain, unless $p = 2$.

The unique Γ_0-invariant probability measure σ on $\partial_S\Omega$ is the normalized surface area (see [22]). Then the *Hardy space* $H^2(\Omega)$ is the space of all complex-valued holomorphic functions f on Ω such that:

$$\|f\|_{H^2(\Omega)} := \left(\sup_{0<r<1} \int_{\partial_S\Omega} |f(r\xi)|^2 \, d\sigma(\xi) \right)^{1/2}$$

is finite (see [22] and [23]). It is known that the integrals in this formula are non-decreasing as r increases to 1, so we can replace the supremum by a limit. The same definition can be given when Ω is a bounded complete Reinhardt domain (see [1]).

The space $H^2(\Omega)$ is a Hilbert space (see [22, Theorem 5]) and for every $z \in \Omega$, the evaluation map $f \in H^2(\Omega) \mapsto f(z)$ is uniformly bounded on compacts subsets of Ω, by a depending only on that compact set, and of Ω ([22, Lemma 3]).

For every $f \in H^2(\Omega)$, there exists a boundary values function f^* such that $\|f_r - f^*\|_{L^2(\partial_S\Omega)} \underset{r\to 1}{\longrightarrow} 0$, where $f_r(z) = f(rz)$ ([9, Theorem 3]), and the map $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial_S\Omega)$ is an isometric embedding ([22, Theorem 6]).

2.2.2 Hardy spaces on hyperconvex domains

For hyperconvex domains, the definition of Hardy spaces is more involved. It was done by E. Poletsky and M. Stessin ([47, Theorem 6]). Those domains are associated to a continuous negative *psh* exhaustion function u on Ω and the definition of the Hardy spaces uses the Demailly-Monge-Ampère measures. The space $H^2_u(\Omega)$ is the space of all holomorphic functions $f\colon \Omega \to \mathbb{C}$ such that:

$$\sup_{r<0} \int_{S_{\Omega,u}(\Omega)} |f|^2 \, d\mu_{u,r} < \infty$$

and its norm is defined by:

$$\|f\|_{H^2_u(\Omega)} = \sup_{r<0} \left(\frac{1}{(2\pi)^N} \int_{S_{\Omega,u}(\Omega)} |f|^2 \, d\mu_{u,r} \right)^{1/2} .$$

We can replace the supremum by a limit since the integrals are non-decreasing as r increases to 0 ([16, Corollaire 1.9].

The space $H^\infty(\Omega)$ of bounded holomorphic functions in Ω is contained in $H^2_u(\Omega)$ (see [47], remark before Lemma 3.4).

These spaces $H^2_u(\Omega)$ are Hilbert spaces ([47, Theorem 4.1]), but depends on the exhaustion function u (even when $N = 1$: see for instance [49]). Nevertheless, they all coincide, with equivalent norms, for the functions u for which the measure $(dd^c u)^N$ is compactly supported ([47, Lemma 3.4]); this is the case when $u(z) = g(z, a)$ is the pluricomplex Green function with pole $a \in \Omega$ (because then $(dd^c u)^N = (2\pi)^N \delta_a$: see [16, Théorème 4.3], or [30, Theorem 6.3.6]).

When Ω is the ball $\mathbb{B}_N$ and $u(z) = \log \|z\|_2$, then $(dd^c u)^N = C\,\delta_0$ and $\mu_{u,r} = (2\pi)^N d\sigma_t$, where $d\sigma_t$ is the normalized surface area on the sphere of radius $t := \mathrm{e}^r$ (see [47, Section 4] or [17, Example 3.3]). When Ω is the polydisk $\mathbb{D}^N$ and $u(z) = \log \|z\|_\infty$, then $(dd^c u)^N = (2\pi)^N \delta_0$ ([18, Corollary 5.4]) and $\frac{1}{(2\pi)^N}\,\mu_{u,r}$ is the Lebesgue measure of the torus $r\mathbb{T}^N$ (see [17, Example 3.10]). Note that in [17] and [18], the operator d^c is defined as $\frac{i}{2\pi}(\bar\partial - \partial)$ instead of $i(\bar\partial - \partial)$, as usually used.

In these two cases, the Hardy spaces are the same as the usual ones (see [2, Remark 5.2.1]).

In the sequel, we only consider the exhausting function $u = g_\Omega$; hence we will write $B_\Omega(r)$, $S_\Omega(r)$ and $H^2(\Omega)$ instead of $B_{\Omega,u}(r)$, $S_{\Omega,u}(r)$ and $H^2_u(\Omega)$.

The two notions of Hardy spaces for a bounded symmetric domain are the same:

Proposition 2.1. *Let Ω be a bounded symmetric domain in $\mathbb{C}^N$. Then the Hardy space $H^2(\Omega)$ coincides with the subspace of the Poletski-Stessin Hardy space $H_{g_\Omega}(\Omega)$, with equality of the norms.*

Proof. First let us note that if $\|\,.\,\|$ is the norm whose open unit ball is Ω, then $g_\Omega(z) = \log \|z\|$ (see [7, Proposition 3.3.2]).

Let μ_Ω be the measure which is the $*$-weak limit of the Demailly-Monge-Ampère measures $\mu_r = \big(dd^c (g_\Omega)_r\big)^N$. We saw that it is supported by $\partial_S \Omega$. By the remark made in [16, pp. 536-537], since the automorphisms of Ω continuously extend on $\partial\Omega$, the measure μ_Ω is Γ-invariant. By unicity, the harmonic measure $\widetilde{\mu}_\Omega = (2\pi)^{-N} \mu_\Omega$ at 0 hence coincides with the normalized area measure on $\partial_S \Omega$. We have, for $f\colon \Omega \to \mathbb{C}$ holomorphic and $0 < s < 1$:

$$\int_{\partial_S \Omega} |f(sz)|^2 \, d\widetilde{\mu}_\Omega(z) = \int_{\partial\Omega} |f(sz)|^2 \, d\widetilde{\mu}_\Omega(z) = \lim_{r\to 0} \frac{1}{(2\pi)^N} \int_{S_\Omega(r)} |f(sz)|^2 \, d\mu_r(z) \,,$$

because $z \mapsto |f(sz)|^2$ is continuous on $\overline{\Omega}$. Now, since $g_\Omega(z) = \log \|z\|$, we have $S_\Omega(r) = \mathrm{e}^r \partial\Omega$ and $(g_\Omega)_r(z) + t = (g_\Omega)_{r+t}(sz)$; hence $\mu_r(sA) = \mu_{r+t}(A)$ for every Borel subset A of $\partial\Omega$, where $t = \log s$. It follows that:

$$\int_{S_\Omega(r)} |f(sz)|^2 \, d\mu_r(z) = \int_{S_\Omega(r+t)} |f(\zeta)|^2 \, d\mu_{r+t}(\zeta) \,.$$

By letting r and t going to 0, we get:

$$\|f\|^2_{H^2(\Omega)} = \lim_{r,t\to 0} \frac{1}{(2\pi)^N} \int_{S_\Omega(r+t)} |f(\zeta)|^2 \, d\mu_{r+t}(\zeta) = \|f\|^2_{H^2_{g_\Omega}} \,;$$

hence $f \in H^2(\Omega)$ if and only if $f \in H^2_{g_0}(\Omega)$, with the same norms. $\square$

We have ([47, Theorem 3.6]):

Proposition 2.2 (Poletsky-Stessin). *For every $z \in \Omega$, the evaluation map $f \in H^2(\Omega) \mapsto f(z)$ is uniformly bounded on compacts subsets of Ω, by a constant depending only on that compact set, and of Ω.*

Hence $H^2(\Omega)$ has a *reproducing kernel*, defined by:

$$f(a) = \langle f, K_a \rangle \,, \quad \text{for } f \in H^2(\Omega)\,, \tag{2.2}$$

and for each $r < 0$:

$$L_r := \sup_{a \in \overline{B_\Omega(r)}} \|K_a\|_2 < \infty\,. \tag{2.3}$$

2.3 Composition operators

A Schur map, associated with the bounded hyperconvex domain Ω, is a *non-constant* analytic map of Ω into itself. It is said *non degenerate* if its Jacobian is not identically null. It is equivalent to say that the differential $\varphi'(a)\colon \mathbb{C}^N \to \mathbb{C}^N$ is an invertible linear map for at least one point $a \in \Omega$. In [4], we used the terminology *truly N-dimensional*. Then, by the implicit function theorem, this is equivalent to saying that $\varphi(\Omega)$ has non-void interior. We say that the Schur map φ is a *symbol* if it defines a *bounded* composition operator $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ by $C_\varphi(f) = f \circ \varphi$.

Let us recall that although any Schur function generates a bounded composition operator on $H^2(\mathbb{D})$, this is no longer the case on $H^2(\mathbb{D}^N)$ as soon as $N \geq 2$, as shown for example by the Schur map $\varphi(z_1, z_2) = (z_1, z_1)$. Indeed (see [3]), if say $N = 2$, taking $f(z) = \sum_{j=0}^n z_1^j z_2^{n-j}$, we see that:

$$\|f\|_2 = \sqrt{n+1} \quad \text{while} \quad \|C_\varphi f\|_2 = \|(n+1) z_1^n\|_2 = n+1\,.$$

The same phenomenon occurs on $H^2(\mathbb{B}_N)$ ([43]; see also [11], [12], and [15]; see also [47]).

2.4 s-numbers of operators on a Hilbert space

We begin by recalling a few operator-theoretic facts. Let H be a Hilbert space. The approximation numbers $a_n(T) = a_n$ of an operator $T\colon H \to H$ are defined as:

$$a_n = \inf_{\operatorname{rank} R < n} \|T - R\|\,, \qquad n = 1, 2, \ldots \tag{2.4}$$

The operator T is compact if and only if $\lim_{n\to\infty} a_n(T) = 0$.

According to a result of Allahverdiev [10, p. 155], $a_n = s_n$, the n-th singular number of T, i.e. the n-th eigenvalue of $|T| := \sqrt{T^*T}$ when those eigenvalues are rearranged in non-increasing order.

The n-th width $d_n(K)$ of a subset K of a Banach space Y measures the defect of flatness of K and is by definition:

$$d_n(K) = \inf_{\dim E < n} \Big[\sup_{f \in K} \operatorname{dist}(f, E) \Big]\,, \tag{2.5}$$

where E runs over all subspaces of Y with dimension $< n$ and where $\operatorname{dist}(f, E)$ denotes the distance of f to E. If $T\colon X \to Y$ is an operator between Banach spaces, the n-th Kolmogorov number $d_n(T)$ of T is the nth-width in Y of $T(B_X)$ where B_X is the closed unit ball of X, namely:

$$d_n(T) = \inf_{\dim E < n} \left[\sup_{f \in B_X} \operatorname{dist}(Tf, E) \right] . \tag{2.6}$$

In the case where $X = Y = H$, a Hilbert space, we have:

$$a_n(T) = d_n(T) \quad \text{for all } n \geq 1 \, , \tag{2.7}$$

and ([40]) the following alternative definition of $a_n(T)$:

$$a_n(T) = \inf_{\dim E < n} \left[\sup_{f \in B_H} \operatorname{dist}(Tf, TE) \right] . \tag{2.8}$$

In this work, we use, for an operator $T\colon H \to H$, the following notation:

$$\beta_N^-(T) = \liminf_{n\to\infty} [a_{n^N}(T)]^{1/n} \tag{2.9}$$

and:

$$\beta_N^+(T) = \limsup_{n\to\infty} [a_{n^N}(T)]^{1/n} \, . \tag{2.10}$$

When these two quantities are equal, we write them $\beta_N(T)$.

3 Pluripotential theory

3.1 Monge-Ampère capacity

Let K be a compact subset of an open subset Ω of $\mathbb{C}^N$. The *Monge-Ampère capacity* of K has been defined by Bedford and Taylor ([5]; see also [30, Part II, Chapter 1]) as:

$$\operatorname{Cap}(K) = \sup \left\{ \int_K (dd^c u)^N \, ; \ u \in \mathcal{PSH}(\Omega) \text{ and } 0 \leq u \leq 1 \text{ on } \Omega \right\} .$$

When Ω is bounded and hyperconvex, we have a more convenient formula ([5, Proposition 5.3], [30, Proposition 4.6.1]):

$$\operatorname{Cap}(K) = \int_\Omega (dd^c u_K^*)^N = \int_K (dd^c u_K^*)^N \, , \tag{3.1}$$

(the positive measure $(dd^c u_K^*)^N$ is supported by K; actually by ∂K: see [17, Properties 8.1 (c)]), where $u_K = u_{K,\Omega}$ is the *relative extremal function* of K, defined, for any subset $E \subseteq \Omega$, as:

$$u_{E,\Omega} = \sup\{v \in \mathcal{PSH}(\Omega) \, ; \ v \leq 0 \text{ and } v \leq -1 \text{ on } E\} \, , \tag{3.2}$$

and $u^*_{E,\Omega}$ is its upper semi-continuous regularization:

$$u^*_{E,\Omega}(z) = \limsup_{\zeta \to z} u_{E,\Omega}(\zeta)\,, \qquad z \in \Omega\,,$$

called the *regularized relative extremal function* of E.

For an open subset ω of Ω, its capacity is defined as:

$$\mathrm{Cap}\,(\omega) = \sup\{\mathrm{Cap}\,(K)\,;\ K \text{ is a compact subset of } \omega\}\,.$$

When $\overline{\omega} \subset \Omega$ is a compact subset of Ω, we have ([5, equation (6.2)], [30, Corollary 4.6.2]):

$$\mathrm{Cap}\,(\omega) = \int_\Omega (dd^c u_\omega)^N\,. \tag{3.3}$$

The *outer capacity* of a subset $E \subseteq \Omega$ is:

$$\mathrm{Cap}^*(E) = \inf\{\mathrm{Cap}\,(\omega)\,;\ \omega \supseteq E \text{ and } \omega \text{ open}\}\,.$$

If Ω is hyperconvex and E relatively compact in Ω, then ([30, Proposition 4.7.2]):

$$\mathrm{Cap}^*(E) = \int_\Omega (dd^c u^*_{E,\Omega})^N\,.$$

Remark. A. Zeriahi ([57]) pointed out to us the following result.

Proposition 3.1. *Let K be a compact subset of Ω. Then:*

$$\mathrm{Cap}\,(K) = \mathrm{Cap}\,(\partial K)\,.$$

Proof. Of course $u_K \leq u_{\partial K}$ since $\partial K \subseteq K$. Conversely, let $v \in \mathcal{PSH}(\Omega)$ non-positive such that $v \leq -1$ on ∂K. By the maximum principle (see [30, Corollary 2.9.6]), we get that $v \leq -1$ on K. Hence $v \leq u_K$. Taking the supremum over all those v, we obtain $u_{\partial K} \leq u_K$, and therefore $u_{\partial K} = u_K$.

By (3.1), it follows that:

$$\mathrm{Cap}\,(K) = \int_\Omega (dd^c u^*_K)^N = \int_\Omega (dd^c u^*_{\partial K})^N = \mathrm{Cap}\,(\partial K)\,. \tag{3.4}$$

□

3.2 Regular sets

Let $E \subseteq \mathbb{C}^N$ be bounded. Recall that the polynomial convex hull of E is:

$$\widehat{E} = \{z \in \mathbb{C}\,;\ |P(z)| \leq \sup_E |P| \text{ for every polynomial } P\}\,.$$

A point $a \in \widehat{E}$ is called *regular* if $u^*_{E,\Omega}(a) = -1$ for an open set $\Omega \supseteq \widehat{E}$ (note that we always have $u_{E,\Omega} = u_{E,\Omega} = -1$ on the interior of E: see [17, Properties 8.1 (c)]). The set E is said to be *regular* if all points of $\widehat{E}$ are regular.

The *pluricomplex Green function* of E, also called the *L-extremal function* of E, is defined, for $z \in \mathbb{C}^N$, as:

$$V_E(z) = \sup\{v(z)\,;\ v \in \mathcal{L}\,, \quad v \leq 0 \text{ on } E\}\,,$$

where $\mathcal{L}$ is the *Lelong class* of all functions $v \in \mathcal{PSH}(\mathbb{C}^N)$ such that, for some constant $C > 0$:

$$v(z) \leq C + \log(1 + |z|) \quad \text{for all } z \in \mathbb{C}^N\,.$$

A point $a \in \widehat{E}$ is called *L-regular* if $V_E^*(a) = 0$, where V_E^* is the upper semi-continuous regularization of V_E. The set E is *L-regular* if all points of $\widehat{E}$ are L-regular.

By [28, Proposition 2.2] (see also [30, Proposition 5.3.3, and Corollary 5.3.4]), for E bounded and non pluripolar, and Ω a bounded open neigbourhood of $\widehat{E}$, we have:

$$m(u_{E,\Omega} + 1) \leq V_E \leq M(u_{E,\Omega} + 1) \tag{3.5}$$

for some positive constants m, M. Hence the regularity of $a \in \widehat{E}$ is equivalent to its L-regularity.

Recall that E is pluripolar if there exists an open set Ω containing E and $v \in \mathcal{PSH}(\Omega)$ such that $E \subseteq \{v = -\infty\}$. This is equivalent to say that there exists a hyperconvex domain Ω of $\mathbb{C}^N$ containing E such that $u_{E,\Omega}^* \equiv 0$ (see [30, Corollary 4.7.3 and Theorem 4.7.5]). By Josefson's theorem ([30, Theorem 4.7.4]), E is pluripolar if and only if there exists $v \in \mathcal{PSH}(\mathbb{C}^N)$ such that $E \subseteq \{v = -\infty\}$. Recall also that E is pluripolar if and only if its outer capacity $\mathrm{Cap}^*(E)$ is null ([30, Theorem 4.7.5]).

When Ω is hyperconvex and E is compact, non pluripolar, the regularity of E implies that $u_{E,\Omega}$ and V_E are continuous, on Ω and $\mathbb{C}^N$ respectively ([30, Proposition 4.5.3 and Corollary 5.1.4]). Conversely, if $u_{E,\Omega}$ is continuous, for some hyperconvex neighbourhood Ω of E, then $u_{E,\Omega}(z) = -1$ for all $z \in E$; hence $V_E(z) = 0$ for all $z \in E$, by (3.5); but $V_E = V_{\widehat{E}}$ when E is compact ([30, Theorem 5.1.7]), so $V_E(z) = 0$ for all $z \in \widehat{E}$; by (3.5) again, we obtain that $u_{E,\Omega}(z) = -1$ for all $z \in \widehat{E}$; therefore E is regular. In the same way, the continuity of V_E implies the regularity of E. These results are due to Siciak ([50, Proposition 6.1 and Proposition 6.2]).

Every closed ball $B = B(a, r)$ of an arbitrary norm $\|\,.\,\|$ on $\mathbb{C}^N$ is regular since its L-extremal function is:

$$V_B(z) = \log^+\big(\|z - a\|/r\big)$$

([50, p. 179, § 2.6]).

3.3 Zakharyuta's formula

We will need a formula that Zakharyuta, in order to solve a problem raised by Kolmogorov, proved, conditionally to a conjecture, called Zakharyuta's conjecture, on the uniform approximation of the relative extremal function $u_{K,\Omega}$

by pluricomplex Green functions. This conjecture has been proved by Nivoche ([45, Theorem A]), in a more general setting that we state below:

Theorem 3.2 (Nivoche)**.** *Let K be a regular compact subset of a bounded hyperconvex domain Ω of $\mathbb{C}^N$. Then for every $\varepsilon > 0$ and δ small enough, there exists a pluricomplex Green function g on Ω with a finite number of logarithmic poles such that:*

1) *the poles of g lie in $W = \{z \in \Omega\,;\ u_K(z) < -1 + \delta\}$;*
2) *we have, for every $z \in \overline{\Omega} \setminus W$:*

$$(1+\varepsilon)\, g(z) \leq u_K(z) \leq (1-\varepsilon)\, g(z)\,.$$

In order to state Zakharyuta's formula, we need some additional notations.

Let K be a compact subset of Ω with non-empty interior, and A_K the set of restrictions to K of those functions that are analytic and bounded by 1, i.e. those functions belonging to the unit ball $B_{H^\infty(\Omega)}$ of the space $H^\infty(\Omega)$ of the bounded analytic functions in Ω, considered as a subset of the space $\mathcal{C}(K)$ of complex functions defined on K, equipped with the sup-norm on K.

Let $d_n(A_K)$ be the nth-width of A_K in $\mathcal{C}(K)$, namely:

$$d_n(A_K) = \inf_L \Big[\sup_{f \in A_K} \mathrm{dist}\,(f, L) \Big]\,, \tag{3.6}$$

where L runs over all k-dimensional subspaces of $\mathcal{C}(K)$, with $k < n$.

Equivalently, $d_n(A_K)$ is the nth-Kolmogorov number of the natural injection J of $H^\infty(\Omega)$ into $\mathcal{C}(K)$ (recall that K has non-empty interior). It is convenient to set, as in [56]:

$$\tau_N(K) = \frac{1}{(2\pi)^N}\, \mathrm{Cap}\,(K) \tag{3.7}$$

and:

$$\Gamma_N(K) = \exp\bigg[- \Big(\frac{N!}{\tau_N(K)} \Big)^{1/N} \bigg]\,, \tag{3.8}$$

i.e.:

$$\Gamma_N(K) = \exp\bigg[- 2\pi \Big(\frac{N!}{\mathrm{Cap}\,(K)} \Big)^{1/N} \bigg]\,. \tag{3.9}$$

Observe that $\mathrm{Cap}\,(K) > 0$ since we assumed that K has non-empty interior. Now, we have ([56, Theorem 5.6]; see also [55, Theorem 5] or [54, pages 30–32], for a detailed proof):

Theorem 3.3 (Zakharyuta-Nivoche)**.** *Let Ω be a bounded hyperconvex domain and K a regular compact subset of Ω with non-empty interior, which is holomorphically convex in Ω (i.e. $K = \widetilde{K}_\Omega$). Then:*

$$- \log d_n(A_K) \sim \Big(\frac{N!}{\tau_N(K)} \Big)^{1/N} n^{1/N}\,. \tag{3.10}$$

Here $\widetilde{K}_\Omega$ is the holomorphic convex hull of K in Ω, that is:

$$\widetilde{K}_\Omega = \{z \in \Omega\,;\ |f(z)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(\Omega)\}\,,$$

where $\mathcal{O}(\Omega)$ is the set of all functions holomorphic in Ω.

Relying on that theorem, which may be seen as the extension of a result of Erokhin, proved in 1958 (see [19]; see also Widom [53] which proved a more general result, with a different proof), to dimension $N > 1$, and as a result on the approximation of functions, we will give an application to the study of approximation numbers of a composition operator on $H^2(\Omega)$ for a bounded symmetric domain of $\mathbb{C}^N$.

4 The spectral radius type formula

In [41, Section 6.2], we proved the following result.

Theorem 4.1. *Let* $\varphi\colon \mathbb{D}^N \to \mathbb{D}^N$ *be given by* $\varphi(z_1, \ldots, z_N) = (r_1 z_1, \ldots, r_N z_N)$ *where* $0 < r_j < 1$. *Then:*

$$\beta_N(C_\varphi) = \Gamma_N\big[\overline{\varphi(\mathbb{D}^N)}\big] = \Gamma_N\big[\varphi(\mathbb{D}^N)\big]\,.$$

The proof was simple, based on result of Blocki [8] on the Monge-Ampère capacity of a cartesian product, and on the estimation, when $A \to \infty$, of the number ν_A of N-tuples $\alpha = (\alpha_1, \ldots, \alpha_N)$ of non-negative integers α_j such that $\sum_{j=1}^N \alpha_j \sigma_j \leq A$, where the numbers $\sigma_j > 0$ are fixed. The estimation was:

$$\nu_A \sim \frac{A^N}{N!\,\sigma_1 \cdots \sigma_N}\,\cdot \tag{4.1}$$

As J. F. Burnol pointed out to us, this is a consequence of the following elementary fact. Let λ_N be the Lebesgue measure on $\mathbb{R}^N$, and let E be a compact subset of $\mathbb{R}^N$ such that $\lambda_N(\partial E) = 0$. Then:

$$\lambda_N(E) = \lim_{A \to \infty} A^{-N} |(A \times E) \cap \mathbb{Z}^N|\,.$$

Then, just take $E = \{(x_1, \ldots, x_N)\,;\ x_j \geq 0 \text{ and } \sum_{j=1}^N x_j \sigma_j \leq 1\}$.

In any case, this lets us suspect that the formula of Theorem 4.1 holds in much more general cases. This is not quite true, as evidenced by our counterexample of [41, Theorem 5.12]. Nevertheless, in good cases, this formula holds, as we will see in the next sections.

In remaining of this section, we consider functions $\varphi\colon \Omega \to \Omega$ such that $\overline{\varphi(\Omega)} \subseteq \Omega$. If ρ is an exhaustion function for Ω, there is some $R_0 < 0$ such that $\overline{\varphi(\Omega)} \subseteq B_\Omega(R_0)$, and that implies that C_φ maps $H^2(\Omega)$ into itself and is a compact operator (see [47, Theorem 8.3], since, with their notations, for $r > R_0$, we have $T(r) = \emptyset$ and hence $\delta_\varphi(r) = 0$) .

4.1 Minoration

Recall that every hyperconvex domain Ω is pseudoconvex. By H. Cartan-Thullen and Oka-Bremermann-Norguet theorems, being pseudoconvex is equivalent to being a domain of holomorphy, and equivalent to being holomorphically convex (meaning that if K is a compact subset in Ω, then its holomorphic hull $\widetilde{K}$ is also contained in Ω): see [33, Corollaire 7.7]. Now (see [32, Chapter 5, Exercise 11], a domain of holomorphy Ω is said a *Runge domain* if every holomorphic function in Ω can be approximated uniformly on its compact subsets by polynomials, and that is equivalent to saying that the polynomial hull and the holomorphic hull of every compact subset of Ω agree. By [32, Chapter 5, Exercise 13], every circled domain (in particular every bounded symmetric domain) is a Runge domain.

Definition 4.2. *A hyperconvex domain Ω is said* strongly regular *if there exists a continuous psh exhaustion function ρ such that all the sub-level sets:*

$$\Omega_c = \{z \in \Omega\,;\ \rho(z) < c\}$$

($c < 0$) have a regular closure.

For example, every bounded symmetric domain Ω is strongly regular since if $\|\,.\,\|$ is the associated norm, its sub-level sets Ω_c (with $\rho(z) = \log \|z\|$) are the open balls $B(0, \mathrm{e}^c)$, and the closed balls are regular, as said above.

Theorem 4.3. *Let Ω be a strongly regular bounded hyperconvex and Runge domain in $\mathbb{C}^N$, and let $\varphi\colon \Omega \to \Omega$ be an analytic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$, and which is non-degenerate. Then:*

$$\Gamma_N\big[\varphi(\Omega)\big] \leq \beta_N^-(C_\varphi)\,. \tag{4.2}$$

Recall that if Ω is a domain in $\mathbb{C}^N$, a holomorphic function $\varphi\colon \Omega \to \mathbb{C}^M$ ($M \leq N$) is *non-degenerate* if there exists $a \in \Omega$ such that $\mathrm{rank}_a\, \varphi = M$. Then $\varphi(\Omega)$ has a non-empty interior.

Proof. Let $(r_j)_{j\geq 1}$ be an increasing sequence of negative numbers tending to 0. The set $H_j = \overline{\Omega_{r_j}}$ is a regular compact subset of Ω, with non-void interior (hence non pluripolar). Let $\widehat{H_j}$ its polynomial convex hull; this compact set is contained in Ω, since Ω being a Runge domain, we have $\widehat{H_j} = \widetilde{H_j}$, and since $\widetilde{H_j} \subseteq \Omega$, because Ω is holomorphically convex (being hyperconvex). Moreover $\widehat{H_j}$ is regular since $V_E = V_{\widehat{E}}$ for every compact subset of $\mathbb{C}^N$ ([50, Corollary 4.14]).

Let $K_j = \varphi\big(\widehat{H_j}\big)$ and let G be a subspace of $H^2(\Omega)$ with dimension $< n^N$.

The set K_j is regular because of the following result (see [30, Theorem 5.3.9], [46, top of page 40], [29, Theorem 1.3], or [44, Theorem 4], with a detailed proof).

Theorem 4.4 (Pleśniak)**.** *Let E be a compact, polynomially convex, regular and non pluripolar, subset of $\mathbb{C}^N$. Then if Ω is a hyperconvex domain such that $E \subseteq \Omega$ and if $\varphi\colon \Omega \to \mathbb{C}^N$ is a non-degenerate holomorphic function, the set $\varphi(E)$ is regular.*

As before, the polynomial convex hull $\widehat{K_j}$ of K_j is contained in Ω and is also regular. Since φ is non-degenerate, K_j has a non-void interior; hence $\widehat{K_j}$ also. We can hence use Zakharyuta's formula (Theorem 3.3) for the compact set $\widehat{K_j}$.

By restriction, the subspace G can be viewed as a subspace of $\mathcal{C}(\widehat{K_j})$. By Zakharyuta's formula, for $0 < \varepsilon < 1$, there is $n_\varepsilon \geq 1$ such that, for $n \geq n_\varepsilon$:

$$d_{n^N}(A_{\widehat{K_j}}) \geq \exp\left[-(1+\varepsilon)\,(2\pi)\,n \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right].$$

Hence, there exists $f \in B_{H^\infty} \subseteq B_{H^2}$ such that, for all $g \in G$:

$$\|g - f\|_{\mathcal{C}(\widehat{K_j})} \geq (1-\varepsilon) \exp\left[-(1+\varepsilon)\,(2\pi)\,n \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right].$$

Since $\widehat{K_j} = \widetilde{K_j}$ and, by definition $\|\,.\,\|_{\mathcal{C}(\widetilde{K_j})} = \|\,.\,\|_{\mathcal{C}(K_j)}$, we have:

$$\|g - f\|_{\mathcal{C}(\widehat{K_j})} = \|g - f\|_{\mathcal{C}(K_j)} = \|C_\varphi(g) - C_\varphi(f)\|_{\mathcal{C}(\widetilde{H_j})}\,.$$

Equivalently, since, by definition $\|\,.\,\|_{\mathcal{C}(\widetilde{H_j})} = \|\,.\,\|_{\mathcal{C}(H_j)}$, we have, for all $g \in G$:

$$\|C_\varphi(g) - C_\varphi(f)\|_{\mathcal{C}(H_j)} \geq (1-\varepsilon) \exp\left[-(1+\varepsilon)\,(2\pi)\,n \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right].$$

This implies, thanks to (2.3), that, for all $g \in G$:

$$\|C_\varphi(g) - C_\varphi(f)\|_{H^2(\Omega)} \geq L_{r_j}^{-1}(1-\varepsilon) \exp\left[-(1+\varepsilon)\,(2\pi)\,n \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right].$$

Using (2.8), we get, since the subspace G is arbitrary:

$$a_{n^N}(C_\varphi) \geq L_{r_j}^{-1}(1-\varepsilon) \exp\left[-(1+\varepsilon)\,(2\pi)\,n \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right].$$

Taking the nth-roots and passing to the limit, we obtain:

$$\beta_N^-(C_\varphi) \geq \exp\left[-(1+\varepsilon)\,(2\pi) \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right].$$

and then, letting ε go to 0:

$$\beta_N^-(C_\varphi) \geq \exp\left[-(2\pi) \left(\frac{N!}{\mathrm{Cap}\,(\widehat{K_j})} \right)^{1/N} \right] = \Gamma_N(\widehat{K_j})\,.$$

Now, the sequence $(\widehat{K_j})_{j\geq 1}$ is increasing and $\bigcup_{j\geq 1} \widehat{K_j} \supseteq \varphi(\Omega)$; hence, by [5, Theorem 8.2 (8.3)], we have $\mathrm{Cap}\,(\widehat{K_j}) \underset{j\to\infty}{\longrightarrow} \mathrm{Cap}\,\big(\bigcup_{j\geq 1} \widehat{K_j}\big) \geq \mathrm{Cap}\,[\varphi(\Omega)]$, so:

$$\beta_N^-(C_\varphi) \geq \Gamma_N[\varphi(\Omega)]\,,$$

and the proof of Theorem 4.3 is finished. □

4.2 Majorization

For the majorization, we assume different hypotheses on the domain Ω. Nevertheless these assumptions agree with that of Theorem 4.3 when Ω is a bounded symmetric domain.

4.2.1 Preliminaries

Recall that a domain Ω of $\mathbb{C}^N$ is a *Reinhardt domain* (resp. *complete Reinhardt domain*) if $z = (z_1, \ldots, z_N) \in \Omega$ implies that $(\zeta_1 z_1, \ldots, \zeta_N z_N) \in \Omega$ for all complex numbers $\zeta_1, \ldots, \zeta_N$ of modulus 1 (resp. of modulus ≤ 1). A complete bounded Reinhardt domain is hyperconvex if and only if $\log j_\Omega$ is *psh* and continuous in $\mathbb{C}^N \setminus \{0\}$, where j_Ω is the Minkowski functional of Ω (see [7, Exercise following Proposition 3.3.3]). In general, the Minkowski functional j_Ω of a bounded complete Reinhardt domain Ω is *usc* and $\log j_\Omega$ is *psh* if and only if Ω is pseudoconvex ([7, Theorem 1.4.8]). Other conditions for a bounded complete Reinhardt domain to being hyperconvex can found in [34, Theorem 3.10].

For a bounded hyperconvex and complete Reinhardt domain Ω, its pluricomplex Green function with pole 0 is $g_\Omega(z) = \log j_\Omega(z)$, where j_Ω is the Minkowski functional of Ω ([7, Proposition 3.3.2]), and $S_\Omega(r) = \mathrm{e}^r \partial\Omega$. Since $\partial\Omega$ is in particular invariant by the pluri-rotations $z = (z_1, \ldots, z_N) \mapsto (\mathrm{e}^{i\theta_1} z_1, \ldots, \mathrm{e}^{i\theta_N} z_N)$, with $\theta_1, \ldots, \theta_N \in \mathbb{R}$, the harmonic measure $\widetilde{\mu}_\Omega$ at 0 (see the proof of Proposition 2.1) is also invariant by the pluri-rotations (note that it is supported by the Shilov boundary of $\overline{\Omega}$: see [51, very end of the paper]). We have, as in the proof of Proposition 2.1, for $f \in H^2(\Omega)$:

$$\sup_{0<s<1} \int_{\partial\Omega} |f(sz)|^2 \, d\widetilde{\mu}_\Omega(z) = \|f\|^2_{H^2(\Omega)} < \infty \, .$$

Since $\widetilde{\mu}_\Omega$ is in particular invariant by the rotations $z \mapsto \mathrm{e}^{i\theta} z$, $\theta \in \mathbb{R}$, there exists, by [9, Theorem 3], a function $f^* \in L^2(\partial\Omega, \widetilde{\mu}_\Omega)$ such that:

$$\int_{\partial\Omega} |f(sz) - f^*(z)|^2 \, d\widetilde{\mu}_\Omega(z) \underset{s \to 1}{\longrightarrow} 0 \, .$$

It ensues that the map $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial\Omega, \widetilde{\mu}_\Omega)$ is an isometric embedding (in fact, f^* is the radial limit of f: see [21, Lemma 2]). Therefore, we can consider $H^2(\Omega)$ as a complemented subspace of $L^2(\partial\Omega, \widetilde{\mu}_\Omega)$, and we call P the orthogonal projection of $L^2(\partial\Omega, \widetilde{\mu}_\Omega)$ onto $H^2(\Omega)$.

Every holomorphic function f in a Reinhardt domain Ω containing 0 (in particular if Ω is a complete Reinhardt domain) has a power series expansion about 0:

$$f(z) = \sum_\alpha b_\alpha z^\alpha$$

which converges normally on compact subsets of Ω ([32, Proposition 2.3.14]). Recall that if $z = (z_1, \ldots, z_N)$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$, then $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $\alpha! = \alpha_1! \cdots \alpha_N!$.

We have:

Proposition 4.5. *Let Ω be a bounded hyperconvex and complete Reinhardt domain, and set $e_\alpha(z) = z^\alpha$. Then the system $(e_\alpha)_\alpha$ is orthogonal in $H^2(\Omega)$.*

Proof. We use the fact that the level sets $S(r)$ and the Demailly-Monge-Ampère measures $\mu_r = \big(dd^c(g_\Omega)_r\big)^N$ are pluri-rotation invariant. For $\alpha \neq \beta$, we choose $\theta_1, \ldots, \theta_N \in \mathbb{R}$ such that $1, (\theta_1/2\pi), \ldots, (\theta_N/2\pi)$ are rationally independent. Then $\exp\big[i\big(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j\big)\big] \neq 1$. Hence, as in [25, p. 78], we have, making the change of variables $z = (\mathrm{e}^{i\theta_1} w_1, \ldots, \mathrm{e}^{i\theta_N} w_N)$:

$$\int_{S(r)} z^\alpha \overline{z^\beta}\, d\mu_r(z) = \exp\left[i\left(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j\right)\right] \int_{S(r)} w^\alpha \overline{w^\beta}\, d\mu_r(w)\,,$$

which implies that:

$$\int_{S(r)} z^\alpha \overline{z^\beta}\, d\mu_r(z) = 0\,,$$

and hence:

$$(z^\alpha \mid z^\beta) := \lim_{r\to 0} \int_{S(r)} z^\alpha \overline{z^\beta}\, d\mu_r(z) = 0\,. \qquad \square$$

For the polydisk, we have $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$, and for the ball (see [48, Proposition 1.4.9]):

$$\|e_\alpha\|^2_{H^2(\mathbb{B}_N)} = \frac{(N-1)!\,\alpha!}{(N-1+|\alpha|)!}\,.$$

Definition 4.6. *We say that Ω is a* good *complete Reinhardt domain if, for some positive constant C_N and some positive integer c, we have, for all $p \geq 0$:*

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\Omega)}} \leq C_N\, p^{cN} [j_\Omega(z)]^{2p}\,,$$

where j_Ω is the Minkowski functional of Ω.

Examples

1. The polydisk $\mathbb{D}^N$ is a good Reinhardt domain because $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$, $|z^\alpha| \leq \|z\|_\infty^{|\alpha|}$, and the number of indices α such that $|\alpha| = p$ is $\binom{N-1+p}{p} \leq C_N p^N$ (see [35, p. 498] or [37, pp. 213–214]).

2. The ball $\mathbb{B}_N$ is a good Reinhardt domain. In fact, observe that:

$$\frac{(N-1+p)!}{(N-1)!} = p!\,\frac{(p+1)(p+2)\cdots(p+N-1)}{1\times 2\times\cdots\times(N-1)} \leq p!\,(p+1)^{N-1} \leq p!\,(p+1)^N\,;$$

hence:

$$\begin{aligned}\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\mathbb{B}_N)}} &= \sum_{|\alpha|=p} |z^\alpha|^2\, \frac{(N-1+|\alpha|)!}{(N-1)!\,\alpha!} \\ &\leq (p+1)^N \sum_{|\alpha|=p} \frac{|\alpha|!}{\alpha!}\, |z_1|^{2\alpha_1}\cdots|z_N|^{2\alpha_N} \\ &= (p+1)^N (|z_1|^2 + \cdots + |z_N|^2)^p\,,\end{aligned}$$

by the multinomial formula, so:

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\mathbb{B}_N)}} \leq (p+1)^N \|z\|_2^{2p} \leq 2^N p^N \|z\|_2^{2p} .$$

3. More generally, if $\Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}$, $l_1 + \cdots + l_m = N$, is a product of balls, we have, writing $\alpha = (\beta_1, \ldots, \beta_m)$, where each β_j is an l_j-tuple:

$$\begin{aligned} \|e_\alpha\|^2_{H^2(\Omega)} &= \int_{\mathbb{S}_{l_1} \times \cdots \times \mathbb{S}_{l_2}} |u_1^{\beta_1}|^2 \ldots |u_m^{\beta_m}|^2 \, d\sigma_{l_1}(u_1) \ldots d\sigma_{l_m}(u_m) \\ &= \prod_{j=1}^m \frac{(l_j - 1)! \, \beta_j!}{(l_j - 1 + |\beta_j|)!} , \end{aligned}$$

and, writing $z = (z_1, \ldots, z_m)$, with $z_j \in \mathbb{B}_{l_j}$:

$$\begin{aligned} \sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\Omega)}} &\leq \sum_{p_1 + \cdots + p_m = p} \prod_{j=1}^m (p_j + 1)^{l_j} \|z_j\|_2^{2p_j} \\ &\leq C_m p^m \, (p+1)^{l_1 + \cdots + l_m} [j_\Omega(z)]^{2(p_1 + \cdots + p_m)} , \end{aligned}$$

since $j_\Omega(z) = \max\{\|z_1\|_2, \ldots, \|z_m\|_2\}$. Hence:

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\Omega)}} \leq C_N p^{2N} [j_\Omega(z)]^{2p} .$$

4.2.2 The result

Theorem 4.7. *Let Ω be a bounded hyperconvex domain which is a good complete Reinhardt domain in $\mathbb{C}^N$, and let $\varphi\colon \Omega \to \Omega$ be an analytic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$. Then, for every compact subset $K \supseteq \varphi(\Omega)$ of Ω with non void interior, we have:*

$$\beta_N^+(C_\varphi) \leq \Gamma_N(K) . \tag{4.3}$$

In particular, if φ is moreover non-degenerate, we have:

$$\beta_N^+(C_\varphi) \leq \Gamma_N\big[\overline{\varphi(\Omega)}\big] . \tag{4.4}$$

The last assertion holds because $\varphi(\Omega)$ is open if φ is non-degenerate.

Corollary 4.8. *Let Ω be a good complete bounded symmetric domain in $\mathbb{C}^N$, and $\varphi\colon \Omega \to \Omega$ a non-degenerate analytic map such that $\overline{\varphi(\Omega)} \subseteq \Omega$. Then:*

$$\Gamma_N\big[\varphi(\Omega)\big] \leq \beta_N^-(C_\varphi) \leq \beta_N^+(C_\varphi) \leq \Gamma_N\big[\overline{\varphi(\Omega)}\big] .$$

For the proof of Theorem 4.7, we will use the following result ([56, Proposition 6.1]), which do not need any regularity condition on the compact set (because it may be written as a decreasing sequence of regular compact sets).

Proposition 4.9 (Zakharyuta). *If K is any compact subset of a bounded hyperconvex domain Ω of $\mathbb{C}^N$ with non-empty interior, we have:*

$$\limsup_{n\to\infty} \frac{\log d_n(A_K)}{n^{1/N}} \leq -\Big(\frac{N!}{\tau_N(K)}\Big)^{1/N} .$$

Proof of Theorem 4.7. In the sequel we write $\| \,.\, \|_{H^2}$ for $\| \,.\, \|_{H^2(\Omega)}$. We set:

$$\Lambda_N = \limsup_{n\to\infty} [d_n(A_K)]^{n^{-1/N}} .$$

Changing n into n^N, Proposition 4.9 means that for every $\varepsilon > 0$, there exists, for n large enough, an $(n^N - 1)$-dimensional subspace F of $\mathcal{C}(K)$ such that, for any $g \in H^\infty(\Omega)$, there exists $h \in F$ such that:

$$\|g - h\|_{\mathcal{C}(K)} \leq (1+\varepsilon)^n \Lambda_N^n \, \|g\|_\infty . \tag{4.5}$$

Let l be an integer to be adjusted later, and

$$f(z) = \sum_\alpha b_\alpha z^\alpha \in H^2(\Omega) \quad \text{with } \|f\|_{H^2} \leq 1 .$$

By Proposition 4.5, we have:

$$\|f\|_{H^2}^2 = \sum_\alpha |b_\alpha|^2 \|e_\alpha\|_{H^2}^2 .$$

We set:

$$g(z) = \sum_{|\alpha|\leq l} b_\alpha z^\alpha .$$

By the Cauchy-Schwarz inequality:

$$|g(z)|^2 \leq \Big(\sum_{|\alpha|\leq l} |b_\alpha|^2 \|e_\alpha\|_{H^2}^2\Big)\Big(\sum_{|\alpha|\leq l} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2}^2}\Big) \leq \sum_{|\alpha|\leq l} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2}^2} .$$

Since Ω is a good complete Reinhardt domain and since $j_\Omega(z) < 1$ for $z \in \Omega$, we have:

$$|g(z)|^2 \leq \sum_{p=0}^{l} p^{cN} [j_\Omega(z)]^{2p} \leq (l+1)^{cN+1} .$$

It follows from (4.5) that there exists $h \in F$ such that:

$$\|g - h\|_{\mathcal{C}(K)} \leq (1+\varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2} .$$

Since $C_\varphi f(z) - C_\varphi g(z) = f\big(\varphi(z)\big) - g\big(\varphi(z)\big)$ and $\overline{\varphi(\Omega)} \subseteq K$, we have $\|C_\varphi f - C_\varphi g\|_\infty \leq \|f - g\|_{\mathcal{C}(K)}$; therefore:

$$\begin{aligned} \|g\circ\varphi - h\circ\varphi\|_{H^2} &\leq \|g\circ\varphi - h\circ\varphi\|_\infty \leq \|g - h\|_{\mathcal{C}(K)} \\ &\leq (1+\varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2} . \end{aligned} \tag{4.6}$$

Now, the subspace $\widetilde{F}$ formed by functions $v \circ \varphi$, for $v \in F$, can be viewed as a subspace of $L^\infty(\partial\Omega, \widetilde{\mu}_\Omega) \subseteq L^2(\partial\Omega, \widetilde{\mu}_\Omega)$ (indeed, since v is continuous, we can write $(v \circ \varphi)^* = v \circ \varphi^*$, where φ^* denotes the almost everywhere existing radial limits of $\varphi(rz)$, which belong to K). Let finally $E = P(\widetilde{F}) \subseteq H^2(\Omega)$ where $P\colon L^2(\partial\Omega, \widetilde{\mu}_\Omega) \to H^2(\Omega)$ is the orthogonal projection. This is a subspace of $H^2(\Omega)$ with dimension $< n^N$, and we have $\operatorname{dist}(C_\varphi g, E) \leq \|g \circ \varphi - P(h \circ \varphi)\|_{H^2}$; hence, by (4.6):

$$\operatorname{dist}(C_\varphi g, E) \leq (1+\varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2} \,. \tag{4.7}$$

Now, the same calculations give that:

$$|f(z) - g(z)|^2 \leq \sum_{p>l} p^{cN} [j_\Omega(z)]^{2p} \,;$$

hence, for some positive constant M_N:

$$|f(z) - g(z)| \leq M_N (l+1)^{(cN+1)/2} \frac{[j_\Omega(z)]^l}{(1 - [j_\Omega(z)]^2)^{(cN+1)/2}} \,,$$

by using the following lemma, whose proof is postponed.

Lemma 4.10. *For every non-negative integer m, there exists a positive constant A_m such that, for all integers $l \geq 0$ and all $0 < x < 1$, we have:*

$$\sum_{p \geq l} p^m x^p \leq A_m l^m \frac{x^l}{(1-x)^{m+1}} \,.$$

Since K is a compact subset of Ω, there is a positive number $r_0 < 1$ such that $j_\Omega(z) \leq r_0$ for $z \in K$. Since $C_\varphi f(z) - C_\varphi g(z) = f\big(\varphi(z)\big) - g\big(\varphi(z)\big)$ and $\overline{\varphi(\Omega)} \subseteq K$, we have $\|C_\varphi f - C_\varphi g\|_\infty \leq \|f - g\|_{\mathcal{C}(K)}$, and we get:

$$\|C_\varphi f - C_\varphi g\|_{H^2} \leq \|C_\varphi f - C_\varphi g\|_\infty \leq M_N \,(l+1)^{(cN+1)/2} \, \frac{r_0^l}{(1-r_0^2)^{(cN+1)/2}} \,. \tag{4.8}$$

Now, (4.7) and (4.8) give:

$$\operatorname{dist}(C_\varphi f, E) \leq M_N \,(l+1)^{(cN+1)/2} \left(\frac{r_0^l}{(1-r_0^2)^{(cN+1)/2}} + (1+\varepsilon)^n \Lambda_N^n \right).$$

It ensues, thanks to (2.7), that:

$$\big[a_{n^N}(C_\varphi)\big]^{1/n} \leq [M_N \,(l+1)^{(cN+1)/2}]^{1/n} \left[\frac{r_0^{l/n}}{(1-r_0^2)^{(cN+1)/2n}} + (1+\varepsilon)\,\Lambda_N \right].$$

Taking now for l the integer part of $n \log n$, and passing to the upper limit as $n \to \infty$, we obtain (since $l/n \to \infty$ and $(\log l)/n \to 0$):

$$\beta_N^+(C_\varphi) \leq (1+\varepsilon)\,\Lambda_N \,,$$

and therefore, since $\varepsilon > 0$ is arbitrary:

$$\beta_N^+(C_\varphi) \leq \Lambda_N \,.$$

That ends the proof, by using Proposition 4.9. $\square$

Proof of Lemma 4.10. We make the proof by induction on m. We set:

$$S_m = \sum_{p\geq l} p^m x^p$$

The result is obvious for $m = 0$, with $A_0 = 1$, since then $S_0 = \sum_{p\geq l} x^p = \frac{x^l}{1-x}$ · Let us assume that it holds till $m-1$ and prove it for m. We observe that, since $p^m - (p-1)^m \leq mp^{m-1}$, we have:

$$\begin{aligned}(1-x)S_m &= \sum_{p\geq l} p^m x^p - \sum_{p\geq l} p^m x^{p+1} = \sum_{p\geq l} p^m x^p - \sum_{p\geq l+1} (p-1)^m x^p \\ &= \sum_{p\geq l+1} (p^m - (p-1)^m) x^p + l^m x^l \leq \sum_{p\geq l+1} mp^{m-1} x^p + l^m x^l \\ &\leq \sum_{p\geq l} mp^{m-1} x^p + l^m x^l \leq mA_{m-1} l^{m-1} \frac{x^l}{(1-x)^m} + l^m x^l \\ &\leq (mA_{m-1} + 1)\, l^m \frac{x^l}{(1-x)^m} \,,\end{aligned}$$

giving the result, with $A_m = mA_{m-1} + 1$. $\square$

4.3 Equality

Proposition 4.11. *Let Ω be a bounded hyperconvex domain and ω a relatively compact open subset of Ω. Assume that:*

(4.9) *For every $a \in \partial\omega$, except on a pluripolar set $E \subseteq \partial\omega$, there exists $z_0 \in \omega$ such that the open segment (z_0, a) is contained in ω.*

Then:

$$\mathrm{Cap}\,(\overline{\omega}) = \mathrm{Cap}\,(\omega) \,.$$

In particular, if $\varphi\colon \Omega \to \Omega$ a non-degenerate holomorphic map such that $\overline{\varphi(\Omega)} \subseteq \Omega$ and $\omega = \varphi(\Omega)$ satisfies (4.9)*, we have:*

$$\mathrm{Cap}\,\big[\varphi(\Omega)\big] = \mathrm{Cap}\,\big[\overline{\varphi(\Omega)}\big] \,.$$

Before proving Proposition 4.11, let us give an example of such a situation.

Proposition 4.12. *Let Ω be a bounded hyperconvex domain with C^1 boundary. Let U be an open neighbourhood of $\overline{\Omega}$ and $\varphi\colon U \to \mathbb{C}^N$ be a non-degenerate holomorphic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$. Then the condition* (4.9) *is satisfied.*

Proof. Let $\omega = \varphi(\Omega)$.

We may assume that U is connected, hyperconvex and bounded. Let B_φ be the set of points $z \in U$ such that the complex Jacobian J_φ is null. Since J_φ is holomorphic in Ω, we have $\log |J_\varphi| \in \mathcal{PSH}(U)$ and hence (see [31, proof of Lemma 10.2]):

$$B_\varphi = \{z \in U\,;\; J_\varphi(z) = 0\} = \{z \in U\,;\; \log |J_\varphi(z)| = -\infty\}$$

is pluripolar . Therefore (see [5, Theorem 6.9]), $\mathrm{Cap}\,(B_\varphi, U) = 0$. It follows (see [5, page 2, line -8]) that $\mathrm{Cap}\,[\varphi(B_\varphi)] := \mathrm{Cap}\,[\varphi(B_\varphi), \Omega] = 0$.

Now, for every $a \in \partial\overline{\omega} \cap [\varphi(U \setminus B_\varphi)]$, there is a tangent hyperplane H_a to $\overline{\omega}$, and hence an inward normal to $\partial\overline{\omega}$ (note that $\partial\overline{\omega} \subseteq \varphi(\partial\Omega) \subseteq \varphi(U)$). It follows that there is $z_0 \in \omega$ such that the open interval (z_0, a) is contained in ω. □

Proof of Proposition 4.11. Let $a \in \partial\omega$ and L be a complex line containing (z_0, a); we have $a \in \overline{\omega \cap L}$. Assume now that this point a is a *fine* ("*effilé*") point of ω, i.e. that there exists $u \in \mathcal{PSH}(V)$, for V a neighbourhood of a, such that:

$$\limsup_{z \to a\,, z \in \omega} u(z) < u(a)\,.$$

By definition, the restriction $\widetilde{u}$ of u to $\omega \cap L$ is subharmonic and we keep the inequality:

$$\limsup_{z \to a\,, z \in \omega \cap L} \widetilde{u}(z) < \widetilde{u}(a) = u(a)\,.$$

That means that a is a fine point of $\omega \cap L$. But $a \in \overline{\omega \cap L}$ and $\omega \cap L$ is connected, so this is not possible, by [40, Lemma 2.4]. Hence no point of $\partial\omega \setminus E$ is fine.

Let now ω^f be the closure of ω for the fine topology (i.e. the coarsest topology on U for which all the functions in $\mathcal{PSH}(U)$ are continuous; it is known: see [6, comment after Theorem 2.3], that it is the trace on U of the fine topology on $\mathbb{C}^N$). It is also known (see [30, Corollary 4.8.10]) that ω^f is the set of points of $\overline{\omega}$ which are not fine. By the above reasoning, we thus have:

$$\overline{\omega} \setminus \omega^f \subseteq E\,.$$

Since $\mathrm{Cap}\,(E) = 0$, we have:

$$\mathrm{Cap}\,(\overline{\omega} \setminus \omega^f) = 0\,,$$

and it follows that:

$$\mathrm{Cap}\,(\overline{\omega}) = \mathrm{Cap}\,[\omega^f \cup (\overline{\omega} \setminus \omega^f)] \leq \mathrm{Cap}\,(\omega^f) + \mathrm{Cap}\,(\overline{\omega} \setminus \omega^f) = \mathrm{Cap}\,(\omega^f)\,,$$

and hence $\mathrm{Cap}\,(\omega^f) = \mathrm{Cap}\,(\overline{\omega})$.

But, since, by definition, the *psh* functions are continuous for the fine topology, it is clear, that the relative extremal functions $u_{\omega,\Omega}$ and $u_{\omega^f,\Omega}$ are equal; hence we have, by [30, Proposition 4.7.2]:

$$\mathrm{Cap}\,(\omega) = \int_\Omega (dd^c u^*_{\omega,\Omega})^N = \int_\Omega (dd^c u^*_{\omega^f,\Omega})^N = \mathrm{Cap}\,(\omega^f)\,.$$

Hence $\mathrm{Cap}\,(\omega) = \mathrm{Cap}\,(\overline{\omega})$. □

4.4 Consequences of the spectral radius type formula

Theorem 4.3 has the following consequence.

Proposition 4.13. *Let Ω be a regular bounded symmetric domain in $\mathbb{C}^N$, and let $\varphi\colon \Omega \to \Omega$ be a non-degenerate analytic function inducing a bounded composition operator C_φ on $H^2(\Omega)$.*

Then, if $\mathrm{Cap}\,[\varphi(\Omega)] = \infty$*, we have $\beta_N(C_\varphi) = 1$.*

In other words, if, for some constants $C, c > 0$, we have $a_n(C_\varphi) \leq C\,\mathrm{e}^{-cn^{1/N}}$ for all $n \geq 1$, then $\mathrm{Cap}\,[\varphi(\Omega)] < \infty$.

As a corollary, we can give a new proof of [41, Theorem 3.1].

Corollary 4.14. *Let $\tau\colon \mathbb{D} \to \mathbb{D}$ be an analytic map such that $\|\tau\|_\infty = 1$ and $\psi\colon \mathbb{D}^{N-1} \to \mathbb{D}^{N-1}$ such that the map $\varphi\colon \mathbb{D}^N \to \mathbb{D}^N$, defined as:*

$$\varphi(z_1, z_2, \ldots, z_N) = \big(\tau(z_1), \psi(z_2, \ldots, z_N)\big)\,,$$

is non-degenerate. Then $\beta_N(C_\varphi) = 1$.

Proof. Since the map φ is non-degenerate, ψ is also non-degenerate. Hence (see [44, Proposition 2] $\psi(\mathbb{D}^{N-1})$ is not pluripolar, i.e. $\mathrm{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] > 0$. On the other hand, it follows from [40, Theorem 3.13 and Theorem 3.14] that $\mathrm{Cap}_1[\tau(\mathbb{D})] = +\infty$. Then, by [8, Theorem 3], we have:

$$\begin{aligned}\mathrm{Cap}_N[\varphi(\mathbb{D}^N)] &= \mathrm{Cap}_N[\tau(\mathbb{D}) \times \psi(\mathbb{D}^{N-1})]\\ &= \mathrm{Cap}_1[\tau(\mathbb{D})] \times \mathrm{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] = +\infty\,.\end{aligned}$$

It follows from Proposition 4.13 that $\beta_N(C_\varphi) = 1$. □

Proof of Proposition 4.13. If $R\colon H^2(\Omega) \to H^2(\Omega)$ is a finite-rank operator, we set, for $t < 0$:

$$(R_t f)(w) = (Rf)(\mathrm{e}^t w)\,, \qquad f \in H^2(\Omega)\,.$$

Then the rank of the operator R_t is less or equal to that of R.

Recall that if $\|\,.\,\|$ is the norm whose unit ball is Ω, then the pluricomplex Green function of Ω is $g_\Omega(z) = \log \|z\|$, and hence the level set $S(r)$ is the sphere $S(0, \mathrm{e}^r) = \mathrm{e}^r \partial\Omega$ for this norm. Since:

$$\int_{S(r)} |f[\varphi(\mathrm{e}^t w)] - (Rf)(\mathrm{e}^t w)|^2 \, d\mu_r(w) = \int_{S(r+t)} |f[\varphi(z)] - (Rf)(z)|^2 \, d\mu_{r+t}(z)\,,$$

we have, setting $\varphi_t(w) = \varphi(\mathrm{e}^t w)$:

$$\|C_{\varphi_t}(f) - R_t(f)\|_{H^2} \leq \|C_\varphi(f) - R(f)\|_{H^2}\,.$$

It follows that $a_n(C_{\varphi_t}) \leq a_n(C_\varphi)$ for every $n \geq 1$. Therefore $\beta_N^-(C_{\varphi_t}) \leq \beta_N^-(C_\varphi)$.

By Theorem 4.3, we have:

$$\exp\left[-2\pi\left(\frac{N!}{\mathrm{Cap}\,[\varphi_t(\Omega)]}\right)^{1/N}\right] \leq \beta_N^-(C_{\varphi_t})\,.$$

Since $\varphi_t(\Omega) = \varphi(\mathrm{e}^t\Omega)$ increases to $\varphi(\Omega)$ as $t \uparrow 0$, we have (see [30, Corollary 4.7.11]):

$$\mathrm{Cap}\,[\varphi(\Omega)] = \lim_{t\to 0} \mathrm{Cap}\,[\varphi_t(\Omega)]\,.$$

As $\mathrm{Cap}\,[\varphi(\Omega)] = \infty$, we get:

$$\beta_N^-(C_\varphi) \geq \limsup_{t\to 0} \beta_N^-(C_{\varphi_t}) = 1\,. \qquad \square$$

Remark 1. In [41, Theorem 5.12], we construct a non-degenerate analytic function $\varphi\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that $\overline{\varphi(\mathbb{D}^2)} \cap \partial\mathbb{D}^2 \neq \emptyset$ and for which $\beta_2^+(C_\varphi) < 1$. We hence have $\mathrm{Cap}\,[\varphi(\mathbb{D}^2)] < \infty$.

Remark 2. The capacity cannot tend to infinity too fast when the compact set approaches the boundary of Ω; in fact, we have the following result, that we state for the ball, but which holds more generally.

Proposition 4.15. *For every compact set K of $\mathbb{B}_N$, we have, for some constant C_N:*

$$\mathrm{Cap}\,(K) \leq \frac{C_N}{[\mathrm{dist}\,(K, \mathbb{S}_N)]^N}\,.$$

Proof. We know that:

$$\mathrm{Cap}\,(K) = \int_{\mathbb{B}_N} (dd^c u_K^*)^N\,.$$

Let $\rho(z) = |z|^2 - 1$ and $a_K := \min_{z\in K}[-\rho(z)] = -\max_{z\in K}\rho(z)$. Then ρ is in $\mathcal{PSH}$ and is non-positive. Since $a_K > 0$, the function:

$$v(z) = \frac{\rho(z)}{a_K}$$

is in $\mathcal{PSH}$, non-positive on $\mathbb{B}_N$, and $v \leq -1$ on K. Hence $v \leq u_K \leq u_K^*$.

Since $v(w) = 0$ for all $w \in \mathbb{S}_N$ and (see [5, Proposition 6.2 (iv)], or [30, Proposition 4.5.2]):

$$\lim_{z\to w} u_K^*(z) = 0\,,$$

for all $w \in \mathbb{S}_N$, the comparison theorem of Bedford and Taylor ([5, Theorem 4.1]; [30, Theorem 3.7.1] gives, since $v \leq u_K^*$ and $v, u_K^* \in \mathcal{PSH}$:

$$\int_{\mathbb{B}_N} (dd^c u_K^*)^N \leq \int_{\mathbb{B}_N} (dd^c v)^N = \frac{1}{a_K^N}\int_{\mathbb{B}_N} (dd^c\rho)^N\,.$$

As $(dd^c\rho)^N = 4^N N!\, d\lambda_{2N}$, we get, with $C_N := 4^N N!\,\lambda_{2N}(\mathbb{B}_N)$:

$$\mathrm{Cap}\,(K) \leq \frac{C_N}{a_K^N}\,.$$

That ends the proof since:

$$a_K = \min_{z\in K}(1-|z|^2) \geq \min_{z\in K}(1-|z|) = \mathrm{dist}\,(K, \mathbb{S}_N) \qquad \square$$

We have assumed that the symbol φ is non-degenerate. For a degenerate symbol φ, we have:

Proposition 4.16. *Let Ω be a bounded hyperconvex and good complete Reinhardt domain in $\mathbb{C}^N$, and let $\varphi\colon \Omega \to \Omega$ be an analytic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$ is pluripolar. Then $\beta_N(C_\varphi) = 0$.*

Recall that $\varphi(\Omega)$ is pluripolar when φ is degenerate (see [44, Proposition 2]); its closure is also pluripolar if it satisfies the condition (4.9).

Proof. Let $K = \overline{\varphi(\Omega)}$. By hypothesis, we have $\mathrm{Cap}\,(K) = 0$. For every $\varepsilon > 0$, let $K_\varepsilon = \{z \in \Omega;\ \mathrm{dist}\,(z, K) \leq \varepsilon\}$. By Theorem 4.7, we have $\beta_N^+(C_\varphi) \leq \Gamma_N(K_\varepsilon)$. As $\lim_{\varepsilon\to 0} \mathrm{Cap}\,(K_\varepsilon) = \mathrm{Cap}\,(K) = 0$ ([30, Proposition 4.7.1(iv)]), we get $\beta_N(C_\varphi) = 0$. $\square$

Remark 1. In [41, Section 4], we construct a degenerate symbol φ on the bi-disk $\mathbb{D}^2$, defined by $\varphi(z_1, z_2) = \big(\lambda_\theta(z_1), \lambda_\theta(z_1)\big)$, where λ_θ is a lens map, for which $\beta^-(C_\varphi) > 0$. For this function $\overline{\varphi(\mathbb{D}^2)} \cap \partial\mathbb{D}^2 \neq \emptyset$ and hence $\overline{\varphi(\mathbb{D}^2)}$ is not a compact subset of $\mathbb{D}^2$.

Remark 2. In the one dimensional case, for any (non constant) analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$, the parameter $\beta(C_\varphi) = \beta_1(C_\varphi)$ is determined by its range $\varphi(\mathbb{D})$, as shown by the formula:

$$\beta(C_\varphi) = \mathrm{e}^{-1/\mathrm{Cap}\,[\varphi(\mathbb{D})]}$$

proved in [40]. This is no longer true in dimension $N \geq 2$. In [42], we construct pairs of (degenerate) symbols $\varphi_1, \varphi_2\colon \mathbb{D}^2 \to \mathbb{D}^2$, such that $\varphi_1(\mathbb{D}^2) = \varphi_2(\mathbb{D}^2)$ and:

1) C_{φ_1} is not bounded, but C_{φ_2} is compact, and even $\beta_2(C_{\varphi_2}) = 0$;

2) C_{φ_1} is bounded but not compact, so $\beta_2(C_{\varphi_1}) = 1$, and C_{φ_2} is compact, with $\beta_2(C_{\varphi_2}) = 0$;

3) C_{φ_1} is compact, with $0 < \beta_2(C_{\varphi_1}) < 1$, and C_{φ_2} is compact, with $\beta_2(C_{\varphi_2}) = 0$.

Acknowledgements. We thank S. Nivoche and A. Zeriahi for useful discussions and informations, and Y. Tiba, who send us his paper [51]. We than specially S. Nivoche, who carefully read a preliminary version of this paper.

The third-named author is partially supported by the project MTM2015-63699-P (Spanish MINECO and FEDER funds).

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Daniel Li
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec
Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n
41 012 SEVILLA, SPAIN
piazza@us.es