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Some examples of composition operators and their approximation numbers on the Hardy space of the bi-disk

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Abstract. We give examples of composition operators $C_\Phi$ on $H^2(\mathbb{D}^2)$ showing that the condition $\|\Phi\|_\infty = 1$ is not sufficient for their approximation numbers $a_n(C_\Phi)$ to satisfy $\lim_{n \to \infty} [a_n(C_\Phi)]^{1/\sqrt{n}} = 1$, contrary to the 1-dimensional case. We also give a situation where this implication holds. We make a link with the Monge-Ampère capacity of the image of $\Phi$.

Key-words: approximation numbers; Bergman space; bidisk; composition operator; Green capacity; Hardy space; Monge-Ampère capacity; weighted composition operator.


1 Introduction and notation

1.1 Introduction

The purpose of this paper is to continue the study of composition operators on the polydisk initiated in [2], and in particular to examine to what extent one of the main results of [21] still holds.

Let $H$ be a Hilbert space and $T: H \to H$ a bounded operator. Recall that the approximation numbers of $T$ are defined as:

$$a_n(T) = \inf_{\text{rank } R < n} \| T - R \|, \quad n \geq 1,$$

and we have:

$$\| T \| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq \cdots$$

The operator $T$ is compact if and only if $a_n(T) \to 0$. 

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For \( d \geq 1 \), we define:

\[
\begin{align*}
\beta_d^- (T) &= \liminf_{n \to \infty} [a_{n^d}(T)]^{1/n} \\
\beta_d^+ (T) &= \limsup_{n \to \infty} [a_{n^d}(T)]^{1/n}
\end{align*}
\]

We have:

\[ 0 \leq \beta_d^- (T) \leq \beta_d^+ (T) \leq 1, \]

and we simply write \( \beta_d(T) \) in case of equality.

It may well happen in general (consider diagonal operators) that \( \beta_d^- (T) = 0 \) and \( \beta_d^+ (T) = 1 \).

When \( H = H^2(\mathbb{D}) \) is the Hardy space on the open unit disk \( \mathbb{D} \) of \( \mathbb{C} \), and \( T = C_\Phi \) is a composition operator, with \( \Phi : \mathbb{D} \to \mathbb{D} \) a non-constant analytic function, we always have ([19]):

\[ \beta_1^- (C_\Phi) > 0, \]

and one of the main results of [19] is the equivalence:

\[ (1.1) \quad \beta_1^+ (C_\Phi) < 1 \iff \| \Phi \|_\infty < 1. \]

An alternative proof was given in [21], as a consequence of a so-called “spectral radius formula”, which moreover shows that:

\[ \beta_1^- (C_\Phi) = \beta_1^+ (C_\Phi). \]

In [2], for \( d \geq 2 \), it is proved that, for a bounded symmetric domain \( \Omega \subseteq \mathbb{C}^d \), if \( \Phi : \Omega \to \Omega \) is analytic, such that \( \Phi(\Omega) \) has a non-void interior, and the composition operator \( C_\Phi : H^2(\Omega) \to H^2(\Omega) \) is compact, then:

\[ \beta_d^- (C_\Phi) > 0. \]

On the other hand, if \( \Omega \) is a product of balls, then:

\[ \| \Phi \|_\infty < 1 \implies \beta_d^+ (C_\Phi) < 1. \]

We do not know whether the converse holds and the purpose of this paper is to study some examples towards an answer.

The paper is organized as follows. Section 1 is this short introduction, as well as some notations and definitions on singular numbers of operators and Hardy spaces of the polydisk to follow. Section 2 contains preliminary results on weighted composition operators in one variable, which surprisingly play an important role in the study of non-weighted composition operators in two variables. Section 3 studies the case of symbols with “separated” variables. Our main one variable result extends in this case. Section 4 studies the “glued case” \( \Phi(z_1, z_2) = (\phi(z_1), \phi(z_1)) \) for which even boundedness is an issue. Here, the
Bergman space $B^2(D)$ enters the picture. Section 5 studies the case of “triangularly separated” variables. This section lets direct Hilbertian sums of weighted composition operators in one variable appear, and it contains our main result: an example of a symbol $\Phi$ satisfying $\|\Phi\|_\infty = 1$ and yet $\beta_+^2(C_\Phi) < 1$. The final Section 6 discusses the role of the Monge-Ampère pluricapacity, which is a multivariate extension of the Green capacity in the disk. Even though, as evidenced by our counterexample of Section 5, this capacity will not capture all the behavior of the parameter $\beta_m(C_\Phi)$, some partial results are obtained, relying on theorems of S. Nivoche and V. Zakharyuta.

1.2 Notation

We denote by $D$ the open unit disk of the complex plane and by $T$ its boundary, the 1-dimensional torus.

The Hardy space $H^2(D^d)$ is the space of holomorphic functions $f: D^d \to \mathbb{C}$ whose boundary values $f^*$ on $T^d$ are square-integrable with respect to the Haar measure $m_d$ of $T^d$, and normed with:

$$\|f\|_2^2 = \|f\|_{H^2(D^d)}^2 = \int_{T^d} |f^*(\xi_1, \ldots, \xi_d)|^2 \, dm_d(\xi_1, \ldots, \xi_d).$$

If $f(z_1, \ldots, z_d) = \sum_{\alpha_1, \ldots, \alpha_d \geq 0} a_{\alpha_1, \ldots, \alpha_d} z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, then:

$$\|f\|_2^2 = \sum_{\alpha_1, \ldots, \alpha_d \geq 0} |a_{\alpha_1, \ldots, \alpha_d}|^2.$$

We say that an analytic map $\Phi: D^d \to D^d$ is a symbol if its associated composition operator $C_\Phi: H^2(D^d) \to H^2(D^d)$, defined by $C_\Phi(f) = f \circ \Phi$, is bounded.

We say that $\Phi$ is truly $d$-dimensional if $\Phi(D^d)$ has a non-void interior.

We will make use of two kinds of symbols defined on $D$.

The lens map $\lambda_\theta: D \to D$ is defined, for $\theta \in (0, 1)$, by:

$$(1.2) \quad \lambda_\theta(z) = \frac{(1 + z)^\theta - (1 - z)^\theta}{(1 + z)^\theta + (1 - z)^\theta}$$

(see [26], p. 27, or [16], for more information), and corresponds to $u \mapsto u^\theta$ in the right half-plane.

The cusp map $\chi: D \to D$ was first defined in [15] and in a slightly different form in [20]; we actually use here the modified form introduced in [17], and then used in [18]. We first define:

$$\chi_0(z) = \frac{(z - i)^{1/2} - i}{(iz - 1)^{1/2} + 1};$$

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we note that $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2}-1$.

Then we set:

$$
\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},
$$

and finally:

$$
\chi(z) = 1 - \chi_3(z),
$$

where:

$$
(1.3) \quad a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)
$$

is chosen in order that $\chi(0) = 0$. The image $\Omega$ of the (univalent) cusp map is formed by the intersection of the inside of the disk $D(1 - \frac{a}{2}, \frac{a}{2})$ and the outside of the two disks $D(1 + \frac{ia}{2}, \frac{a}{2})$ and $D(1 - \frac{ia}{2}, \frac{a}{2})$.

Besides the approximation numbers, we need other singular numbers for an operator $S : X \to Y$ between Banach spaces $X$ and $Y$.

The **Bernstein numbers** $b_n(S)$, $n \geq 1$, which are defined by:

$$
(1.4) \quad b_n(S) = \sup_{E} \min_{x \in S_E} \|Sx\|,
$$

where the supremum is taken over all $n$-dimensional subspaces of $X$ and $S_E$ is the unit sphere of $E$.

The **Gelfand numbers** $c_n(S)$, $n \geq 1$, which are defined by:

$$
(1.5) \quad c_n(S) = \inf\{\|S|_M\| : \text{codim } M < n\}.
$$

The **Kolmogorov numbers** $d_n(S)$, $n \geq 1$, which are defined by:

$$
(1.6) \quad d_n(S) = \inf_{\dim E < n} \left[\sup_{x \in B_X} \text{dist}(Sx, E)\right].
$$

Pietsch showed that all $s$-numbers on Hilbert spaces are equal (see [24], § 2, Corollary, or [25], Theorem 11.3.4); hence:

$$
(1.7) \quad a_n(S) = b_n(S) = c_n(S) = d_n(S).
$$

We denote $m$ the normalized Lebesgue measure on $T = \partial \mathbb{D}$. If $\varphi : \mathbb{D} \to \mathbb{D}$, $m_\varphi$ is the pull-back measure on $\overline{\mathbb{D}}$ defined by $m_\varphi(E) = m[\varphi^{-1}(E)]$, where $\varphi^*$ stands for the non-tangential boundary values of $\varphi$.

The notation $A \lesssim B$ means that $A \leq C B$ for some positive constant $C$ and we write $A \approx B$ if we have both $A \lesssim B$ and $B \lesssim A$.  

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2 Preliminary results on weighted composition operators on $H^2(\mathbb{D})$

We see in this section that the presence of a “rapidly decaying” weight allows simpler estimates for the approximation numbers of a corresponding weighted composition operator. Such a study, but a bit different, is made in [14].

Let $\varphi: \mathbb{D} \to \mathbb{D}$ a non-constant analytic self-map in the disk algebra $A(\mathbb{D})$ such that, for some constant $C > 1$ and for all $z \in \mathbb{D}$:

\begin{equation}
\varphi(1) = 1, \quad |1 - \varphi(z)| \leq 1, \quad |1 - \varphi(z)| \leq C \left(1 - |\varphi(z)| \right)
\end{equation}

as well as $\varphi(z) \neq 1$ for $z \neq 1$. We can take for example $\varphi = \frac{1 + \lambda \theta}{2}$ where $\lambda \theta$ is the lens map with parameter $\theta$.

Let $w \in H^\infty$ and let $T$ be the weighted composition operator $T = M_w \circ \varphi C : H^2 \to H^2$.

Note that $M_w \circ \varphi C = C \circ M_w$. We first show that:

**Theorem 2.1.** Let $T = M_w \circ \varphi C : H^2 \to H^2$ be as above and let $B$ be a Blaschke product with length $< N$. Then, with the implied constant depending only on the number $C$ in (2.1) (and of $\varphi$):

$$a_N(T) \lesssim \sup_{|z - 1| \leq 1, z \in \varphi(\mathbb{D})} |B(z)||w(z)|.$$

**Proof.** The following preliminary observation (see also [16], p. 809), in which we denote by $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}$ the Carleson window with center $\xi \in \mathbb{T}$ and size $h$, and by $K_\varphi$ the support of the pull-back measure $m_\varphi$, will be useful.

\begin{equation}
u \in S(\xi, h) \cap K_\varphi \implies \nu \in S(1, Ch) \cap K_\varphi.
\end{equation}

Indeed, if $|u - \xi| \leq h$ and $u \in K_\varphi$, (2.1) implies:

$$1 - |u| \leq |u - \xi| \leq h \quad \text{and} \quad |u - 1| \leq C(1 - |u|) \leq Ch.$$

Set $E = BH^2$. This is a subspace of codimension $< N$. If $f = Bg \in E$, with $\|g\| = \|f\|$ (isometric division by $B$ in $BH^2$), we have $Tf = (wBg) \circ \varphi$, whence:

$$\|T(f)\|^2 = \int_{\mathbb{D}} |B|^2|w|^2|g|^2dm_\varphi,$$

implying $\|T(f)\|^2 \leq \|f\|^2\|J\|^2$ where $J: H^2 \to L^2(\sigma)$ is the natural embedding and where

$$\sigma = |B|^2|w|^2dm_\varphi.$$
Now, Carleson’s embedding theorem for the measure $\sigma$ and (2.2) show that (the implied constants being absolute):

$$
\|J\|_2 \lesssim \sup_{0<h<1} \frac{1}{h} \int_{S(\xi,h) \cap K_\varphi} |B|^2 |w|^2 \, dm_\varphi
$$

$$
\lesssim \sup_{0<h<1} \frac{1}{h} \int_{S(1,h) \cap K_\varphi} |B|^2 |w|^2 \, dm_\varphi
$$

$$
\lesssim \left( \sup_{|z-1| \leq 1, \ z \in \varphi(D)} |B(z)|^2 |w(z)|^2 \right) \left( \sup_{0<h<1} \frac{1}{h} \int_{S(1,h) \cap K_\varphi} dm_\varphi \right)
$$

since $m_\varphi$ is a Carleson measure for $H^2$ and where we used that, according to (2.1):

$$
K_\varphi \subseteq \overline{\varphi(D)} \subseteq \{ z \in \mathbb{D}; \ |z-1| \leq 1 \}.
$$

This ends the proof of Theorem 2.1 with help of the equality of $a_N(T)$ with the Gelfand number $c_N(T)$ recalled in (1.7).

In order to specialize efficiently the general Theorem 2.1, we recall the following simple Lemma 2.3 of [16], where:

(2.3) \[ \rho(a,b) = \left| \frac{a-b}{1-\overline{a}b} \right|, \quad a,b \in \mathbb{D}, \]

is the pseudo-hyperbolic distance:

**Lemma 2.2 ([16]).** Let $a, b \in \mathbb{D}$ such that $|a-b| \leq L \min(1-|a|, 1-|b|)$. Then:

$$
\rho(a,b) \leq \frac{L}{\sqrt{L^2 + 1}} =: \kappa < 1.
$$

We can now state:

**Theorem 2.3.** Assume that $\varphi$ is as in (2.1) and that the weight $w$ satisfies, for some parameters $0 < \theta \leq 1$ and $R > 0$:

$$
|w(z)| \leq \exp \left( -\frac{R}{|1-z|^\theta} \right), \quad \forall z \in \mathbb{D} \text{ with } \Re z \geq 0.
$$

Then, the approximation numbers of $T = M_{\omega \circ \varphi} C_\varphi$ satisfy:

$$
a_{nm+1}(T) \lesssim \max \left[ \exp(-an), \exp(-R 2^m \theta) \right],
$$

for all integers $n, m \geq 1$, where $a = \log[\sqrt{16C^2 + 1}/(4C)] > 0$ and $C$ is as in (2.1).
Proof. Let \( p_l = 1 - 2^{-l} \), \( 0 \leq l < m \) and let \( B \) be the Blaschke product:

\[
B(z) = \prod_{0 \leq l < m} \left( \frac{z - p_l}{1 - p_l z} \right)^n.
\]

Let \( z \in K_\varphi \cap \mathbb{D} \) so that \( 0 < \lvert z - 1 \rvert \leq 1 \). Let \( l \) be the non-negative integer such that \( 2^{-l-1} < \lvert z - 1 \rvert \leq 2^{-l} \). We separate two cases:

Case 1: \( l \geq m \). Then, the weight does the job. Indeed, majorizing \( |B(z)| \) by 1 and using the assumption on \( w \), we get:

\[
|B(z)|^2 |w(z)|^2 \leq |w(z)|^2 \leq \exp \left( - \frac{2R}{1 - |z|} \right) \leq \exp(-2R 2^{j_l}) \leq \exp(-2R 2^{m_l}).
\]

Case 2: \( l < m \). Then, the Blaschke product does the job. Indeed, majorize \( |w(z)| \) by 1 and estimate \( |B(z)| \) more accurately with help of Lemma 2.2; we observe that

\[
\lvert z - p_l \rvert \leq \lvert z - 1 \rvert + 1 - p_l \leq 2 \times 2^{-l} = 2(1 - p_l) \leq 4C(1 - p_l)
\]

and then, since \( z \in K_\varphi \), we can write with \( C \geq 1 \) as in (2.1):

\[
1 - |z| \geq \frac{1}{C} |1 - z| \geq \frac{1}{2C} 2^{-l} \geq \frac{1}{4C} |z - p_l|,
\]

so that the assumptions of Lemma 2.2 are verified with \( L = 4C \), giving:

\[
\rho(z, p_l) \leq \frac{4C}{\sqrt{16C^2 + 1}} = \exp(-a) < 1.
\]

Hence, by definition, since \( l < m \):

\[
|B(z)| \leq |\rho(z, p_l)|^n \leq \exp(-an).
\]

Putting both cases together, and observing that our Blaschke product has length \( nm < nm + 1 \), we get the result by applying Theorem 2.1 with \( N = nm + 1 \).

2.1 Some remarks

1. Twisting a composition operator by a weight may improve the compactness of this composition operator, or even may make this weighted composition operator compact though the non-weighted was not (see [8] or [14]). However, this is not possible for all symbols, as seen in the following proposition.

Proposition 2.4. Let \( w \in H^\infty \). If \( \varphi \) is inner, or more generally if \( |\varphi| = 1 \) on a subset of \( \mathbb{T} \) of positive measure, then \( M_w C_\varphi \) is never compact (unless \( w \equiv 0 \)).
Proof. Indeed, suppose $T = M_wC_\phi$ compact. Since $(z^n)_n$ converges weakly to 0 in $H^2$ and since $T(z^n) = w\varphi^n$, we should have, since $|\varphi| = 1$ on $E$, with $m(E) > 0$:
\[
\int_E |w|^2 \, dm = \int_E |w|^2 |\varphi|^{2n} \, dm \leq \int_T |w|^2 |\varphi|^{2n} \, dm = \|T(z^n)\|^2 \rightarrow 0,
\]
but this would imply that $w$ is null a.e. on $E$ and hence $w \equiv 0$ (see [7], Theorem 2.2), which was excluded.

Note that É. Amar and A. Lederer proved in [1] that $|\varphi| = 1$ on a set of positive measure if and only if $\varphi$ is an exposed point of the unit ball of $H^\infty$; hence the following proposition can be viewed as the (almost) opposite case.

**Proposition 2.5.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\|\varphi\|_\infty = 1$. Assume that:
\[
\int_T \log(1 - |\varphi|) \, dm > -\infty
\]
(meaning that $\varphi$ is not an extreme point of the unit ball of $H^\infty$: see [7], Theorem 7.9). Then, if $w$ is an outer function such that $|w| = 1 - |\varphi|$, the weighted composition operator $T = M_wC_\varphi$ is Hilbert-Schmidt.

Proof. We have:
\[
\sum_{n=0}^{\infty} \|T(z^n)\|^2 = \sum_{n=0}^{\infty} \int_T (1 - |\varphi|)^2 |\varphi|^{2n} \, dm = \int_T \frac{1 - |\varphi|}{1 + |\varphi|} \, dm < +\infty,
\]
and $T$ is Hilbert-Schmidt, as claimed. □

2. In [14], Theorem 2.5, it is proved that we always have, for some constants $\delta, \rho > 0$:
\[
(2.4) \quad a_n(M_wC_\varphi) \geq \delta \rho^n, \quad n = 1, 2, \ldots
\]
(if $w \not\equiv 0$). We give here an alternative proof, based on a result of Gunatillake ([9]), this result holding in a wider context.

**Theorem 2.6** (Gunatillake). Let $T = M_wC_\varphi$ be a compact weighted composition operator on $H^2$ and assume that $\varphi$ has a fixed point $a \in \mathbb{D}$. Then the spectrum of $T$ is the set:
\[
\sigma(T) = \{0, w(a), w(a)\varphi'(a), w(a)[\varphi'(a)]^2, \ldots, w(a)[\varphi'(a)]^n, \ldots\}
\]

Proof of (2.4). First observe that, in view of Proposition 2.4, $\varphi$ cannot be an automorphism of $\mathbb{D}$ so that the point $a$ is the Denjoy-Wolff point of $\varphi$ and is attractive. Theorem 2.6 is interesting only when $w(a) \varphi'(a) \not= 0$. 

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Now, we can give a new proof Theorem 2.5 of [14] as follows. Let $a \in \mathbb{D}$ be such that $w(a) \varphi'(a) \neq 0$ ($H(\mathbb{D})$ is a division ring and $\varphi' \neq 0$, $w \neq 0$). Let $b = \varphi(a)$ and $\tau \in \text{Aut} \mathbb{D}$ with $\tau(b) = a$. We set:

$$\psi = \tau \circ \varphi \quad \text{and} \quad S = M_wC_\psi = TC_\tau.$$ 

This operator $S$ is compact because $T$ is.

Since $\psi(a) = a$ and $\psi'(a) = \tau'(b)\varphi'(a) \neq 0$, Theorem 2.6 says that the non-zero eigenvalues of $S$, arranged in non-increasing order, are the numbers $\lambda_n = w(a)|\psi'(a)|^{n-1}$, $n \geq 1$. As a consequence of Weyl’s inequalities, we know that:

$$a_1(S)a_n(S) \geq |\lambda_{2n}|^2 \geq \delta \rho^n,$$

with:

$$\delta = |w(a)|^2 > 0 \quad \text{and} \quad \rho = |\psi'(a)|^4 > 0.$$

To finish, it is enough to observe that $a_n(S) \leq a_n(T)\|C_\tau\|$ by the ideal property of approximation numbers. 

3 The splitted case

**Theorem 3.1.** Let $\Phi = (\varphi, \psi) : \mathbb{D}^d \to \mathbb{D}^d$ be a truly $d$-dimensional symbol with $\varphi : \mathbb{D} \to \mathbb{D}$ depending only on $z_1$ and $\psi : \mathbb{D}^{d-1} \to \mathbb{D}^{d-1}$ only on $z_2, \ldots, z_d$, i.e. $\Phi(z_1, z_2, \ldots, z_d) = (\varphi(z_1), \psi(z_2, \ldots, z_d))$. Then, whatever $\psi$ behaves:

$$\|\varphi\|_\infty = 1 \implies \beta_d(C_\Phi) = 1.$$

**Proof.** The proof is based on the following simple lemma, certainly well-known.

**Lemma 3.2.** Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two compact linear operators, where $H_1$ and $H_2$ are Hilbert spaces. Let $S \otimes T$ be their tensor product, acting on the tensor product $H_1 \otimes H_2$. Then:

$$a_{mn}(S \otimes T) \geq a_m(S)a_n(T)$$

for all positive integers $m,n$.

We postpone the proof of the lemma and show how to conclude.

We can assume $C_\Phi$ to be compact, so that $C_\varphi$ is compact as well. Since $\|\varphi\|_\infty = 1$, we have, thanks to (1.1):

$$a_m(C_\varphi) \geq e^{-m}\varepsilon_m \quad \text{with} \quad \varepsilon_m \underset{m \to \infty}{\to} 0.$$

Replacing $\varepsilon_m$ by $\delta_m := \sup_{p \geq m} \varepsilon_p$, we can assume that $(\varepsilon_m)_m$ is non-increasing. Moreover,

$$m \varepsilon_m \to \infty$$

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since $C_\phi$ is compact and hence $a_m(C_\phi) \to 0$. We next observe that, due to the separation of variables in the definition of $\phi$ and $\psi$, we can write:

\[ C_\Phi = C_\phi \otimes C_\psi. \tag{3.1} \]

Indeed, write $z = (z_1, w)$ with $z_1 \in \mathbb{D}$ and $w \in \mathbb{D}^{d-1}$. If $f \in H^2(\mathbb{D})$ and $g \in H^2(\mathbb{D}^{d-1})$, we see that:

\[ C_\Phi(f \otimes g)(z) = (f \otimes g)(\phi(z_1), \psi(w)) = f(\phi(z_1))g(\psi(w)) = [C_\phi f(z_1)] [C_\psi g(w)] = (C_\phi f \otimes C_\psi g)(z). \]

Since tensor products $f \otimes g$ generate $H^2(\mathbb{D}^{d}) = H^2(\mathbb{D}) \otimes H^2(\mathbb{D}^{d-1})$, this proves (3.1).

Let now $m$ be a large positive integer. Set $[\cdot]$ stands for the integer part:

\[ n_m = [m \varepsilon_m]^{d-1} \quad \text{and} \quad N_m = m n_m. \tag{3.2} \]

From what we know in dimension $d-1$ (see [2], Theorem 3.1) and from the preceding, we can write (observe that $\psi$ has to be truly $(d-1)$-dimensional since $\Phi$ is truly $d$-dimensional):

\[ a_m(C_\phi) \geq \exp(-m \varepsilon_m) \quad \text{and} \quad a_n(C_\psi) \geq a \exp(-C n^{1/(d-1)}), \]

for some positive constant $C$, which will be allowed to vary from one formula to another. Lemma 3.2 implies:

\[ a_{N_m}(C_\Phi) \geq a \exp[-C (m \varepsilon_m + n_m^{1/(d-1)})]. \]

Since $n_m \lesssim (m \varepsilon_m)^{d-1}$, we get:

\[ a_{N_m}(C_\Phi) \geq a \exp(-C m \varepsilon_m). \]

Observe that $N_m = m n_m \sim m^d \varepsilon_m^{d-1}$ and so $N_m^{1/d} \sim m^{1-1/d} \varepsilon_m$. As a consequence:

\[ a_{N_m}(C_\Phi) \geq a \exp(-C m \varepsilon_m) = a \exp \left[ - (C \varepsilon_m^{1/d}) (m \varepsilon_m^{1-1/d}) \right] \geq a \exp(-\eta_m N_m^{1/d}) \]

with $\eta_m := C \varepsilon_m^{1/d}$.

Now, for $N > N_1$, let $m$ be the smallest integer satisfying $N_m \geq N$ (so that $N_{m-1} < N \leq N_m$), and set $\delta_N = \eta_m$. We have $\lim_{N \to \infty} \delta_N = 0$. Next, we note that $\lim_{m \to \infty} N_m/N_{m-1} = 1$, because $N_m \geq N_{m-1}$ and:

\[ \frac{N_m}{N_{m-1}} \leq \frac{m}{m-1} \left( \frac{m \varepsilon_m + 1}{(m-1) \varepsilon_{m-1}} \right)^{d-1} \sim \left( \frac{\varepsilon_m}{\varepsilon_{m-1}} \right)^{d-1} \leq 1. \]

Finally, if $N$ is an arbitrary integer and $N_{m-1} < N \leq N_m$, we obtain:

\[ a_N(C_\Phi) \geq a_{N_m}(C_\Phi) \geq a \exp(-\eta_m N_m^{1/d}) \geq a \exp(-C \delta_N N^{1/d}), \]

since we observed that $\lim_{m \to \infty} N_m/N_{m-1} = 1$.

This amounts to say that $\beta_d(C_\Phi) = 1$.  \[\square\]
Proof of Lemma 3.2. It is rather formal. Start from the Schmidt decompositions of $S$ and $T$ respectively (recall that Hilbert spaces, the approximation numbers are equal to the singular ones):

$$S = \sum_{m=1}^{\infty} a_m(S) u_m \odot v_m, \quad T = \sum_{n=1}^{\infty} a_n(T) u'_n \odot v'_n,$$

where $(u_m), (v_m)$ are two orthonormal sequences of $H_1$, $(u'_n), (v'_n)$ two orthonormal sequences of $H_2$, and $u_m \odot v_m$ and $u'_n \odot v'_n$ denote the rank one operators defined by $(u_m \odot v_m)(x) = \langle x, v_m \rangle u_m, \ x \in H_1$, and $(u'_n \odot v'_n)(x) = \langle x, v'_n \rangle u'_n, \ x \in H_2$.

We clearly have:

$$(u_m \odot v_m) \odot (u'_n \odot v'_n) = (u_m \odot u'_n) \odot (v_m \odot v'_n),$$

so that the Schmidt decomposition of $S \otimes T$ is (with SOT-convergence):

$$S \otimes T = \sum_{m,n \geq 1} a_m(S) a_n(T) (u_m \odot u'_n) \odot (v_m \odot v'_n),$$

since the two sequences $(u_m \odot u'_n)_{m,n}$ and $(v_m \odot v'_n)_{m,n}$ are orthonormal: for instance, we have by definition:

$$\langle u_{m_1} \odot u'_{n_1}, u_{m_2} \odot u'_{n_2} \rangle = \langle u_{m_1}, u_{m_2} \rangle \langle u'_{n_1}, u'_{n_2} \rangle.$$

This shows that the singular values of $S \otimes T$ are the non-increasing rearrangement of the positive numbers $a_m(S) a_n(T)$ and ends the proof of the lemma: the $mn$ numbers $a_k(S) a_l(T)$, for $1 \leq k \leq m$, $1 \leq l \leq n$ all satisfy $a_k(S) a_l(T) \geq a_m(S) a_n(T)$, so that $a_{mn}(S \otimes T) \geq a_m(S) a_n(T)$. \hfill $\square$

4 The glued case

Here we consider symbols of the form:

$$(4.1) \quad \Phi(z_1, z_2) = \{ \phi(z_1), \phi(z_1) \},$$

where $\phi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic map.

Note that such maps $\Phi$ are not truly 2-dimensional.

4.1 Preliminary

We begin by remarking the following fact.

Let $B^2(\mathbb{D})$ be the Bergman space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{B^2}^2 := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where $dA$ is the normalized area measure on $\mathbb{D}$. 

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Proposition 4.1. Assume that the composition operator $C\phi$ maps boundedly $B^2(D)$ into $H^2(D)$. Then $C\Phi : H^2(D^2) \to H^2(D^2)$, defined by (4.1), is bounded.

Proof. If we write $f \in H^2(D^2)$ as:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k,$$

with $\sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|^2_{H^2}$,

we formally (or assuming that $f$ is a polynomial) have:

$$[C\Phi f](z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\phi(z_1)]^k = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) [\phi(z_1)]^n.$$

Hence, if we set $g(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n$, we get:

$$[C\Phi(f)](z_1, z_2) = [C\phi(g)](z_1),$$

so that, by integrating:

$$\|C\Phi(f)\|_{H^2(D^2)} = \|C\phi(g)\|_{H^2(D)}.$$

By hypothesis, there is a positive constant $M$ such that:

$$\|C\phi(g)\|_{H^2(D)} \leq M \|g\|_{B^2(D)}.$$

But, by the Cauchy-Schwarz inequality:

$$\|g\|^2_{H^2(D)} = \sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{j+k=n} c_{j,k} \right|^2 \leq \sum_{n=0}^{\infty} \left( \sum_{j+k=n} |c_{j,k}|^2 \right) = \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|^2_{H^2(D^2)},$$

and we obtain $\|C\Phi(f)\|_{H^2(D^2)} \leq M \|f\|_{H^2(D^2)}$. \hfill \Box

4.2 Lens maps

Let $\lambda_\theta$ be a lens map of parameter $\theta, 0 < \theta < 1$. We consider $\Phi_\theta : D^2 \to D^2$ defined by:

$$\Phi_\theta(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_1)).$$

We have the following result.

Theorem 4.2. The composition operator $C\phi : H^2(D^2) \to H^2(D^2)$ is:

1) not bounded for $\theta > 1/2$;
2) bounded, but not compact for $\theta = 1/2$;
3) compact, and even Hilbert-Schmidt, for $0 < \theta < 1/2$.  

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Proof. The reproducing kernel of \( H^2(\mathbb{D}^2) \) is, for \((a, b) \in \mathbb{D}^2\):

\[
K_{a,b}(z_1, z_2) = \frac{1}{1 - a \bar{z}_1} \frac{1}{1 - b \bar{z}_2}, \quad (z_1, \bar{z}_2) \in \mathbb{D}^2,
\]

and:

\[
\|K_{a,b}\|^2 = \frac{1}{(1 - |a|^2)(1 - |b|^2)}.
\]

1) If \( C_{\Phi_\theta} \) were bounded, we should have, for some \( M < \infty \):

\[
\|C_{\Phi_\theta}^*(K_{a,b})\|_{H^2} \leq M \|K_{a,b}\|_{H^2}, \quad \text{for all } a, b \in \mathbb{D}.
\]

Since \( C_{\Phi_\theta}^*(K_{a,b}) = K_{\lambda_\theta(a), \lambda_\theta(a)} \), we would get, with \( b = 0 \):

\[
\left( \frac{1}{1 - |\lambda_\theta(a)|^2} \right)^2 \leq M^2 \frac{1}{1 - |a|^2};
\]

but this is not possible for \( \theta > 1/2 \), since \( 1 - |\lambda_\theta(a)|^2 \approx 1 - |\lambda_\theta(a)| \sim (1 - a)^\theta \)
when \( a \) goes to 1, with \( 0 < a < 1 \).

For 2) and 3), let us consider the pull-back measure \( m_\theta \) of the normalized Lebesgue measure on \( \mathbb{T} = \partial \mathbb{D} \) by \( \lambda_\theta \). It is easy to see that:

\[
\sup_{\xi \in \mathbb{T}} m_\theta(D(\xi, h) \cap \mathbb{D}) = m_\theta[D(1, h) \cap \mathbb{D}] \approx h^{1/\theta}.
\]

In particular, for \( \theta \leq 1/2 \), \( m_\theta \) is a 2-Carleson measure, and hence (see [15], Theorem 2.1, for example) the canonical injection \( j : B^2(\mathbb{D}) \rightarrow L^2(m_\theta) \) is bounded, meaning that, for some positive constant \( M < \infty \):

\[
\int_{\mathbb{D}} |f(z)|^2 \, dm_\theta(z) \leq M^2 \|f\|^2_{H^2}.
\]

Since

\[
\int_{\mathbb{D}} |f(z)|^2 \, dm_\theta(z) = \int_{\mathbb{T}} |f[\lambda_\theta(u)]|^2 \, dm(u) = \|C_{\lambda_\theta}(f)\|^2_{H^2},
\]

we get that \( C_{\lambda_\theta} \) maps boundedly \( B^2(\mathbb{D}) \) into \( H^2(\mathbb{D}) \).

It follows from Proposition 4.1 that \( C_{\Phi_\theta} : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2) \) is bounded.

However, \( C_{\Phi_{1/2}} \) is not compact since \( C_{\Phi_{1/2}}^*(K_{a,b})/\|K_{a,b}\| \) does not converge to 0 as \( a, b \rightarrow 1 \), by the calculations made in 1).

For 3), let \( e_{j,k}(z_1, z_2) = z_1^j \bar{z}_2^k \), \( j, k \geq 0 \), be the canonical orthonormal basis of \( H^2(\mathbb{D}^2) \); we have \( |C_{\Phi_\theta}(e_{j,k})|(z_1, z_2) = |\lambda_\theta(z_1)|^{j+k} \). Hence:

\[
\sum_{j, k \geq 0} \|C_{\Phi_\theta}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \leq \sum_{n=0}^{\infty} (2n + 1) \int_{\mathbb{T}} |\lambda_\theta|^2n \, dm \leq \int_{\mathbb{T}} \frac{2}{(1 - |\lambda_\theta|^2)^2} \, dm.
\]

Since, by Lemma 4.3 below, \( 1 - |\lambda_\theta(e^{it})|^2 \geq |1 - e^{it}|^\theta \geq t^\theta \) for \( |t| \leq \pi/2 \), we get:

\[
\sum_{j, k \geq 0} \|C_{\Phi_\theta}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \leq \int_{0}^{\pi/2} \frac{dt}{t^{2\theta}} < \infty,
\]

since \( \theta < 1/2 \). Therefore \( C_{\Phi_\theta} \) is Hilbert-Schmidt for \( \theta < 1/2 \).
For sake of completeness, we recall the following elementary fact (see [26], p. 28, or also [16], Lemma 2.5)).

**Lemma 4.3.** With \( \delta = \cos(\theta \pi/2) \), we have, for \(|z| \leq 1\) and \( \Re z \geq 0 \):

\[
1 - |\lambda_\theta(z)|^2 \geq \frac{\delta}{2} |1 - z|^\theta.
\]

**Proof.** We can write:

\[
\lambda_\theta(z) = \frac{1 - w}{1 + w} \quad \text{with} \quad w = \left(\frac{1 - z}{1 + z}\right)^\theta \quad \text{and} \quad |w| \leq 1.
\]

Then:

\[
\Re w \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta.
\]

Hence:

\[
1 - |\lambda_\theta(z)|^2 = \frac{4 \Re w}{|1 + w|^2} \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta,
\]

as announced. 

We now improve the result 3) of Theorem 4.2 by estimating the approximation numbers of \( C_{\Phi_\theta} \) and get that \( C_{\Phi_\theta} \) is in all Schatten classes of \( H^2(\mathbb{D}^2) \) when \( \theta < 1/2 \).

**Theorem 4.4.** For \( 0 < \theta < 1/2 \), there exists \( b = b_\theta > 0 \) such that:

\[
a_n(C_{\Phi_\theta}) \lesssim e^{-b \sqrt{n}}.
\]

In particular \( \beta^+_n(C_{\Phi_\theta}) \leq e^{-b} < 1 \), though \( \|\Phi_\theta\|_\infty = 1 \), and even \( \Phi_\theta(T^2) \cap T^2 \neq \emptyset \).

**Proof.** Proposition 4.1 (and its proof) can be rephrased in the following way: if \( C_\phi \) maps boundedly \( B^2(\mathbb{D}) \) into \( H^2(\mathbb{D}) \), then, we have the following factorization:

\[
C_\Phi: H^2(\mathbb{D}^2) \to B^2(\mathbb{D}) \xrightarrow{C_\phi} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2),
\]

where \( I: H^2(\mathbb{D}) \to H^2(\mathbb{D}^2) \) is the canonical injection given by \((I f)(z_1, z_2) = f(z_1)\) for \( f \in H^2(\mathbb{D}) \), and \( J: H^2(\mathbb{D}^2) \to B^2(\mathbb{D}) \) is the contractive map defined by:

\[
(J f)(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n,
\]

for \( f \in H^2(\mathbb{D}^2) \) with \( f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k \).

In the proof of Theorem 4.2, we have seen that, for \( 0 < \theta \leq 1/2 \), the composition operator \( C_{\lambda_\theta} \) is bounded from \( B^2(\mathbb{D}) \) into \( H^2(\mathbb{D}) \); we get hence the factorization:

\[
C_{\Phi_\theta}: H^2(\mathbb{D}^2) \to B^2(\mathbb{D}) \xrightarrow{C_{\lambda_\theta}} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2),
\]

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Now, the lens maps have a semi-group property:

\[ \lambda_{\theta_1 \theta_2} = \lambda_{\theta_1} \lambda_{\theta_2}, \]

giving \( C_{\lambda_{\theta_1 \theta_2}} = C_{\lambda_{\theta_1}} \circ C_{\lambda_{\theta_2}} \).

For \( 0 < \theta < 1/2 \), we therefore can write \( C_{\lambda_{\theta}} = C_{\lambda_{2\theta}} \circ C_{\lambda_{1/2}} \) (note that \( 2\theta < 1 \), so \( C_{\lambda_{2\theta}} : H^2(D) \to H^2(D) \) is bounded), and we get:

\[ C_{\Phi_{\theta}} = I C_{\lambda_{2\theta}} C_{\lambda_{1/2}} J. \]

Consequently:

\[ a_n(C_{\Phi_{\theta}}) \leq \|I\| \|J\| \|C_{\lambda_{1/2}}\|_{B^2 \to H^2} a_n(C_{\lambda_{2\theta}}). \]

Now, we know ([16], Theorem 2.1) that \( a_n(C_{\lambda_{2\theta}}) \lesssim e^{-b\sqrt{n}} \), so we get that \( a_n(C_{\Phi_{\theta}}) \lesssim e^{-b\sqrt{n}}. \)

**Remark.** In [2], we saw that for a truly 2-dimensional symbol \( \Phi \), we have \( \beta_2(C_{\Phi}) > 0 \). Here the symbol \( \Phi_{\theta} \) is not truly 2-dimensional, but we nevertheless have \( \beta_2(C_{\Phi_{\theta}}) > 0 \). In fact, let \( E = \{ f \in H^2(D^2) : \frac{\partial f}{\partial z_2} \equiv 0 \} \); \( E \) is isometrically isomorphic to \( H^2(D) \) and the restriction of \( C_{\Phi_{\theta}} \) to \( E \) behaves as the 1-dimensional composition operator \( C_{\lambda_{\theta}} : H^2(D) \to H^2(D) \); hence ([19], Proposition 6.3):

\[ e^{-b_0\sqrt{n}} \lesssim a_n(C_{\lambda_{\theta}}) = a_n(C_{\Phi_{\theta}|E}) \leq a_n(C_{\Phi_{\theta}}) \]

and \( \beta_2^{-}(C_{\Phi_{\theta}}) \geq e^{-b_0} > 0. \)

### 5 Triangularly separated variables

In this section, we consider symbols of the form:

\[ (5.1) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2), \]

where \( \phi, \psi : D \to D \) are non-constant analytic maps.

Such maps \( \Phi \) are truly 2-dimensional.

More generally, if \( h \in H^\infty \), with \( h(0) = 0 \) and \( \|h\|_\infty \leq 1 \), has its powers \( h^k, k \geq 0 \), orthogonal in \( H^2 \) (for convenience, we shall say that \( h \) is a Rudin function), we can consider:

\[ (5.2) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)) \]

For such \( h \) we can take for example an inner function vanishing at the origin, but there are other such functions, as shown by C. Bishop:
Theorem (Bishop [4]). The function $h$ is a Rudin function if and only if the pull-back measure $\mu = \mu_h$ is radial and Jensen, i.e for every Borel set $E$:

$$\mu(e^{i\theta} E) = \mu(E) \quad \text{and} \quad \int_D \log(1/|z|) \, d\mu(z) < \infty.$$ 

Conversely, for every probability measure $\mu$ supported by $D$, which is radial and Jensen, there exists $h$ in the unit ball of $H^\infty$, with $h(0) = 0$, such that $\mu = \mu_h$.

If we take for $\mu$ the Lebesgue measure of $T$, we get an inner function. But, as remarked in [4], we can take for $\mu$ the Lebesgue measure on the union $T \cup (1/2)T$, normalized in order that $\mu(T) = \mu((1/2)T) = 1/2$. Then the corresponding $h$ is not inner since $|h| = 1/2$ on a subset of $T$ of positive measure. He also showed that $h(z)/z$ may be a non-constant outer function. Also, P. Bourdon ([6]) showed that the powers of $h$ are orthogonal if and only if its Nevanlinna counting function is almost everywhere constant on each circle centered on the origin.

5.1 General facts

We first observe that if $f \in H^2(D^2)$ and:

$$f(z_1, z_2) = \sum_{j, k \geq 0} c_{j,k} z_1^j z_2^k,$$

then we can write:

$$f(z_1, z_2) = \left( \sum_{k \geq 0} f_k(z_1) \right) z_2^k$$

with:

$$f_k(z_1) = \sum_{j \geq 0} c_{j,k} z_1^j,$$

and:

$$\|f\|^2_{H^2(D^2)} = \sum_{j, k \geq 0} |c_{j,k}|^2 = \sum_{k \geq 0} \|f_k\|^2_{H^2(D)}.$$

That means that we have an isometric isomorphism:

$$J : H^2(D^2) \longrightarrow \bigoplus_{k=0}^\infty H^2(D),$$

defined by $Jf = (f_k)_{k \geq 0}$.

Now, for symbols $\Phi$ as in (5.1), we have:

$$(C\Phi f)(z_1, z_2) = \sum_{j, k \geq 0} c_{j,k} [\phi(z_1)]^j [\psi(z_1)]^k z_2^k,$$
so that $JC_\Phi J^{-1}$ appears as the operator $\bigoplus_k M_{\psi^k}C_\phi$ on $\bigoplus_k H^2(\mathbb{D})$, where $M_{\psi^k}$ is the multiplication operator by $\psi^k$: 

$$[(M_{\psi^k}C_\phi)f_k](z_1) = [\psi(z_1)]^k [(f_k \circ \phi)(z_1)].$$

When $\Phi$ is as in (5.2), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\psi(z_1)]^k [h(z_2)]^k,$$

with:

$$\|C_\Phi f\|^2 \leq \sum_{k=0}^\infty \|T_k f_k\|^2$$

and:

$$T_k = M_{\psi^k}C_\phi;$$

hence $JC_\Phi J^{-1}$ appears as pointwise dominated by the operator $T = \bigoplus_k T_k$ on $\bigoplus_k H^2(\mathbb{D})$. This implies a factorization $C_\Phi = AT$ with $\|A\| \leq 1$, so that $a_n(C_\Phi) \leq a_n(T)$ for all $n \geq 1$.

We recall the following elementary fact.

**Lemma 5.1.** Let $(H_k)_{k \geq 0}$ be a sequence of Hilbert spaces and $T_k: H_k \to H_k$ be bounded operators. Let $H = \bigoplus_{k=0}^\infty H_k$ and $T: H \to H$ defined by $Tx = (T_k x_k)_k$. Then:

1) $T$ is bounded on $H$ if and only if $\sup_k \|T_k\| < \infty$;
2) $T$ is compact on $H$ if and only if each $T_k$ is compact and $\|T_k\| \to 0$ as $k \to \infty$.

Going back to the symbols of the form (5.1), we have $\|M_{\psi^k}\| \leq \|\psi^k\|_\infty \leq 1$, since $\|\psi\|_\infty \leq 1$; hence $\|M_{\psi^k}C_\phi\| \leq \|C_\phi\|$ and the operator $(M_{\psi^k}C_\phi)_k$ is bounded on $\bigoplus_k H^2(\mathbb{D})$. Therefore $C_\phi$ is bounded on $H^2(\mathbb{D}^2)$.

For approximation numbers, we have the following two facts.

**Lemma 5.2.** Let $T_k: H_k \to H_k$ be bounded linear operators between Hilbert spaces $H_k$, $k \geq 0$. Let $H = \bigoplus_{k=0}^\infty H_k$ and $T = (T_k)_k: H \to H$, assumed to be compact. Then, for every $n_1, \ldots, n_K \geq 1$, and $0 \leq m_1 < \cdots < m_K$, $K \geq 1$, we have:

$$a_N(T) \geq \inf_{1 \leq k \leq K} a_{n_k}(T_{m_k}),$$

where $N = n_1 + \cdots + n_K$.

**Proof.** We use the Bernstein numbers $b_n$ (see (1.4)), which are equal to the approximation numbers (see (1.7)).

For $k = 1, \ldots, K$, there is an $n_k$-dimensional subspace $E_k$ of $H_{m_k}$ such that:

$$b_{n_k}(T_{m_k}) \leq \|T_{m_k}x\|, \quad \text{for all } x \in S_{E_k}.$$
Then $E = \bigoplus_{k=1}^{K} E_k$ is an $N$-dimensional subspace of $H$ and for every $x = (x_1, x_2, \ldots) \in E$, we have:

$$
\|Tx\|^2 = \sum_{k \leq K} \|T_{mk}x_{mk}\|^2 \geq \sum_{k \leq K} [b_{nk}(T_{mk})]^2 \|x_{mk}\|^2
$$

$$
\geq \inf_{k \leq K} [b_{nk}(T_{mk})]^2 \sum_{k \leq K} \|x_{mk}\|^2 = \inf_{k \leq K} [b_{nk}(T_{mk})]^2 \|x\|^2;
$$

hence $b_N(T) \geq \inf_{k \leq K} b_{nk}(T_{mk})$, and we get the announced result. \qed

**Lemma 5.3.** Let $T = \bigoplus_{k=0}^{\infty} R_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^{\infty} H_k$. Let $n_0, \ldots, n_K$ be positive integers and $N = n_0 + \cdots + n_K - K$. Then, the approximation numbers of $T$ satisfy:

$$
(5.4) \quad a_N(T) \leq \max \left( \max_{0 \leq k \leq K} a_{n_k}(T_k), \sup_{k > K} \|T_k\| \right).
$$

**Proof.** Denote by $S$ the right-hand side of (5.4). Let $R_k$, $0 \leq k \leq K$ be operators on $H$ of respective rank $< n_k$ such that $\|T_k - R_k\| = a_{n_k}(T_k)$ and let $R = \bigoplus_{k=0}^{K} R_k$. Then $R$ is an operator of rank $\leq n_0 + \cdots + n_K - K - 1 < N$. If $f = \sum_{k=0}^{\infty} f_k \in H$, we see that:

$$
\|Tf - Rf\|^2 = \sum_{k=0}^{K} \|T_kf_k - R_kf_k\|^2 + \sum_{k > K} \|T_kf_k\|^2
$$

$$
\leq \sum_{k=0}^{K} a_{n_k}(T_k)^2 \|f_k\|^2 + \sum_{k > K} \|T_kf_k\|^2 \leq S^2 \sum_{k=0}^{\infty} \|f_k\|^2 = S^2 \|f\|^2,
$$

hence the result. \qed

We give now two corollaries of Lemma 5.3.

**Example 1.** We first use lens maps. We get:

**Theorem 5.4.** Let $\lambda_\theta$ the lens map of parameter $\theta$ and let $\psi: \mathbb{D} \to \mathbb{D}$ such that $\|\psi\|_{\infty} := c < 1$ and $h$ a Rudin function. We consider:

$$
\Phi(z_1, z_2) = (\lambda_\theta(z_1), \psi(z_1) h(z_2)).
$$

Then, for some positive constant $\beta$, we have, for all $N \geq 1$:

$$
(5.5) \quad a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}.
$$

**Proof.** Let $T_k = M_{\psi} C_{\lambda_\theta}$. We have $\|T_k\| \leq e^k$, so $\sup_{k \geq K} \|T_k\| \leq e^K$. On the other hand, we have $a_n(T_k) \leq e^k a_n(C_{\lambda_\theta}) \leq a_n(C_{\lambda_\theta}) \lesssim e^{-\beta \sqrt{n}}$ ([16], Theorem 2.1). Taking $n_0 = n_1 = \cdots = n_K = K^2$ in Lemma 5.3, we get:

$$
\max_{0 \leq k \leq K} a_{n_k}(T_k) \lesssim e^{-\beta K^3}.
$$

Since $n_0 + \cdots + n_K - K \approx K^3$, we obtain $a_{K^3} \lesssim e^{-\beta K}$, which gives the claimed result, by taking $\beta = \max (\beta_\theta, \log(1/c))$. \qed
Example 2. We consider the cusp map $\chi$. We have:

**Theorem 5.5.** Let $\chi$ be the cusp map, $h$ a Rudin function, and $\psi$ in the unit ball of $H^\infty$, with $\|\psi\|_\infty := c < 1$. Let:

$$
\Phi(z_1, z_2) = (\chi(z_1), \psi(z_1) h(z_2)).
$$

Then, for positive constant $\beta$, we have, for all $N \geq 1$:

$$
a_N(C_\Phi) \lesssim e^{-\beta \sqrt{N}/\sqrt{\log N}}.
$$

**Proof.** Let $T_k = M_{\psi_k}C_\chi$. As above, we have $\sup_{k > K} \|T_k\| \leq c^K$. For the cusp map, we have $a_n(C_\chi) \lesssim e^{-\alpha n/\log n}$ ([20], Theorem 4.3); hence $a_n(T_k) \lesssim e^{-\alpha n/\log n}$. We take $n_0 = n_1 = \cdots = n_K = K \lfloor \log K \rfloor$ (where $\lfloor \log K \rfloor$ is the integer part of $\log K$). Since $n_0 + \cdots + n_K \approx K^2 \lfloor \log K \rfloor$, we get, for another $\alpha > 0$:

$$
a_{K^2 \lfloor \log K \rfloor}(C_\Phi) \lesssim e^{-\alpha K},
$$

which reads: $a_N(C_\Phi) \lesssim e^{-\beta \sqrt{N}/\sqrt{\log N}}$, as claimed. \(\square\)

5.2 Lower bounds

In this subsection, we give lower bounds for approximation numbers of composition operators on $H^2$ of the bidisk, attached to a symbol $\Phi$ of the previous form $\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$ where $h$ is a Rudin function. The sharpness of those estimates will be discussed in the next subsection. We first need some lemmas in dimension one.

**Lemma 5.6.** Let $u, v: \mathbb{D} \to \mathbb{D}$ be two non-constant analytic self-maps and $T = M_v C_u: H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the associated weighted composition operator. For $0 < r < 1$, we set $A = u(r \mathbb{D})$ and $\Gamma = \exp \left( -1/\text{Cap} (A) \right)$. Then, for $0 < \delta \leq \inf_{|z| = r} |v(z)|$, we have:

$$
a_n(T) \gtrsim \sqrt{1 - r^2} \delta^n.
$$

In this lemma, $\text{Cap} (A)$ denotes the Green capacity of the compact subset $A \subseteq \mathbb{D}$ (see [21], § 2.3 for the definition).

For the proof, we need the following result ([27], Theorem 7, p. 353).

**Theorem 5.7 (Widom).** Let $A$ be a compact subset of $\mathbb{D}$ and $C(A)$ be the space of continuous functions on $A$ with its natural norm. Set:

$$
\tilde{d}_n(A) = \inf_{E} \left[ \sup_{f \in B_{n^\infty}} \text{dist} (f, E) \right],
$$

where $E$ runs over all $(n-1)$-dimensional subspaces of $C(A)$ and $\text{dist} (f, E) = \inf_{h \in E} \|f - h\|_{C(A)}$. Then

$$
\tilde{d}_n(A) \geq \alpha e^{-n/\text{Cap} (A)}
$$

for some positive constant $\alpha$. 

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Proof of Lemma 5.6. We apply Theorem 5.7 to the compact set \( A = u(r \overline{D}) \).

Let \( E \) be an \((n-1)\)-dimensional subspace of \( H^2 = H^2(D) \); it can be viewed as a subspace of \( C(A) \), so, by Theorem 5.7, there exists \( f \in H^\infty \subseteq H^2 \) with \( \|f\|_2 \leq \|f\|_\infty \leq 1 \) such that:

\[
\|f - h\|_{C(A)} \geq \alpha \Gamma^n, \quad \forall h \in E.
\]

Then:

\[
\|v(f \circ u - h \circ u)\|_{C(rT)} \geq \delta \|f - h\|_{C(rT)} = \delta \|f - h\|_{C(A)} \geq \alpha \delta \Gamma^n.
\]

But:

\[
\|v(f \circ u - h \circ u)\|_{C(rT)} \leq \frac{1}{\sqrt{1 - r^2}} \|v(f \circ u - h \circ u)\|_{H^2}.
\]

Hence:

\[
\|Tf - Th\|_{H^2} \geq \alpha \sqrt{1 - r^2} \delta \Gamma^n \geq \alpha \sqrt{1 - r} \delta \Gamma^n.
\]

Since \( h \) is an arbitrary function of \( E \), we get (\( B_{H^2} \) being the unit ball of \( H^2 \)):

\[
\inf_{\text{dim } E < n} \left[ \sup_{f \in B_{H^2}} \text{dist} \left( Tf, T(E) \right) \right] \geq \alpha \sqrt{1 - r} \delta \Gamma^n.
\]

But the left-hand side is equal to the Kolmogorov number \( d_n(T) \) of \( T \) (see [21], Lemma 3.12), and, as recalled in (1.7), in Hilbert spaces, the Kolmogorov numbers are equal to the approximation numbers; hence we obtain:

\[
a_n(T) \geq \alpha \sqrt{1 - r} \delta \Gamma^n, \quad n = 1, 2, \ldots,
\]

as announced.

The next lemma shows that some Blaschke products are far away from \( 0 \) on some circles centered at \( 0 \).

We consider a strongly interpolating sequence \((z_j)_{j \geq 1}\) of \( D \) in the sense that, if \( \varepsilon_j := 1 - |z_j| \), then:

\[
\varepsilon_{j+1} \leq \sigma \varepsilon_j
\]

and so \( \varepsilon_j \leq \sigma^{j-1} \varepsilon_1 \), where \( 0 < \sigma < 1 \) is fixed. Equivalently, the sequence \((|z_j|)_{j \geq 1}\) is interpolating. We consider the corresponding interpolating Blaschke product:

\[
B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - z_j z}.
\]

The following lemma is probably well-known, but we could find no satisfactory reference (see yet [10] for related estimates) and provide a simple proof.
Lemma 5.8. Let \((z_j)_{j \geq 1}\) be a strongly interpolating sequence as in (5.9) and \(B\) the associated Blaschke product (5.10).

Then there exists a sequence \(r_l := 1 - \rho_l\) such that:

\[
C_1 \sigma^l \leq \rho_l \leq C_2 \sigma^l,
\]

where \(C_1, C_2\) are positive constants, and for which:

\[
|z| = r_l \implies |B(z)| \geq \delta,
\]

where \(\delta > 0\) does not depend on \(l\).

Proof. Let us denote by \(p_l, 1 \leq p_l \leq l\), the biggest integer such that \(\epsilon_{p_l} \geq \sigma^{l-1} \epsilon_1\).

We separate two cases.

Case 1: \(\epsilon_{p_l} \geq 2 \sigma^{l-1} \epsilon_1\).

Then, we choose \(\rho_l = \alpha \sigma^{l-1} \epsilon_1\) with \(\alpha\) fixed, \(1 < \alpha < 2\). Since \(\rho(\xi, \zeta) \geq \rho(|\xi|, |\zeta|)\) for all \(\xi, \zeta \in \mathbb{D}\) (recall that \(\rho\) is the pseudo-hyperbolic distance on \(\mathbb{D}\)), we have the following lower bound for \(|z| = r_l\):

\[
|B(z)| = \prod_{j=1}^{\infty} \rho(r_l, |z_j|) \geq \prod_{j \leq p_l} \rho(r_l, |z_j|) \times \prod_{j > p_l} \rho(r_l, |z_j|) := P_1 \times P_2,
\]

and we estimate \(P_1\) and \(P_2\) separately.

We first observe that \(\frac{\rho_l}{\epsilon_{p_l}} \leq \frac{\alpha \sigma^{l-1} \epsilon_1}{2 \sigma^{l-1} \epsilon_1} \leq \frac{\alpha}{2}\), and then:

\[
\frac{\rho_l}{\epsilon_j} = \frac{\rho_l}{\epsilon_{p_l}} \frac{\epsilon_{p_l}}{\epsilon_j} \leq \frac{\alpha}{2} \sigma^{p_l-j}.
\]

The inequality \(\rho(1-u, 1-v) \geq \frac{1-u}{(u+v)}\) for \(0 < u, v \leq 1\) now gives us:

\[
\rho(r_l, |z_j|) \geq \frac{\epsilon_j - \rho_l}{\epsilon_j + \rho_l} = \frac{1 - \rho_l/\epsilon_j}{1 + \rho_l/\epsilon_j} \geq \frac{1 - (\alpha/2) \sigma^{p_l-j}}{1 + (\alpha/2) \sigma^{p_l-j}}, \text{ for } j \leq p_l,
\]

and:

\[
P_1 \geq \prod_{k=0}^{\infty} \left( 1 - \frac{(\alpha/2) \sigma^k}{1 + (\alpha/2) \sigma^k} \right).
\]

Similarly:

\[
\frac{\epsilon_{p_l+1}}{\rho_l} \leq \frac{\sigma^{l-1} \epsilon_1}{\alpha \sigma^{l-1} \epsilon_1} \leq \frac{1}{\alpha}
\]

and:

\[
\frac{\epsilon_j}{\rho_l} \leq \frac{1}{\alpha} \sigma^{p_l-1} \text{ for } j > p_l;
\]

so that:

\[
\rho(r_l, |z_j|) \geq \frac{\rho_l - \epsilon_j}{\rho_l + \epsilon_j} = \frac{1 - \epsilon_j/\rho_l}{1 + \epsilon_j/\rho_l} \geq \frac{1 - \alpha \sigma^{p_l-1}}{1 + \alpha \sigma^{p_l-1}}, \text{ for } j > p_l,
\]
and
\[
P_2 \geq \prod_{k=0}^{\infty} \left( \frac{1 - \alpha^{-1} \sigma^k}{1 + \alpha^{-1} \sigma^k} \right).
\]

Finally, the condition of lower and upper bound for \( \rho_i \) is fulfilled by construction.

**Case 2:** \( \varepsilon_{p_i} \leq 2 \sigma^{l-1} \varepsilon_1 \).

Then, we choose \( \rho_i = a \varepsilon_{p_i} \) with \( \sigma < a < 1 \) fixed. Computations exactly similar to those of Case 1 give us:

\[
|B(z)| \geq \prod_{k=0}^{\infty} \left( \frac{1 - a \sigma^k}{1 + a \sigma^k} \right) \times \prod_{k=0}^{\infty} \left( \frac{1 - \alpha^{-1} \sigma^k}{1 + \alpha^{-1} \sigma^k} \right) =: \delta > 0, \quad \text{for } |z| = r_i.
\]

Moreover, in this case:
\[
a \sigma^{l-1} \varepsilon_1 \leq \rho_i \leq 2 a \sigma^{l-1} \varepsilon_1,
\]
and the proof is ended.

Now, we have the following estimation.

**Theorem 5.9.** Let \( \phi, \psi : \mathbb{D} \to \mathbb{D} \) be two non-constant analytic self-maps and \( \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)) \), where \( h \) is inner.

Let \( (r_i)_{i \geq 1} \) be an increasing sequence of positive numbers with limit 1 such that:
\[
\inf_{|z| = r_i} |\psi(z)| \geq \delta_i > 0,
\]
with \( \delta_i \leq e^{-1/\text{Cap}(A_i)} \), where \( A_i = \phi(r_i \mathbb{D}) \).

Then the approximation numbers \( a_N(C_{\Phi}) \), \( N \geq 1 \), of the composition operator \( C_{\Phi} : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \) satisfy:
\[
a_N(C_{\Phi}) \geq \sup_{l \geq 1} \left[ \sqrt{1 - r_l} \exp \left( -8 \sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)} \right) \right],
\]
where:
\[
\Gamma_l = e^{-1/\text{Cap}(A_i)}.
\]

**Proof.** Since \( h \) is inner, the sequence \( (h^k)_{k \geq 0} \) is orthonormal in \( H^2 \) and hence \( a_n(C_{\Phi}) = a_n(T) \) for all \( n \geq 1 \), where \( T = \bigoplus_{k=0}^{\infty} T_k \) and \( T_k = M_{\psi^k} C_{\phi} \). Then Lemma 5.6 gives:
\[
a_n(T_k) \geq \sqrt{1 - r_l} \delta_l^{k} \Gamma_l^n
\]
for all \( n \geq 1 \) and all \( k \geq 0 \).

Let now:
\[
p_l = \left\lfloor \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)} \right\rfloor,
\]

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where \( [\cdot] \) stands for the integer part, and:

\begin{equation}
(5.22) \quad n_k = p_k, \quad \text{for } k = 1, \ldots, K.
\end{equation}

By Lemma 5.2, applied with \( m_k = k \) (i.e. to \( H_1, \ldots, H_K \)), we have, if \( N = n_1 + \cdots + n_K \):

\[ a_N(T) \geq \inf_{1 \leq k \leq K} \alpha \sqrt{1 - r_i} \delta_i^k \Gamma_i^n = \alpha \sqrt{1 - r_i} \delta_i^K \Gamma_i^{n_K}. \]

But, since \( p_l \leq \log(1/\delta_l)/\log(1/\Gamma_l) \):

\[ \delta_i^K \Gamma_i^{n_K} = \exp \left[ - (K \log(1/\delta_l) + p_l K \log(1/\Gamma_l)) \right] \geq \exp[-2K \log(1/\delta_l)]. \]

Since:

\[ N = p_l \frac{K(K + 1)}{2} \geq p_l \frac{K^2}{4} \geq \frac{K^2 \log(1/\delta_l)}{16 \log(1/\Gamma_l)}, \]

we get:

\[ \delta_i^K \Gamma_i^{n_K} \geq \exp \left[ -8 \sqrt{N} \sqrt{\log(1/\delta_l) \log(1/\Gamma_l)} \right], \]

and the result ensues.

**Example 1.** We take \( \phi = \lambda \theta \), a lens map, and \( \psi = B \), a Blaschke product associated to a strongly regular sequence, as defined in (5.10); then we get:

**Theorem 5.10.** Let \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \) be defined by:

\[ \Phi(z_1, z_2) = (\lambda \theta(z_1), c B(z_1) h(z_2)), \]

where \( B \) is a Blaschke product as in (5.10), \( 0 < c < 1 \), and \( h \) is an arbitrary inner function, we have, for some positive constant \( b \), for all \( N \geq 1 \):

\begin{equation}
(5.23) \quad a_N(C_\Phi) \gtrsim \exp(-b N^{1/3}) = \exp(-b \sqrt{N}/N^{1/6}).
\end{equation}

In particular \( \beta_2(C_\Phi) = \beta_2^2(C_\Phi) = 1. \)

**Remark.** We saw in Theorem 5.4 that this is the exact size, since we have:

\[ a_N(C_\Phi) \lesssim e^{-b N^{1/3}}. \]

**Proof.** By Lemma 5.8, there is a sequence of numbers \( r_l \approx \sigma^l \) such that \(|B(z)| \geq \delta \) for \(|z| = r_l \), where \( \delta \) is a positive constant (depending on \( \sigma \)). Since \( \lambda \theta(0) = 0 \), we have:

\[ \text{diam}_\rho(A_l) \gtrsim \lambda \theta(r_l) \gtrsim 1 - (1 - r_l)^\theta; \]

hence, by [21], Theorem 3.13, we have:

\[ \text{Cap}(A_l) \gtrsim \log \frac{1}{1 - r_l} \gtrsim l, \]

or, equivalently: \( \Gamma_l \geq e^{-b/l} \), some some \( b > 0 \). Then (5.18) gives, for all \( l \geq 1 \) (with another \( b \)):

\[ a_N(C_\Phi) \gtrsim \exp \left[ -b \left( l + \frac{\sqrt{N}}{\sqrt{l}} \right) \right]. \]

Taking \( l = N^{1/3} \), we get the result. \( \square \)
Example 2. By taking the cusp instead of a lens map, we obtain a better result, close to the extremal one.

**Theorem 5.11.** Let \( \Phi(z_1, z_2) = \left( \chi(z_1), cB(z_1) h(z_2) \right) \), where \( \chi \) is the cusp map, \( B \) a Blaschke product as in (5.10), \( 0 < c < 1 \), and \( h \) an arbitrary inner function. Then, for all \( N \geq 1 \):

\[
a_N(C_\Phi) \gtrsim e^{-b \sqrt{N/\log N}}.
\]

In particular \( \beta_2(C_\Phi) = 1 \).

**Remark.** We saw in Theorem 5.5 that this is the exact size, since we have:

\[
a_N(C_\phi) \lesssim e^{-\beta \sqrt{N/\log N}}.
\]

**Proof.** The proof is the same as that of Proposition 5.10, except that, for the cusp map, we have (note that \( \chi(0) = 0 \)):

\[
diam(\rho(A_l)) \geq \chi(r_l).
\]

But when \( r \) goes to 1:

\[
1 - \chi(r) \sim \frac{\pi (\sqrt{2} - 1)}{2} \frac{1}{\log (1/(1 - r))}
\]

(see [20], Lemma 4.2). Hence, by [21], Theorem 3.13, again, we have:

\[
\text{Cap}(A_l) \gtrsim \log \left( \log \left( \frac{1}{1 - r_l} \right) \right),
\]

so \( \Gamma_1 \geq e^{-b/\log l} \). Then, (5.18) gives (with another \( b \)):

\[
a_N(C_\Phi) \gtrsim \exp \left[ -b \left( l + \sqrt{\frac{N}{\log l}} \right) \right].
\]

In taking \( l = \sqrt{N/\log N} \), we get the announced result. \( \square \)

### 5.3 Upper bounds

All previous results point in the direction that, if \( \| \Phi \|_\infty = 1 \), then however small \( a_n(C_\Phi) \) is, it will always be larger than \( \alpha e^{-\beta \varepsilon_n \sqrt{n}} \) with \( \varepsilon_n \to 0^+ \), as this is the case in dimension one (with \( n \) instead of \( \sqrt{n} \)). But Theorem 5.12 to follow shows that we cannot hope, in full generality, to get the same result in dimension \( d \geq 2 \), and that other phenomena await to be understood. Here is our main result. It shows that, even for a truly 2-dimensional symbol \( \Phi \), we can have \( \| \Phi \|_\infty = 1 \) and nevertheless \( \beta_2^+(C_\Phi) < 1 \), in contrast to the 1-dimensional case where (1.1) holds.

**Theorem 5.12.** There exist a map \( \Phi : \mathbb{D}^2 \to \mathbb{D}^2 \) such that:

1. the composition operator \( C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \) is bounded and compact;
2. we have \( \| \Phi \|_\infty = 1 \) and \( \Phi \) is truly 2-dimensional, so that \( \beta_2^-(C_\Phi) > 0 \);
3. the singular numbers satisfy \( a_n(C_\Phi) \leq \alpha e^{-\beta \sqrt{n}} \) for some positive constants \( \alpha, \beta \); in particular \( \beta_2^+(C_\Phi) < 1 \).
Proof. Let $0 < \theta < 1$ be fixed, and $\lambda_\theta$ be the corresponding lens map. We set:

$$
\begin{align*}
\phi &= \frac{1 + \lambda_\theta}{2} \\
w(z) &= \exp \left[ - \left( 1 + \frac{z}{1 - z} \right)^\theta \right] \\
\psi &= w \circ \phi.
\end{align*}
$$

Note that $\|\phi\|_\infty = 1$.

Setting $\delta = \cos(\theta \pi / 2) > 0$, we have for $z \in \mathbb{D}$:

$$
|1 - \phi(z)| = \frac{1}{2} |1 - \lambda_\theta(z)| = \left| \frac{(1 - z)^\theta}{(1 - z)^\theta + (1 + z)^\theta} \right| \leq \frac{|1 - z|^\theta}{\delta}.
$$

Indeed, the argument $\alpha$ of $(1 \pm z)^\theta$ satisfies $|\alpha| \leq \theta \pi / 2$ for $z \in \mathbb{D}$, and we get:

$$
|(1 - z)^\theta + (1 + z)^\theta| \geq \Re [(1 - z)^\theta + (1 + z)^\theta] \geq \delta(|1 + z|^\theta + |1 - z|^\theta) \geq \delta.
$$

We also see that $\phi(\mathbb{D})$ touches the boundary $\partial \mathbb{D}$ only at 1 in a non-tangential way, meaning that for some constant $C > 1$:

$$
1 - |\phi(z)| \geq \frac{1}{C} |1 - \phi(z)|, \quad \forall z \in \mathbb{D}.
$$

Now, we have the following two inequalities:

$$
\Re z \geq 0 \quad \Rightarrow \quad |w(z)| \leq \exp \left( - \frac{\delta}{|1 - z|^\theta} \right)
$$

$$
z \in \mathbb{D} \quad \Rightarrow \quad |\psi(z)| \leq \exp \left( - \frac{\delta^2}{|1 - z|^{\theta \pi / 2}} \right).
$$

Indeed, with $S(z) = \left( \frac{1 + z}{1 - z} \right)^\theta$, we have $\Re S(z) \geq \delta |S(z)| \geq \delta |1 - z|^{-\theta}$ when $\Re z \geq 0$, giving (5.25), and (5.24) and (5.25) imply, since $\Re \phi(z) \geq 0$:

$$
|\psi(z)| = |w(\phi(z))| \leq \exp \left( - \frac{\delta}{|1 - \phi(z)|^\theta} \right) \leq \exp \left( - \frac{\delta^2}{|1 - z|^{\theta \pi / 2}} \right).
$$

We now set:

$$
\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)),
$$

with $h$ a Rudin function.

Observe that $\phi \in A(\mathbb{D})$ and $\psi = w \circ \phi \in A(\mathbb{D})$ as well ($w \in A(\mathbb{D})$ with $w(1) = 0$; this is due to the presence of the parameter $\theta < 1$). Hence if we take for $h$ a finite Blaschke product, the two components of $\Phi$ are in the bidisk algebra $A(\mathbb{D}^2)$. 

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We have $\|\psi\|_{\infty} := \rho < 1$. In fact, for $\Re u \geq 0$, we have:

$$\frac{1 + u}{1 - u} \geq 2^{-\theta}|1 + u|^{\theta} \geq 2^{-\theta}(1 + \Re u)^{\theta} \geq 2^{-\theta},$$

hence:

$$\Re \left[ \frac{1 + u}{1 - u} \right]^{\theta} \geq \left( \cos \frac{\theta \pi}{2} \right) \frac{1 + u}{1 - u} \geq \left( \cos \frac{\theta \pi}{2} \right) 2^{-\theta} = \delta 2^{-\theta},$$

and $\|w \circ \phi\|_{\infty} \leq e^{2^{-\theta} \delta}$.

Now, 1) follows from the orthogonal model presented in Section 5.1, because $\|\psi\|_{\infty} < 1$.

The assertion 2) follows from [2], Theorem 3.1, since $\|\phi\|_{\infty} = 1$.

We now prove 3).

As observed, $C_\Phi$ can be viewed as a direct sum $T = \bigoplus_{k=0}^{\infty} T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^{\infty} H_k$, where $T_k$ acts on a copy $H_k$ of $H^2(D)$ with:

$$T_k = M_{\psi_k} C_{\phi}.$$ 

We fix the positive integer $n$. The rest of the proof will consist of three lemmas.

**Lemma 5.13.** We have $\|T_k\| \leq 2 \rho^{-k} \leq 2 \rho^{-n}$ for $k > n$.

**Proof.** Indeed, since $\phi(0) = 1/2$, we know that $\|C_\phi\| \leq \sqrt{\frac{1 + \phi(0)}{1 - \phi(0)}} = \sqrt{3} \leq 2$, so that $\|T_k\| \leq \|\psi_k\|_{\infty} \|C_\phi\| \leq \rho^{-k} \times 2$. □

**Lemma 5.14.** Set $b = a/\delta^2$ where $a > 0$ is given by $e^{-a} = 4C/\sqrt{16C^2 + 1}$ and $C$ is as in (2.1). Let $m_k$ be the smallest integer such that $k \delta^2 2^{m_k} \theta^2 \geq an$; namely:

$$(5.28) \quad m_k = \left\lfloor \frac{\log(b n/k)}{\theta^2 \log 2} \right\rfloor + 1,$$

where $\lfloor . \rfloor$ stands for the integer part. Then, with $a' = \min(\log 2, a)$:

$$a_{nm_k+1}(T_k) \leq e^{-a'n}.$$ 

**Proof.** This follows from Theorem 2.3 applied with $w = \psi^k$, $R = k \delta^2$ and $\theta$ changed into $\theta^2$. This is possible thanks to (5.26) and to Lemma 5.13. Moreover we have adjusted $m_k$ so as to make the two terms in Theorem 2.3 of the same order. □

**Lemma 5.15.** The dimension $d := \sum_{k=0}^{n} n m_k$ satisfies, for some positive constant $\alpha$:

$$d \leq \alpha n^2.$$ 

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Proof. Indeed, it is well-known that:

\[ \sum_{k=1}^{n} \log k = n \log n - n + O(\log n), \]

and, in view of (5.28), we have \( m_k \leq \alpha'_\theta \log(b n/k) \leq \alpha''_\theta (\log n - \log k); \) hence:

\[ \sum_{k=1}^{n} m_k \leq \alpha''_\theta \left[ n \log n - (n \log n - n + O(\log n)) \right] = \alpha''_\theta n + O(\log n), \]

and we get \( d \leq \alpha''_\theta n^2 + O(n \log n) \leq \alpha_\theta n^2. \)

Alternatively, we could have used a Riemann sum for the function \( \log(1/x) \) on \((0, 1]\).

Finally, putting things together and using as well Proposition 5.3 with \( K = n \) and \( n_k = n m_k + 1 \) so that \( (\sum_{k=0}^{n} n_k) - n = (\sum_{k=0}^{n} n m_k) + 1 = d + 1 \), we get ignoring once more multiplicative constants:

\[ a_n(T) \lesssim a_d(T) \leq \alpha e^{-\beta n} \]

with positive constants \( \alpha, \beta \). This ends the proof of Theorem 5.12. \( \square \)

6 Monge-Ampère capacity and applications

6.1 Definition

Let \( K \) be a compact subset of \( \mathbb{D}^m \) (in this section, for notational reasons, we denote the dimension by \( m \) instead of \( d \)). The Monge-Ampère capacity of \( K \) has been defined by Bedford and Taylor ([3]; see also [13], § 5 or [11], Chapter 1) as:

\[ \text{Cap}_m (K) = \sup \left\{ \int_K (dd^c u)^m ; u \in PSH \text{ and } 0 \leq u \leq 1 \right\}, \]

where \( PSH \) is the set of plurisubharmonic functions on \( \mathbb{D}^m \), \( dd^c = 2i\partial\bar{\partial} \), and \((dd^c)^m = dd^c \wedge \cdots \wedge dd^c \) (\( m \) times). When \( u \in PSH \cap C^2(\mathbb{D}^m) \), we have:

\[ (dd^c u)^m = 4^m m! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV(z), \]

where \( dV(z) = (i/2)^m dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m \) is the usual volume in \( \mathbb{C}^m \). A more convenient formula (because \( \mathbb{D}^m \) is bounded and hyperconvex: see [11], p. 80, for the definition) is:

\[ \text{Cap}_m (K) = \int_K (dd^c u^*_K)^m, \]

where \( u^*_K \) is called the extremal function of \( K \) and is the upper semi-continuous regularization of:

\[ u_K = \sup\{ v \in PSH ; v \leq 0 \text{ and } v \leq -1 \text{ on } K \}. \]
but we will not need that.

As in [28], we set:

\[
\tau_m(K) = \frac{1}{(2\pi)^m} \text{Cap}_m(K).
\]

For \(m = 1\), \(\tau(K) := \tau_1(K)\) is equal to the Green capacity \(\text{Cap}(K)\) of \(K\) with respect to \(D\), with the definition used in [21] (see [13], Theorem 8.1, where a factor \(2\pi\) is introduced).

We further set:

\[
\Gamma_m(K) = \exp \left[ -\frac{m!}{\tau_m(K)} \right]^{1/m}.
\]

We proved in [21] that, for \(m = 1\), and \(\varphi: \mathbb{D} \to r\mathbb{D}\), with \(0 < r < 1\), we have:

\[
\beta_1(C\varphi) = \Gamma_1(K) = \Gamma_1(\varphi(\mathbb{D})).
\]

The goal of this section is to see that Theorem 5.12 shows that this no longer holds for \(m = 2\).

### 6.2 A seminal example

In one variable, our initial motivation had been the simple-minded example \(\varphi(z) = rz, 0 < r < 1\), for which \(C\varphi(z^n) = r^n z^n\), implying \(a_n(C\varphi) = r^{n-1}\) and \(\beta_1(C\varphi) = r\). If \(K = \varphi(D) = D(0, r)\), we have \(\text{Cap}(K) = \frac{1}{\log(1/\sqrt{r})}\) and \(\Gamma_1(K) = r\), so that \(\beta_1(C\varphi) = \Gamma_1(K)\). Let us examine the multivariate example (where \(0 < r_j < 1\)):

\[
\Phi(z_1, z_2, \ldots, z_m) = (r_1 z_1, r_2 z_2, \ldots, r_m z_m).
\]

If \(K = \Phi(\mathbb{D}^m)\), we have \(K = \prod_{k=1}^m D(0, r_k)\), and hence ([5], Theorem 3):

\[
\tau_m(K) = \prod_{k=1}^m \frac{1}{\log(1/r_k)}.
\]

On the other hand, \(C\Phi(z_1^{n_1}, z_2^{n_2}, \ldots, z_m^{n_m}) = r_1^{n_1} r_2^{n_2} \ldots r_m^{n_m} z_1^{n_1} z_2^{n_2} \ldots z_m^{n_m}\) so that the sequence \((a_n)\) of approximation numbers of \(C\Phi\) is the non-increasing rearrangement of the numbers \(r_1^{n_1} r_2^{n_2} \ldots r_m^{n_m}\). It is convenient to state the following simple lemma.

**Lemma 6.1.** Let \(\lambda_1, \ldots, \lambda_m\) be positive numbers. Let \(N_A\) be the number of \(m\)-tuples of non-negative integers \((n_1, \ldots, n_m)\) such that \(\sum_{k=1}^m \lambda_k n_k \leq A\). Then, as \(A \to \infty\):

\[
N_A \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}.
\]
Indeed, just apply Karamata’s tauberian theorem (see [12] p. 30) to the
generalized Dirichlet series:

\[ S(\varepsilon) := \prod_{k=1}^{m} \frac{1}{1-e^{-\lambda_k \varepsilon}} = \sum_{n_1, \ldots, n_m \geq 0} e^{-(\sum_{k=1}^{m} \lambda_k n_k) \varepsilon}; \]

we have \( S(\varepsilon) \sim \frac{\varepsilon^{-m}}{(\lambda_1 \cdots \lambda_m) m!} \) as \( \varepsilon \to 0^+. \)

Let now \( N \) be a positive integer and \( \varepsilon = a_N. \) Setting \( \lambda_k = \log(1/r_k) \) and
\( A = \log(1/\varepsilon), \) we see that \( N \) is the number of \( m \)-tuples \((n_1, \ldots, n_m)\) of non-negative integers such that \( r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \geq \varepsilon, \) i.e. such that \( \sum_{k=1}^{m} \lambda_k n_k \leq A. \)
This number \( N \) is hence nothing but the number \( N_A \) of the previous lemma, so that:

\[ N \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}. \]

Inverting this formula, we get:

\[ a_N(C_\Phi) = \exp \left[ - (1 + o(1)) (m!(\lambda_1 \lambda_2 \cdots \lambda_m) N)^{1/m} \right] \]

and:

\[ \beta_m(C_\Phi) = \exp \left[ - (m!\lambda_1 \lambda_2 \cdots \lambda_m)^{1/m} \right] = \Gamma_m(K), \]
in view of (6.2) and (6.4).

On the view of the simple-minded previous example, the extension of the
spectral radius formula (6.3) to the multivariate case holds, and it is tempting to
conjecture that this is a general phenomenon as in dimension one, all the more
as the following extension of Widom’s theorem was proved by Zakharyuta, based
on the solution by S. Nivoche of Zakharyuta’s conjecture ([23]); see also [28],
Theorem 5.4. A compact subset \( K \) of \( \mathbb{D}^m \) is said to be regular if its extremal
function \( u_K^* \) is continuous on \( \mathbb{D}^m. \)

**Theorem 6.2** ([28], Theorem 5.6). Let \( K \) be a regular compact subset of \( \mathbb{D}^m \)
and \( J: H^\infty(\mathbb{D}^m) \to C(K) \) the canonical injection; then the Kolmogorov numbers
\( d_n(J) \) satisfy:

\[ \lim_{n \to \infty} [d_n(J)]^{1/n} = \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right]. \]

Note that the right side is nothing but \( \Gamma_m(K). \)
We will see consequences of this result in a forthcoming paper ([22]).

### 6.3 Upper bound

For the upper bound, the situation behaves better, as stated in the following
theorem.
Theorem 6.3 ([28], Proposition 6.1). Let $K$ be a compact subset of $\mathbb{D}^m$ with non-void interior. Then:

$$\limsup_{n \to \infty} \left[ d_n(J) \right]^{1/n^{1/m}} \leq \exp \left[ - \left( \frac{m!}{r_m(K)} \right)^{1/m} \right].$$

Note that $(K, \mathbb{D}^m)$ is a condenser since $K$ has non-void interior. We deduce the following upper bound.

Theorem 6.4. Let $\Phi$ be an analytic self-map of $\mathbb{D}^m$ with $\|\Phi\|_\infty = \rho < 1$, thus inducing a compact composition operator on $H^2(\mathbb{D}^m)$. Then we have:

$$\beta_{m,c}(C_\Phi) \leq \Gamma_m(\Phi(\mathbb{D}^m)).$$

Proof. This proof provides in particular a simplification of that given in [21] in dimension $m = 1$.

Changing $n$ into $n^m$, Theorem 6.3 means that for every $\varepsilon > 0$, there exists an $(n^m - 1)$-dimensional subspace $V$ of $C(K)$ such that, for any $g \in H^\infty(\mathbb{D}^m)$, there exists $h \in V$ such that:

$$\|g - h\|_{C(K)} \leq C(1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n \|g\|_\infty.$$

Let $l$ be an integer to be adjusted later, and $f(z) = \sum_{|\alpha| \leq l} b_\alpha z^\alpha \in B_{H^2}$, as well as $g(z) = \sum_{|\alpha| \leq l} b_\alpha z^\alpha$. We first note that (with $M_m$ depending only on $m$ and $\rho$, and since the number of $\alpha$’s such that $|\alpha| \leq p$ is $O(p^m)$):

$$\sum_{|\alpha| > l} \rho^{2|\alpha|} \leq M_m \sum_{p > l} p^m \rho^{2p} \leq M_m l^m \frac{\rho^{2l}}{(1 - \rho^2)^{m+1}}.$$

We next observe that, by the Cauchy-Schwarz and Parseval inequalities:

$$\|g\|_\infty \leq M_m l^{m/2},$$

and

$$|f(z) - g(z)| \leq M_m l^{m/2} \frac{|z|^l}{(1 - |z|^2)^{(m+1)/2}}, \quad \forall z \in \mathbb{D}^m,$$

where $|z|_\infty := \max_{j \leq m} |z_j|$ if $z = (z_1, \ldots, z_m)$.

The subspace $F$ formed by functions $v \circ \Phi$, for $v \in V$, can be viewed as a subspace of $L^\infty(\mathbb{T}^m) \subseteq L^2(\mathbb{T}^m)$ with respect to the Haar measure of $\mathbb{T}^m$, the distinguished boundary of $\mathbb{D}^m$ (indeed, we can write $(v \circ \Phi)^* = v \circ \Phi^*$, where $\Phi^*$ denotes the almost everywhere existing radial limits of $\Phi(rz)$, which belong to $K$). Let finally $E = P(F) \subseteq H^2(\mathbb{D}^m)$ where $P: L^2(\mathbb{T}^m) \to H^2(\mathbb{T}^m) = H^2(\mathbb{D}^m)$ is the orthogonal projection. This is a subspace of $H^2$ with dimension $< n^m$. Set temporarily $\eta = C(1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n$. It follows from (6.7) and (6.8) that, for some $h \in V$:

$$\|g - h\|_{C(K)} \leq \eta \|g\|_\infty \leq \eta M_m l^{m/2}$$

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and hence:
\[
\|g \circ \Phi - h \circ \Phi\|_2 \leq \|g \circ \Phi - h \circ \Phi\|_\infty \leq \eta M_m l^{m/2},
\]
implying by orthogonal projection:
\[
\text{dist} (C\Phi g, E) \leq \|g \circ \Phi - P(h \circ \Phi)\|_2 \leq \eta M_m l^{m/2}.
\]
Now, since \(C\Phi f(z) - C\Phi g(z) = f(\Phi(z)) - g(\Phi(z))\), (6.9) gives:
\[
\|C\Phi f - C\Phi g\|_2 \leq \|C\Phi f - C\Phi g\|_\infty \leq M_m l^{m/2} \frac{\rho^l}{(1 - \rho^2)^{(m+1)/2}}
\]
and hence:
\[
\text{dist} (C\Phi f, E) \leq M_m l^{m/2} \left( \frac{\rho^l}{(1 - \rho^2)^{(m+1)/2}} + C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n \right).
\]
It ensues, since \(a_N(C\Phi) = d_N(C\Phi)\), that:
\[
\left[ a_{n^m}(C\Phi) \right]^{1/n} \leq (M_m l^{m/2})^{1/n} \left[ \frac{\rho^l / n}{(1 - \rho^2)^{(m+1)/2n}} + C_\varepsilon^{1/n} (1 + \varepsilon) \Gamma_m(K) \right].
\]
Taking now for \(l\) the integer part of \(n \log n\), and passing to the upper limit as \(n \to \infty\), we obtain (since \(l/n \to \infty\) and \((\log l)/n \to 0\):
\[
\beta_m^+(C\Phi) \leq (1 + \varepsilon) \Gamma_m(K),
\]
and Theorem 6.4 follows. \(\Box\)

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