Some examples of composition operators and their approximation numbers on the Hardy space of the bi-disk

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We give examples of composition operators C_Φ on $H^2(\mathbb{D}^2)$ showing that the condition $\|\Phi\|_\infty = 1$ is not sufficient for their approximation numbers $a_n(C_\Phi)$ to satisfy $\lim_{n\to\infty}[a_n(C_\Phi)]^{1/\sqrt{n}} = 1$, contrary to the 1-dimensional case. We also give a situation where this implication holds. We make a link with the Monge-Ampère capacity of the image of Φ.

Key-words: approximation numbers; Bergman space; bidisk; composition operator; Green capacity; Hardy space; Monge-Ampère capacity; weighted composition operator.

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1 Introduction and notation

1.1 Introduction

The purpose of this paper is to continue the study of composition operators on the polydisk initiated in [2], and in particular to examine to what extent one of the main results of [21] still holds.

Let H be a Hilbert space and $T\colon H \to H$ a bounded operator. Recall that the *approximation numbers* of T are defined as:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\|\,, \quad n \geq 1\,,$$

and we have:

$$\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq \cdots$$

The operator T is compact if and only if $a_n(T) \underset{n\to\infty}{\longrightarrow} 0$.

For $d \geq 1$, we define:

$$\begin{cases} \beta_d^-(T) & = \liminf_{n \to \infty} \big[a_{n^d}(T) \big]^{1/n} \\ \beta_d^+(T) & = \limsup_{n \to \infty} \big[a_{n^d}(T) \big]^{1/n} \end{cases}$$

We have:

$$0 \leq \beta_d^-(T) \leq \beta_d^+(T) \leq 1 \, ,$$

and we simply write $\beta_d(T)$ in case of equality.

It may well happen in general (consider diagonal operators) that $\beta_d^-(T) = 0$ and $\beta_d^+(T) = 1$.

When $H = H^2(\mathbb{D})$ is the Hardy space on the open unit disk $\mathbb{D}$ of $\mathbb{C}$, and $T = C_\Phi$ is a composition operator, with $\Phi \colon \mathbb{D} \to \mathbb{D}$ a non-constant analytic function, we always have ([19]):

$$\beta_1^-(C_\Phi) > 0 \, ,$$

and one of the main results of [19] is the equivalence:

$$\beta_1^+(C_\Phi) < 1 \quad \Longleftrightarrow \quad \|\Phi\|_\infty < 1 \, . \tag{1.1}$$

An alternative proof was given in [21], as a consequence of a so-called "spectral radius formula", which moreover shows that:

$$\beta_1^-(C_\Phi) = \beta_1^+(C_\Phi) \, .$$

In [2], for $d \geq 2$, it is proved that, for a bounded symmetric domain $\Omega \subseteq \mathbb{C}^d$, if $\Phi \colon \Omega \to \Omega$ is analytic, such that $\Phi(\Omega)$ has a non-void interior, and the composition operator $C_\Phi \colon H^2(\Omega) \to H^2(\Omega)$ is compact, then:

$$\beta_d^-(C_\Phi) > 0 \, .$$

On the other hand, if Ω is a product of balls, then:

$$\|\Phi\|_\infty < 1 \quad \Longrightarrow \quad \beta_d^+(C_\Phi) < 1 \, .$$

We do not know whether the converse holds and the purpose of this paper is to study some examples towards an answer.

The paper is organized as follows. Section 1 is this short introduction, as well as some notations and definitions on singular numbers of operators and Hardy spaces of the polydisk to follow. Section 2 contains preliminary results on weighted composition operators in one variable, which surprisingly play an important role in the study of non-weighted composition operators in two variables. Section 3 studies the case of symbols with "separated" variables. Our main one variable result extends in this case. Section 4 studies the "glued case" $\Phi(z_1, z_2) = \big(\phi(z_1), \phi(z_1)\big)$ for which even boundedness is an issue. Here, the

Bergman space $B^2(\mathbb{D})$ enters the picture. Section 5 studies the case of "triangularly separated" variables. This section lets direct Hilbertian sums of weighted composition operators in one variable appear, and it contains our main result: an example of a symbol Φ satisfying $\|\Phi\|_\infty = 1$ and yet $\beta_2^+(C_\Phi) < 1$. The final Section 6 discusses the role of the Monge-Ampère pluricapacity, which is a multivariate extension of the Green capacity in the disk. Even though, as evidenced by our counterexample of Section 5, this capacity will not capture all the behavior of the parameter $\beta_m(C_\Phi)$, some partial results are obtained, relying on theorems of S. Nivoche and V. Zakharyuta.

1.2 Notation

We denote by $\mathbb{D}$ the open unit disk of the complex plane and by $\mathbb{T}$ its boundary, the 1-dimensional torus.

The Hardy space $H^2(\mathbb{D}^d)$ is the space of holomorphic functions $f\colon \mathbb{D}^d \to \mathbb{C}$ whose boundary values f^* on $\mathbb{T}^d$ are square-integrable with respect to the Haar measure m_d of $\mathbb{T}^d$, and normed with:

$$\|f\|_2^2 = \|f\|_{H^2(\mathbb{D}^d)}^2 = \int_{\mathbb{T}^d} |f^*(\xi_1, \ldots, \xi_d)|^2 \, dm_d(\xi_1, \ldots, \xi_d) \, .$$

If $f(z_1, \ldots, z_d) = \sum_{\alpha_1, \ldots, \alpha_d \geq 0} a_{\alpha_1, \ldots, \alpha_d} \, z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, then:

$$\|f\|_2^2 = \sum_{\alpha_1, \ldots, \alpha_d \geq 0} |a_{\alpha_1, \ldots, \alpha_d}|^2 \, .$$

We say that an analytic map $\Phi\colon \mathbb{D}^d \to \mathbb{D}^d$ is a *symbol* if its associated composition operator $C_\Phi\colon H^2(\mathbb{D}^d) \to H^2(\mathbb{D}^d)$, defined by $C_\Phi(f) = f \circ \Phi$, is bounded.

We say that Φ is *truly d-dimensional* if $\Phi(\mathbb{D}^d)$ has a non-void interior.

We will make use of two kinds of symbols defined on $\mathbb{D}$.

The *lens map* $\lambda_\theta\colon \mathbb{D} \to \mathbb{D}$ is defined, for $\theta \in (0, 1)$, by:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta} \tag{1.2}$$

(see [26], p. 27, or [16], for more information), and corresponds to $u \mapsto u^\theta$ in the right half-plane.

The *cusp map* $\chi\colon \mathbb{D} \to \mathbb{D}$ was first defined in [15] and in a slightly different form in [20]; we actually use here the modified form introduced in [17], and then used in [18]. We first define:

$$\chi_0(z) = \frac{\left(\dfrac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\dfrac{z-i}{iz-1}\right)^{1/2} + 1} \, ;$$

we note that $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2}-1$. Then we set:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi}\,\chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)}\,,$$

and finally:

$$\chi(z) = 1 - \chi_3(z)\,,$$

where:

$$a = 1 - \frac{2}{\pi}\log(\sqrt{2} - 1) \in (1, 2) \tag{1.3}$$

is chosen in order that $\chi(0) = 0$. The image Ω of the (univalent) cusp map is formed by the intersection of the inside of the disk $D\big(1 - \frac{a}{2}, \frac{a}{2}\big)$ and the outside of the two disks $D\big(1 + \frac{ia}{2}, \frac{a}{2}\big)$ and $D\big(1 - \frac{ia}{2}, \frac{a}{2}\big)$.

Besides the approximation numbers, we need other singular numbers for an operator $S\colon X \to Y$ between Banach spaces X and Y.

The *Bernstein numbers* $b_n(S)$, $n \geq 1$, which are defined by:

$$b_n(S) = \sup_E \min_{x \in S_E} \|Sx\|\,, \tag{1.4}$$

where the supremum is taken over all n-dimensional subspaces of X and S_E is the unit sphere of E.

The *Gelfand numbers* $c_n(S)$, $n \geq 1$, which are defined by:

$$c_n(S) = \inf\{\|S_{|M}\|\,;\ \operatorname{codim} M < n\}\,. \tag{1.5}$$

The *Kolmogorov numbers* $d_n(S)$, $n \geq 1$, which are defined by:

$$d_n(S) = \inf_{\dim E < n} \Big[\sup_{x \in B_X} \operatorname{dist}(Sx, E)\Big]\,. \tag{1.6}$$

Pietsch showed that all s-numbers on Hilbert spaces are equal (see [24], § 2, Corollary, or [25], Theorem 11.3.4); hence:

$$a_n(S) = b_n(S) = c_n(S) = d_n(S)\,. \tag{1.7}$$

We denote m the normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$. If $\varphi\colon \mathbb{D} \to \mathbb{D}$, m_φ is the pull-back measure on $\overline{\mathbb{D}}$ defined by $m_\varphi(E) = m[{\varphi^*}^{-1}(E)]$, where φ^* stands for the non-tangential boundary values of φ.

The notation $A \lesssim B$ means that $A \leq C\,B$ for some positive constant C and we write $A \approx B$ if we have both $A \lesssim B$ and $B \lesssim A$.

2 Preliminary results on weighted composition operators on $H^2(\mathbb{D})$

We see in this section that the presence of a "rapidly decaying" weight allows simpler estimates for the approximation numbers of a corresponding weighted composition operator. Such a study, but a bit different, is made in [14].

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ a non-constant analytic self-map in the disk algebra $A(\mathbb{D})$ such that, for some constant $C > 1$ and for all $z \in \mathbb{D}$:

$$\varphi(1) = 1\,, \quad |1 - \varphi(z)| \leq 1\,, \quad |1 - \varphi(z)| \leq C\,(1 - |\varphi(z)|) \tag{2.1}$$

as well as $\varphi(z) \neq 1$ for $z \neq 1$. We can take for example $\varphi = \frac{1+\lambda_\theta}{2}$ where λ_θ is the lens map with parameter θ.

Let $w \in H^\infty$ and let T be the weighted composition operator

$$T = M_{w\circ\varphi} C_\varphi \colon H^2 \to H^2\,.$$

Note that $M_{w\circ\varphi} C_\varphi = C_\varphi M_w$. We first show that:

Theorem 2.1. *Let $T = M_{w\circ\varphi} C_\varphi \colon H^2 \to H^2$ be as above and let B be a Blaschke product with length $< N$. Then, with the implied constant depending only on the number C in* (2.1) *(and of φ):*

$$a_N(T) \lesssim \sup_{|z-1|\leq 1,\ z\in\varphi(\mathbb{D})} |B(z)|\,|w(z)|\,.$$

Proof. The following preliminary observation (see also [16], p. 809), in which we denote by $S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| \leq h\}$ the Carleson window with center $\xi \in \mathbb{T}$ and size h, and by K_φ the support of the pull-back measure m_φ, will be useful.

$$u \in S(\xi, h) \cap K_\varphi \quad \Longrightarrow \quad u \in S(1, Ch) \cap K_\varphi\,. \tag{2.2}$$

Indeed, if $|u - \xi| \leq h$ and $u \in K_\varphi$, (2.1) implies:

$$1 - |u| \leq |u - \xi| \leq h \quad \text{and} \quad |u - 1| \leq C(1 - |u|) \leq Ch\,.$$

Set $E = BH^2$. This is a subspace of codimension $< N$. If $f = Bg \in E$, with $\|g\| = \|f\|$ (isometric division by B in BH^2), we have $Tf = (wBg)\circ\varphi$, whence:

$$\|T(f)\|^2 = \int_{\mathbb{D}} |B|^2 |w|^2 |g|^2 dm_\varphi\,,$$

implying $\|T(f)\|^2 \leq \|f\|^2 \|J\|^2$ where $J\colon H^2 \to L^2(\sigma)$ is the natural embedding and where

$$\sigma = |B|^2 |w|^2 dm_\varphi\,.$$

Now, Carleson's embedding theorem for the measure σ and (2.2) show that (the implied constants being absolute):

$$\begin{aligned}
\|J\|^2 &\lesssim \sup_{\xi\in\mathbb{T},\ 0<h<1} \frac{1}{h} \int_{S(\xi,h)\cap K_\varphi} |B|^2|w|^2 dm_\varphi \\
&\lesssim \sup_{0<h<1} \frac{1}{h} \int_{S(1,Ch)\cap K_\varphi} |B|^2|w|^2 dm_\varphi \\
&\lesssim \bigg(\sup_{|z-1|\leq 1,\ z\in\overline{\varphi(\mathbb{D})}} |B(z)|^2|w(z)|^2 \bigg) \bigg(\sup_{0<h<1} \frac{1}{h} \int_{S(1,Ch)\cap K_\varphi} dm_\varphi \bigg) \\
&\lesssim \sup_{|z-1|\leq 1,\ z\in\overline{\varphi(\mathbb{D})}} |B(z)|^2|w(z)|^2 \,,
\end{aligned}$$

since m_φ is a Carleson measure for H^2 and where we used that, according to (2.1):

$$K_\varphi \subseteq \overline{\varphi(\mathbb{D})} \subseteq \{z\in\mathbb{D}\,;\ |z-1|\leq 1\}\,.$$

This ends the proof of Theorem 2.1 with help of the equality of $a_N(T)$ with the Gelfand number $c_N(T)$ recalled in (1.7). □

In order to specialize efficiently the general Theorem 2.1, we recall the following simple Lemma 2.3 of [16], where:

$$\rho(a,b) = \left| \frac{a-b}{1-\bar{a}b} \right|, \qquad a,b\in\mathbb{D}\,, \tag{2.3}$$

is the *pseudo-hyperbolic distance*:

Lemma 2.2 ([16])**.** *Let $a,b\in\mathbb{D}$ such that $|a-b|\leq L\min(1-|a|,1-|b|)$. Then:*

$$\rho(a,b) \leq \frac{L}{\sqrt{L^2+1}} =: \kappa < 1\,.$$

We can now state:

Theorem 2.3. *Assume that φ is as in* (2.1) *and that the weight w satisfies, for some parameters $0<\theta\leq 1$ and $R>0$:*

$$|w(z)| \leq \exp\bigg(-\frac{R}{|1-z|^\theta} \bigg), \quad \forall z\in\mathbb{D} \text{ with } \Re\mathrm{e}\, z\geq 0\,.$$

Then, the approximation numbers of $T=M_{w\circ\varphi}C_\varphi$ satisfy:

$$a_{nm+1}(T) \lesssim \max\big[\exp(-an), \exp(-R\,2^{m\theta})\big]\,,$$

for all integers $n,m\geq 1$, where $a=\log[\sqrt{16C^2+1}/(4C)]>0$ and C is as in (2.1).

Proof. Let $p_l = 1 - 2^{-l}$, $0 \le l < m$ and let B be the Blaschke product:

$$B(z) = \prod_{0 \le l < m} \left(\frac{z - p_l}{1 - p_l z} \right)^n .$$

Let $z \in K_\varphi \cap \mathbb{D}$ so that $0 < |z - 1| \le 1$. Let l be the non-negative integer such that $2^{-l-1} < |z - 1| \le 2^{-l}$. We separate two cases:

Case 1: $l \ge m$. Then, *the weight does the job*. Indeed, majorizing $|B(z)|$ by 1 and using the assumption on w, we get:

$$\begin{aligned} |B(z)|^2 |w(z)|^2 \le |w(z)|^2 &\le \exp\Big(- \frac{2R}{|1 - z|^\theta} \Big) \\ &\le \exp(-2R\, 2^{l\theta}) \le \exp(-2R\, 2^{m\theta}) . \end{aligned}$$

Case 2: $l < m$. Then, *the Blaschke product does the job*. Indeed, majorize $|w(z)|$ by 1 and estimate $|B(z))|$ more accurately with help of Lemma 2.2; we observe that

$$|z - p_l| \le |z - 1| + 1 - p_l \le 2 \times 2^{-l} = 2(1 - p_l) \le 4C(1 - p_l)$$

and then, since $z \in K_\varphi$, we can write with $C \ge 1$ as in (2.1):

$$1 - |z| \ge \frac{1}{C}\, |1 - z| \ge \frac{1}{2C}\, 2^{-l} \ge \frac{1}{4C}\, |z - p_l| ,$$

so that the assumptions of Lemma 2.2 are verified with $L = 4C$, giving:

$$\rho(z, p_l) \le \frac{4C}{\sqrt{16C^2 + 1}} = \exp(-a) < 1 .$$

Hence, by definition, since $l < m$:

$$|B(z)| \le [\rho(z, p_l)]^n \le \exp(-an) .$$

Putting both cases together, and observing that our Blaschke product has length $nm < nm + 1$, we get the result by applying Theorem 2.1 with $N = nm + 1$. □

2.1 Some remarks

1. Twisting a composition operator by a weight may improve the compactness of this composition operator, or even may make this weighted composition operator compact though the non-weighted was not (see [8] or [14]). However, this is not possible for all symbols, as seen in the following proposition.

Proposition 2.4. *Let $w \in H^\infty$. If φ is inner, or more generally if $|\varphi| = 1$ on a subset of $\mathbb{T}$ of positive measure, then $M_w C_\varphi$ is never compact (unless $w \equiv 0$).*

Proof. Indeed, suppose $T = M_w C_\varphi$ compact. Since $(z^n)_n$ converges weakly to 0 in H^2 and since $T(z^n) = w\,\varphi^n$, we should have, since $|\varphi| = 1$ on E, with $m(E) > 0$:

$$\int_E |w|^2\, dm = \int_E |w|^2 |\varphi|^{2n}\, dm \leq \int_{\mathbb{T}} |w|^2 |\varphi|^{2n}\, dm = \|T(z^n)\|^2 \underset{n\to\infty}{\longrightarrow} 0\,,$$

but this would imply that w is null a.e. on E and hence $w \equiv 0$ (see [7], Theorem 2.2), which was excluded. $\square$

Note that É. Amar and A. Lederer proved in [1] that $|\varphi| = 1$ on a set of positive measure if and only if φ is an exposed point of of the unit ball of H^∞; hence the following proposition can be viewed as the (almost) opposite case.

Proposition 2.5. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that $\|\varphi\|_\infty = 1$. Assume that:*

$$\int_{\mathbb{T}} \log(1 - |\varphi|)\, dm > -\infty$$

(meaning that φ is not an extreme point of the unit ball of H^∞: see [7], Theorem 7.9). Then, if w is an outer function such that $|w| = 1 - |\varphi|$, the weighted composition operator $T = M_w C_\varphi$ is Hilbert-Schmidt.

Proof. We have:

$$\sum_{n=0}^{\infty} \|T(z^n)\|^2 = \sum_{n=0}^{\infty} \int_{\mathbb{T}} (1 - |\varphi|)^2 |\varphi|^{2n}\, dm = \int_{\mathbb{T}} \frac{1 - |\varphi|}{1 + |\varphi|}\, dm < +\infty\,,$$

and T is Hilbert-Schmidt, as claimed. $\square$

2. In [14], Theorem 2.5, it is proved that we always have, for some constants $\delta, \rho > 0$:

$$a_n(M_w C_\varphi) \geq \delta\, \rho^n\,, \quad n = 1, 2, \ldots \tag{2.4}$$

(if $w \not\equiv 0$). We give here an alternative proof, based on a result of Gunatillake ([9]), this result holding in a wider context.

Theorem 2.6 (Gunatillake)**.** *Let $T = M_w C_\varphi$ be a compact weighted composition operator on H^2 and assume that φ has a fixed point $a \in \mathbb{D}$. Then the spectrum of T is the set:*

$$\sigma(T) = \{0, w(a), w(a)\,\varphi'(a), w(a)\,[\varphi'(a)]^2, \ldots, w(a)\,[\varphi'(a)]^n, \ldots\}$$

Proof of (2.4). First observe that, in view of Proposition 2.4, φ cannot be an automorphism of $\mathbb{D}$ so that the point a is the Denjoy-Wolff point of φ and is attractive. Theorem 2.6 is interesting only when $w(a)\,\varphi'(a) \neq 0$.

Now, we can give a new proof Theorem 2.5 of [14] as follows. Let $a \in \mathbb{D}$ be such that $w(a)\, \varphi'(a) \neq 0$ ($H(\mathbb{D})$ is a division ring and $\varphi' \not\equiv 0$, $w \not\equiv 0$). Let $b = \varphi(a)$ and $\tau \in \operatorname{Aut} \mathbb{D}$ with $\tau(b) = a$. We set:

$$\psi = \tau \circ \varphi \quad \text{and} \quad S = M_w C_\psi = T C_\tau\,.$$

This operator S is compact because T is.

Since $\psi(a) = a$ and $\psi'(a) = \tau'(b)\varphi'(a) \neq 0$, Theorem 2.6 says that the non-zero eigenvalues of S, arranged in non-increasing order, are the numbers $\lambda_n = w(a)\, [\psi'(a)]^{n-1}$, $n \geq 1$. As a consequence of Weyl's inequalities, we know that:

$$a_1(S)\, a_n(S) \geq |\lambda_{2n}|^2 \geq \delta\, \rho^n\,,$$

with:

$$\delta = |w(a)|^2 > 0 \quad \text{and} \quad \rho = |\psi'(a)|^4 > 0\,.$$

To finish, it is enough to observe that $a_n(S) \leq a_n(T)\, \|C_\tau\|$ by the ideal property of approximation numbers. □

3 The splitted case

Theorem 3.1. *Let $\Phi = (\phi, \psi) \colon \mathbb{D}^d \to \mathbb{D}^d$ be a truly d-dimensional symbol with $\phi \colon \mathbb{D} \to \mathbb{D}$ depending only on z_1 and $\psi \colon \mathbb{D}^{d-1} \to \mathbb{D}^{d-1}$ only on $z_2, \ldots, z_d$, i.e. $\Phi(z_1, z_2, \ldots, z_d) = \big(\phi(z_1), \psi(z_2, \ldots, z_d)\big)$. Then, whatever ψ behaves:*

$$\|\phi\|_\infty = 1 \quad \Longrightarrow \quad \beta_d(C_\Phi) = 1\,.$$

Proof. The proof is based on the following simple lemma, certainly well-known.

Lemma 3.2. *Let $S \colon H_1 \to H_1$ and $T \colon H_2 \to H_2$ be two compact linear operators, where H_1 and H_2 are Hilbert spaces. Let $S \otimes T$ be their tensor product, acting on the tensor product $H_1 \otimes H_2$. Then:*

$$a_{mn}(S \otimes T) \geq a_m(S)\, a_n(T)$$

for all positive integers m, n.

We postpone the proof of the lemma and show how to conclude.

We can assume C_Φ to be compact, so that C_ϕ is compact as well. Since $\|\phi\|_\infty = 1$, we have, thanks to (1.1):

$$a_m(C_\phi) \geq \mathrm{e}^{-m\, \varepsilon_m} \quad \text{with} \quad \varepsilon_m \underset{m \to \infty}{\longrightarrow} 0\,.$$

Replacing ε_m by $\delta_m := \sup_{p \geq m} \varepsilon_p$, we can assume that $(\varepsilon_m)_m$ is non-increasing. Moreover,

$$m\, \varepsilon_m \to \infty$$

since C_ϕ is compact and hence $a_m(C_\phi) \underset{m\to\infty}{\longrightarrow} 0$. We next observe that, due to the separation of variables in the definition of ϕ and ψ, we can write:

$$C_\Phi = C_\phi \otimes C_\psi\,. \tag{3.1}$$

Indeed, write $z = (z_1, w)$ with $z_1 \in \mathbb{D}$ and $w \in \mathbb{D}^{d-1}$. If $f \in H^2(\mathbb{D})$ and $g \in H^2(\mathbb{D}^{d-1})$, we see that:

$$\begin{aligned} C_\Phi(f\otimes g)(z) &= (f\otimes g)\big(\phi(z_1), \psi(w)\big) = f\big(\phi(z_1)\big)\, g\big(\psi(w)\big) \\ &= [C_\phi f(z_1)]\,[C_\psi g(w)] = (C_\phi f \otimes C_\psi g)(z)\,. \end{aligned}$$

Since tensor products $f\otimes g$ generate $H^2(\mathbb{D}^d) = H^2(\mathbb{D})\otimes H^2(D^{d-1})$, this proves (3.1).

Let now m be a large positive integer. Set ($[\,.\,]$ stands for the integer part):

$$n_m = [m\varepsilon_m]^{d-1} \quad \text{and} \quad N_m = m\, n_m\,. \tag{3.2}$$

From what we know in dimension $d-1$ (see [2], Theorem 3.1) and from the preceding, we can write (observe that ψ has to be truly $(d-1)$-dimensional since Φ is truly d-dimensional):

$$a_m(C_\phi) \geq \exp(-m\,\varepsilon_m) \quad \text{and} \quad a_n(C_\psi) \geq a\, \exp(-C\, n^{1/(d-1)})\,,$$

for some positive constant C, which will be allowed to vary from one formula to another. Lemma 3.2 implies:

$$a_{N_m}(C_\Phi) \geq a\, \exp[-C\,(m\,\varepsilon_m + {n_m}^{1/(d-1)})]\,.$$

Since $n_m \lesssim (m\varepsilon_m)^{d-1}$, we get:

$$a_{N_m}(C_\Phi) \geq a\, \exp(-C\, m\,\varepsilon_m)\,.$$

Observe that $N_m = m\, n_m \sim m^d {\varepsilon_m}^{d-1}$ and so ${N_m}^{1/d} \sim m\,{\varepsilon_m}^{1-1/d}$. As a consequence:

$$\begin{aligned} a_{N_m}(C_\Phi) \geq a\, \exp(-C\, m\,\varepsilon_m) &= a\, \exp\big[-(C\,{\varepsilon_m}^{1/d})\,(m\,{\varepsilon_m}^{1-1/d})\big] \\ &\geq a\, \exp(-\eta_m\, {N_m}^{1/d}) \end{aligned}$$

with $\eta_m := C\,{\varepsilon_m}^{1/d}$.

Now, for $N > N_1$, let m be the smallest integer satisfying $N_m \geq N$ (so that $N_{m-1} < N \leq N_m$), and set $\delta_N = \eta_m$. We have $\lim_{N\to\infty} \delta_N = 0$. Next, we note that $\lim_{m\to\infty} N_m/N_{m-1} = 1$, because $N_m \geq N_{m-1}$ and:

$$\frac{N_m}{N_{m-1}} \leq \frac{m}{m-1}\left(\frac{m\,\varepsilon_m + 1}{(m-1)\,\varepsilon_{m-1}}\right)^{d-1} \sim \left(\frac{\varepsilon_m}{\varepsilon_{m-1}}\right)^{d-1} \leq 1\,.$$

Finally, if N is an arbitrary integer and $N_{m-1} < N \leq N_m$, we obtain:

$$a_N(C_\Phi) \geq a_{N_m}(C_\Phi) \geq a\, \exp(-\eta_m\, {N_m}^{1/d}) \geq a\, \exp(-C\,\delta_N N^{1/d})\,,$$

since we observed that $\lim_{m\to\infty} N_m/N_{m-1} = 1$.

This amounts to say that $\beta_d(C_\Phi) = 1$. $\square$

Proof of Lemma 3.2. It is rather formal. Start from the Schmidt decompositions of S and T respectively (recall that Hilbert spaces, the approximation numbers are equal to the singular ones):

$$S = \sum_{m=1}^{\infty} a_m(S)\, u_m \odot v_m\,, \qquad T = \sum_{n=1}^{\infty} a_n(T)\, u'_n \odot v'_n\,,$$

where (u_m), (v_m) are two orthonormal sequences of H_1, (u'_n), (v'_n) two orthonormal sequences of H_2, and $u_m \odot v_m$ and $u'_n \odot v'_n$ denote the rank one operators defined by $(u_m \odot v_m)(x) = \langle x, v_m\rangle\, u_m$, $x \in H_1$, and $(u'_n \odot v'_n)(x) = \langle x, v'_n\rangle\, u'_n$, $x \in H_2$.

We clearly have:

$$(u_m \odot v_m) \otimes (u'_n \odot v'_n) = (u_m \otimes u'_n) \odot (v_m \otimes v'_n)\,,$$

so that the Schmidt decomposition of $S \otimes T$ is (with SOT-convergence):

$$S \otimes T = \sum_{m,n \geq 1} a_m(S)\, a_n(T)\, (u_m \otimes u'_n) \odot (v_m \otimes v'_n)\,,$$

since the two sequences $(u_m \otimes u'_n)_{m,n}$ and $(v_m \otimes v'_n)_{m,n}$ are orthonormal: for instance, we have by definition:

$$\langle u_{m_1} \otimes u'_{n_1}, u_{m_2} \otimes u'_{n_2}\rangle = \langle u_{m_1}, u_{m_2},\rangle\, \langle u'_{n_1}, u'_{n_2}\rangle.$$

This shows that the singular values of $S \otimes T$ are the non-increasing rearrangement of the positive numbers $a_m(S)\, a_n(T)$ and ends the proof of the lemma: the mn numbers $a_k(S)\, a_l(T)$, for $1 \leq k \leq m$, $1 \leq l \leq n$ all satisfy $a_k(S)\, a_l(T) \geq a_m(S)\, a_n(T)$, so that $a_{mn}(S \otimes T) \geq a_m(S)\, a_n(T)$. $\square$

4 The glued case

Here we consider symbols of the form:

$$\Phi(z_1, z_2) = \big(\phi(z_1), \phi(z_1)\big)\,, \tag{4.1}$$

where $\phi\colon \mathbb{D} \to \mathbb{D}$ is a non-constant analytic map.

Note that such maps Φ are not truly 2-dimensional.

4.1 Preliminary

We begin by remarking the following fact.

Let $B^2(\mathbb{D})$ be the Bergman space of all analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{B^2}^2 := \int_{\mathbb{D}} |f(z)|^2\, dA(z) < \infty\,,$$

where dA is the normalized area measure on $\mathbb{D}$.

Proposition 4.1. *Assume that the composition operator C_ϕ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$. Then $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$, defined by* (4.1)*, is bounded.*

Proof. If we write $f \in H^2(\mathbb{D}^2)$ as:

$$f(z_1, z_2) = \sum_{j,k\geq 0} c_{j,k} z_1^j z_2^k\,, \quad \text{with} \quad \sum_{j,k\geq 0} |c_{j,k}|^2 = \|f\|_{H^2}^2\,,$$

we formally (or assuming that f is a polynomial) have:

$$[C_\Phi f](z_1, z_2) = \sum_{j,k\geq 0} c_{j,k} [\phi(z_1)]^j [\phi(z_1)]^k = \sum_{n=0}^{\infty} \bigg(\sum_{j+k=n} c_{j,k} \bigg) [\phi(z_1)]^n\,.$$

Hence, if we set $g(z) = \sum_{n=0}^{\infty} \big(\sum_{j+k=n} c_{j,k} \big) z^n$, we get:

$$[C_\Phi(f)](z_1, z_2) = [C_\phi(g)](z_1)\,,$$

so that, by integrating:

$$\|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} = \|C_\phi(g)\|_{H^2(\mathbb{D})}\,.$$

By hypothesis, there is a positive constant M such that:

$$\|C_\phi(g)\|_{H^2(\mathbb{D})} \leq M\, \|g\|_{B^2(\mathbb{D})}\,.$$

But, by the Cauchy-Schwarz inequality:

$$\begin{aligned} \|g\|_{B^2(\mathbb{D})}^2 &= \sum_{n=0}^{\infty} \frac{1}{n+1} \bigg| \sum_{j+k=n} c_{j,k} \bigg|^2 \\ &\leq \sum_{n=0}^{\infty} \bigg(\sum_{j+k=n} |c_{j,k}|^2 \bigg) = \sum_{j,k\geq 0} |c_{j,k}|^2 = \|f\|_{H^2(\mathbb{D}^2)}^2\,, \end{aligned}$$

and we obtain $\|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} \leq M\, \|f\|_{H^2(\mathbb{D}^2)}$. $\square$

4.2 Lens maps

Let λ_θ be a lens map of parameter θ, $0 < \theta < 1$. We consider $\Phi_\theta \colon \mathbb{D}^2 \to \mathbb{D}^2$ defined by:

$$\Phi_\theta(z_1, z_2) = \big(\lambda_\theta(z_1), \lambda_\theta(z_1) \big)\,. \tag{4.2}$$

We have the following result.

Theorem 4.2. *The composition operator $C_{\Phi_\theta} \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is:*

1) *not bounded for $\theta > 1/2$;*
2) *bounded, but not compact for $\theta = 1/2$;*
3) *compact, and even Hilbert-Schmidt, for $0 < \theta < 1/2$.*

Proof. The reproducing kernel of $H^2(\mathbb{D}^2)$ is, for $(a,b) \in \mathbb{D}^2$:

$$K_{a,b}(z_1,z_2) = \frac{1}{1-\bar{a}z_1}\,\frac{1}{1-\bar{b}z_2}\,, \qquad (z_1,z_2)\in\mathbb{D}^2\,, \tag{4.3}$$

and:

$$\|K_{a,b}\|^2 = \frac{1}{(1-|a|^2)(1-|b|^2)}\,.$$

1) If C_{Φ_θ} were bounded, we should have, for some $M<\infty$:

$$\|C^*_{\Phi_\theta}(K_{a,b})\|_{H^2} \leq M\,\|K_{a,b}\|_{H^2}\,, \quad \text{for all } a,b\in\mathbb{D}\,.$$

Since $C^*_{\Phi_\theta}(K_{a,b}) = K_{\Phi_\theta(a,b)} = K_{\lambda_\theta(a),\lambda_\theta(a)}$, we would get, with $b=0$:

$$\left(\frac{1}{1-|\lambda_\theta(a)|^2}\right)^2 \leq M^2\,\frac{1}{1-|a|^2}\,;$$

but this is not possible for $\theta > 1/2$, since $1-|\lambda_\theta(a)|^2 \approx 1-|\lambda_\theta(a)| \sim (1-a)^\theta$ when a goes to 1, with $0<a<1$.

For 2) and 3), let us consider the pull-back measure m_θ of the normalized Lebesgue measure on $\mathbb{T}=\partial\mathbb{D}$ by λ_θ. It is easy to see that:

$$\sup_{\xi\in\mathbb{T}} m_\theta[D(\xi,h)\cap\mathbb{D})] = m_\theta[D(1,h)\cap\mathbb{D}] \approx h^{1/\theta}\,. \tag{4.4}$$

In particular, for $\theta\leq 1/2$, m_θ is a 2-Carleson measure, and hence (see [15], Theorem 2.1, for example) the canonical injection $j\colon B^2(\mathbb{D})\to L^2(m_\theta)$ is bounded, meaning that, for some positive constant $M<\infty$:

$$\int_{\mathbb{D}} |f(z)|^2\,dm_\theta(z) \leq M^2\|f\|^2_{B^2}\,.$$

Since

$$\int_{\mathbb{D}} |f(z)|^2\,dm_\theta(z) = \int_{\mathbb{T}} |f[\lambda_\theta(u)]|^2\,dm(u) = \|C_{\lambda_\theta}(f)\|^2_{H^2}\,,$$

we get that C_{λ_θ} maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$.

It follows from Proposition 4.1 that $C_{\Phi_\theta}\colon H^2(\mathbb{D}^2)\to H^2(\mathbb{D}^2)$ is bounded.

However, $C_{\Phi_{1/2}}$ is not compact since $C^*_{\Phi_{1/2}}(K_{a,b})/\|K_{a,b}\|$ does not converge to 0 as $a,b\to 1$, by the calculations made in 1).

For 3), let $e_{j,k}(z_1,z_2) = z_1^j z_2^k$, $j,k\geq 0$, be the canonical orthonormal basis of $H^2(\mathbb{D}^2)$; we have $[C_{\phi_\theta}(e_{j,k})](z_1,z_2) = [\lambda_\theta(z_1)]^{j+k}$. Hence:

$$\sum_{j,k\geq 0}\|C_{\phi_\theta}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \leq \sum_{n=0}^{\infty}(2n+1)\int_{\mathbb{T}}|\lambda_\theta|^{2n}\,dm \leq \int_{\mathbb{T}}\frac{2}{(1-|\lambda_\theta|^2)^2}\,dm\,.$$

Since, by Lemma 4.3 below, $1-|\lambda_\theta(\mathrm{e}^{it})|^2 \gtrsim |1-\mathrm{e}^{it}|^\theta \geq t^\theta$ for $|t|\leq\pi/2$, we get:

$$\sum_{j,k\geq 0}\|C_{\phi_\theta}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \lesssim \int_0^{\pi/2}\frac{dt}{t^{2\theta}} < \infty,$$

since $\theta<1/2$. Therefore C_{ϕ_θ} is Hilbert-Schmidt for $\theta<1/2$. $\square$

For sake of completeness, we recall the following elementary fact (see [26], p. 28, or also [16], Lemma 2.5)).

Lemma 4.3. *With $\delta = \cos(\theta\pi/2)$, we have, for $|z| \leq 1$ and $\mathfrak{Re}\, z \geq 0$:*

$$1 - |\lambda_\theta(z)|^2 \geq \frac{\delta}{2}\, |1 - z|^\theta \,.$$

Proof. We can write:

$$\lambda_\theta(z) = \frac{1-w}{1+w} \quad \text{with} \quad w = \Big(\frac{1-z}{1+z}\Big)^\theta \quad \text{and } |w| \leq 1 \,.$$

Then:

$$\mathfrak{Re}\, w \geq \delta\, |w| \geq \frac{\delta}{2} |1 - z|^\theta \,.$$

Hence:

$$1 - |\lambda_\theta(z)|^2 = \frac{4\, \mathfrak{Re}\, w}{|1+w|^2} \geq \delta\, |w| \geq \frac{\delta}{2}\, |1 - z|^\theta \,,$$

as announced ◻

We now improve the result 3) of Theorem 4.2 by estimating the approximation numbers of C_{Φ_θ} and get that C_{Φ_θ} is in all Schatten classes of $H^2(\mathbb{D}^2)$ when $\theta < 1/2$.

Theorem 4.4. *For $0 < \theta < 1/2$, there exists $b = b_\theta > 0$ such that:*

$$a_n(C_{\Phi_\theta}) \lesssim \mathrm{e}^{-b\sqrt{n}} \,. \tag{4.5}$$

In particular $\beta_2^+(C_{\Phi_\theta}) \leq \mathrm{e}^{-b} < 1$, though $\|\Phi_\theta\|_\infty = 1$, and even $\Phi_\theta(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$.

Proof. Proposition 4.1 (and its proof) can be rephrased in the following way: if C_ϕ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$, then, we have the following factorization:

$$C_\Phi \colon H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_\phi} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2) \,, \tag{4.6}$$

where $I \colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is the canonical injection given by $(If)(z_1, z_2) = f(z_1)$ for $f \in H^2(\mathbb{D})$, and $J \colon H^2(\mathbb{D}^2) \to B^2(\mathbb{D})$ is the contractive map defined by:

$$(Jf)(z) = \sum_{n=0}^\infty \Big(\sum_{j+k=n} c_{j,k} \Big) z^n \,,$$

for $f \in H^2(\mathbb{D}^2)$ with $f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k$.

In the proof of Theorem 4.2, we have seen that, for $0 < \theta \leq 1/2$, the composition operator C_{λ_θ} is bounded from $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$; we get hence the factorization:

$$C_{\Phi_\theta} \colon H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_{\lambda_\theta}} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2) \,,$$

Now, the lens maps have a semi-group property:

$$\lambda_{\theta_1\theta_2} = \lambda_{\theta_1}\lambda_{\theta_2} \,, \tag{4.7}$$

giving $C_{\lambda_{\theta_1\theta_2}} = C_{\lambda_{\theta_1}} \circ C_{\lambda_{\theta_2}}$.

For $0 < \theta < 1/2$, we therefore can write $C_{\lambda_\theta} = C_{\lambda_{2\theta}} \circ C_{\lambda_{1/2}}$ (note that $2\theta < 1$, so $C_{\lambda_{2\theta}} \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded), and we get:

$$C_{\Phi_\theta} = I \, C_{\lambda_{2\theta}} C_{\lambda_{1/2}} \, J \,.$$

Consequently:

$$a_n(C_{\Phi_\theta}) \leq \|I\| \, \|J\| \, \|C_{\lambda_{1/2}}\|_{B^2 \to H^2} \, a_n(C_{\lambda_{2\theta}}) \,.$$

Now, we know ([16], Theorem 2.1) that $a_n(C_{\lambda_{2\theta}}) \lesssim \mathrm{e}^{-b\sqrt{n}}$, so we get that $a_n(C_{\Phi_\theta}) \lesssim \mathrm{e}^{-b\sqrt{n}}$. $\square$

Remark. In [2], we saw that for a truly 2-dimensional symbol Φ, we have $\beta_2^-(C_\phi) > 0$. Here the symbol Φ_θ is not truly 2-dimensional, but we nevertheless have $\beta_2(C_{\Phi_\theta}) > 0$. In fact, let $E = \{f \in H^2(\mathbb{D}^2) \, ; \, \frac{\partial f}{\partial z_2} \equiv 0\}$; E is isometrically isomorphic to $H^2(\mathbb{D})$ and the restriction of C_{Φ_θ} to E behaves as the 1-dimensional composition operator $C_{\lambda_\theta} \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$; hence ([19], Proposition 6.3):

$$\mathrm{e}^{-b_0\sqrt{n}} \lesssim a_n(C_{\lambda_\theta}) = a_n(C_{\Phi_\theta \, | E}) \leq a_n(C_{\Phi_\theta}) \,,$$

and $\beta_2^-(C_{\Phi_\theta}) \geq \mathrm{e}^{-b_0} > 0$.

5 Triangularly separated variables

In this section, we consider symbols of the form:

$$\Phi(z_1, z_2) = \big(\phi(z_1), \psi(z_1)\, z_2\big) \,, \tag{5.1}$$

where $\phi, \psi \colon \mathbb{D} \to \mathbb{D}$ are non-constant analytic maps.

Such maps Φ are truly 2-dimensional.

More generally, if $h \in H^\infty$, with $h(0) = 0$ and $\|h\|_\infty \leq 1$, has its powers h^k, $k \geq 0$, orthogonal in H^2 (for convenience, we shall say that h is a *Rudin function*), we can consider:

$$\Phi(z_1, z_2) = \big(\phi(z_1), \psi(z_1)\, h(z_2)\big) \tag{5.2}$$

For such h we can take for example an inner function vanishing at the origin, but there are other such functions, as shown by C. Bishop:

Theorem (Bishop [4]). *The function h is a Rudin function if and only if the pull-back measure $\mu = \mu_h$ is radial and Jensen, i.e for every Borel set E:*

$$\mu(\mathrm{e}^{i\theta} E) = \mu(E) \quad \textit{and} \quad \int_{\mathbb{D}} \log(1/|z|)\, d\mu(z) < \infty\,.$$

Conversely, for every probability measure μ supported by $\overline{\mathbb{D}}$, which is radial and Jensen, there exists h in the unit ball of H^∞, with $h(0) = 0$, such that $\mu = \mu_h$.

If we take for μ the Lebesgue measure of $\mathbb{T}$, we get an inner function. But, as remarked in [4], we can take for μ the Lebesgue measure on the union $\mathbb{T} \cup (1/2)\mathbb{T}$, normalized in order that $\mu(T) = \mu\big((1/2)\mathbb{T}\big) = 1/2$. Then the corresponding h is not inner since $|h| = 1/2$ on a subset of $\mathbb{T}$ of positive measure. He also showed that $h(z)/z$ may be a non-constant outer function. Also, P. Bourdon ([6]) showed that the powers of h are orthogonal if and only if its Nevanlinna counting function is almost everywhere constant on each circle centered on the origin.

5.1 General facts

We first observe that if $f \in H^2(\mathbb{D}^2)$ and:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k}\, z_1^j z_2^k\,,$$

then we can write:

$$f(z_1, z_2) = \Bigg(\sum_{k \geq 0} f_k(z_1) \Bigg)\, z_2^k$$

with:

$$f_k(z_1) = \sum_{j \geq 0} c_{j,k}\, z_1^j\,,$$

and:

$$\|f\|^2_{H^2(\mathbb{D}^2)} = \sum_{j,k \geq 0} |c_{j,k}|^2 = \sum_{k \geq 0} \|f_k\|^2_{H^2(\mathbb{D})}\,.$$

That means that we have an isometric isomorphism:

$$J \colon H^2(\mathbb{D}^2) \longrightarrow \bigoplus_{k=0}^{\infty} H^2(\mathbb{D})\,,$$

defined by $Jf = (f_k)_{k \geq 0}$.

Now, for symbols Φ as in (5.1), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k}\, [\phi(z_1)]^j [\psi(z_1)]^k z_2^k\,,$$

so that $J\,C_\Phi\,J^{-1}$ appears as the operator $\bigoplus_k M_{\psi^k} C_\phi$ on $\bigoplus_k H^2(\mathbb{D})$, where M_{ψ^k} is the multiplication operator by ψ^k:

$$[(M_{\psi^k} C_\phi) f_k](z_1) = [\psi(z_1)]^k \, [(f_k \circ \phi)(z_1)] \, .$$

When Φ is as in (5.2), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} \, [\phi(z_1)]^j [\psi(z_1)]^k [h(z_2)]^k \, ,$$

with:

$$\| C_\Phi f \|^2 \leq \sum_{k=0}^{\infty} \| T_k f_k \|^2$$

and:

$$T_k = M_{\psi^k} C_\phi \, ;$$

hence $J\,C_\Phi\,J^{-1}$ appears as pointwise dominated by the operator $T = \oplus_k T_k$ on $\bigoplus_k H^2(\mathbb{D})$. This implies a factorization $C_\Phi = AT$ with $\|A\| \leq 1$, so that $a_n(C_\Phi) \leq a_n(T)$ for all $n \geq 1$.

We recall the following elementary fact.

Lemma 5.1. *Let $(H_k)_{k \geq 0}$ be a sequence of Hilbert spaces and $T_k \colon H_k \to H_k$ be bounded operators. Let $H = \bigoplus_{k=0}^\infty H_k$ and $T \colon H \to H$ defined by $Tx = (T_k x_k)_k$. Then:*

1) *T is bounded on H if and only if $\sup_k \|T_k\| < \infty$;*
2) *T is compact on H if and only if each T_k is compact and $\|T_k\| \underset{k \to \infty}{\longrightarrow} 0$.*

Going back to the symbols of the form (5.1), we have $\|M_{\psi^k}\| \leq \|\psi^k\|_\infty \leq 1$, since $\|\psi\|_\infty \leq 1$; hence $\|M_{\psi^k} C_\phi\| \leq \|C_\phi\|$ and the operator $(M_{\psi^k} C_\phi)_k$ is bounded on $\bigoplus_k H^2(\mathbb{D})$. Therefore C_Φ is bounded on $H^2(\mathbb{D}^2)$.

For approximation numbers, we have the following two facts.

Lemma 5.2. *Let $T_k \colon H_k \to H_k$ be bounded linear operators between Hilbert spaces H_k, $k \geq 0$. Let $H = \bigoplus_k H_k$ and $T = (T_k)_k \colon H \to H$, assumed to be compact. Then, for every $n_1, \ldots, n_K \geq 1$, and $0 \leq m_1 < \cdots < m_K$, $K \geq 1$, we have:*

$$a_N(T) \geq \inf_{1 \leq k \leq K} a_{n_k}(T_{m_k}) \, , \tag{5.3}$$

where $N = n_1 + \cdots + n_K$.

Proof. We use the Bernstein numbers b_n (see (1.4)), which are equal to the approximation numbers (see (1.7)).

For $k = 1, \ldots, K$, there is an n_k-dimensional subspace E_k of H_{m_k} such that:

$$b_{n_k}(T_{m_k}) \leq \|T_{m_k} x\| \, , \quad \text{for all } x \in S_{E_k} \, .$$

Then $E = \bigoplus_{k=1}^{K} E_k$ is an N-dimensional subspace of H and for every $x = (x_1, x_2, \ldots) \in E$, we have:

$$\begin{aligned} \|Tx\|^2 &= \sum_{k\leq K} \|T_{m_k} x_{m_k}\|^2 \geq \sum_{k\leq K} [b_{n_k}(T_{m_k})]^2 \, \|x_{m_k}\|^2 \\ &\geq \inf_{k\leq K} [b_{n_k}(T_{m_k})]^2 \sum_{k\leq K} \|x_{m_k}\|^2 = \inf_{k\leq K} [b_{n_k}(T_{m_k})]^2 \|x\|^2 \,; \end{aligned}$$

hence $b_N(T) \geq \inf_{k\leq K} b_{n_k}(T_{m_k})$, and we get the announced result. $\square$

Lemma 5.3. *Let $T = \bigoplus_{k=0}^{\infty} T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^{\infty} H_k$. Let $n_0, \ldots, n_K$ be positive integers and $N = n_0 + \cdots + n_K - K$. Then, the approximation numbers of T satisfy:*

$$a_N(T) \leq \max \Big(\max_{0\leq k\leq K} a_{n_k}(T_k), \sup_{k>K} \|T_k\| \Big) \,. \tag{5.4}$$

Proof. Denote by S the right-hand side of (5.4). Let R_k, $0 \leq k \leq K$ be operators on H_k of respective rank $< n_k$ such that $\|T_k - R_k\| = a_{n_k}(T_k)$ and let $R = \bigoplus_{k=0}^{K} R_k$. Then R is an operator of rank $\leq n_0 + \cdots + n_K - K - 1 < N$. If $f = \sum_{k=0}^{\infty} f_k \in H$, we see that:

$$\begin{aligned} \|Tf - Rf\|^2 &= \sum_{k=0}^{K} \|T_k f_k - R_k f_k\|^2 + \sum_{k>K} \|T_k f_k\|^2 \\ &\leq \sum_{k=0}^{K} a_{n_k}(T_k)^2 \|f_k\|^2 + \sum_{k>K} \|T_k f_k\|^2 \leq S^2 \sum_{k=0}^{\infty} \|f_k\|^2 = S^2 \|f\|^2 \,, \end{aligned}$$

hence the result. $\square$

We give now two corollaries of Lemma 5.3.

Example 1. We first use lens maps. We get:

Theorem 5.4. *Let λ_θ the lens map of parameter θ and let $\psi \colon \mathbb{D} \to \mathbb{D}$ such that $\|\psi\|_\infty := c < 1$ and h a Rudin function. We consider:*

$$\Phi(z_1, z_2) = \big(\lambda_\theta(z_1), \psi(z_1)\, h(z_2) \big) \,.$$

Then, for some positive constant β, we have, for all $N \geq 1$:

$$a_N(C_\Phi) \lesssim \mathrm{e}^{-\beta\, N^{1/3}} \,. \tag{5.5}$$

Proof. Let $T_k = M_{\psi^k} C_{\lambda_\theta}$. We have $\|T_k\| \leq c^k$, so $\sup_{k>K} \|T_k\| \leq c^K$. On the other hand, we have $a_n(T_k) \leq c^k \, a_n(C_{\lambda_\theta}) \leq a_n(C_{\lambda_\theta}) \lesssim \mathrm{e}^{-\beta_\theta \sqrt{n}}$ ([16], Theorem 2.1). Taking $n_0 = n_1 = \cdots = n_K = K^2$ in Lemma 5.3, we get:

$$\max_{0\leq k\leq K} a_{n_k}(T_k) \lesssim \mathrm{e}^{-\beta_\theta K} \,.$$

Since $n_0 + \cdots + n_K - K \approx K^3$, we obtain $a_{K^3} \lesssim \mathrm{e}^{-\beta K}$, which gives the claimed result, by taking $\beta = \max \big(\beta_\theta, \log(1/c) \big)$. $\square$

Example 2. We consider the cusp map χ. We have:

Theorem 5.5. *Let χ be the cusp map, h a Rudin function, and ψ in the unit ball of H^∞, with $\|\psi\|_\infty := c < 1$. Let:*

$$\Phi(z_1, z_2) = \big(\chi(z_1), \psi(z_1)\, h(z_2)\big)\,.$$

Then, for positive constant β, we have, for all $N \geq 1$:

$$a_N(C_\Phi) \lesssim \mathrm{e}^{-\beta\sqrt{N}/\sqrt{\log N}}\,.$$

Proof. Let $T_k = M_{\psi^k} C_\chi$. As above, we have $\sup_{k>K} \|T_k\| \leq c^K$. For the cusp map, we have $a_n(C_\chi) \lesssim \mathrm{e}^{-\alpha n/\log n}$ ([20], Theorem 4.3); hence $a_n(T_k) \lesssim \mathrm{e}^{-\alpha n/\log n}$. We take $n_0 = n_1 = \cdots = n_K = K\,[\log K]$ (where $[\log K]$ is the integer part of $\log K$). Since $n_0 + \cdots + n_K \approx K^2[\log K]$, we get, for another $\alpha > 0$:

$$a_{K^2[\log K]}(C_\Phi) \lesssim \mathrm{e}^{-\alpha K}\,,$$

which reads: $a_N(C_\Phi) \lesssim \mathrm{e}^{-\beta\sqrt{N/\log N}}$, as claimed. □

5.2 Lower bounds

In this subsection, we give lower bounds for approximation numbers of composition operators on H^2 of the bidisk, attached to a symbol Φ of the previous form $\Phi(z_1, z_2) = \big(\phi(z_1), \psi(z_1)\, h(z_2)\big)$ where h is a Rudin function. The sharpness of those estimates will be discussed in the next subsection. We first need some lemmas in dimension one.

Lemma 5.6. *Let $u, v\colon \mathbb{D} \to \mathbb{D}$ be two non-constant analytic self-maps and $T = M_v C_u\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the associated weighted composition operator. For $0 < r < 1$, we set $A = u(r\,\overline{\mathbb{D}})$ and $\Gamma = \exp\big(-1/\mathrm{Cap}\,(A)\big)$. Then, for $0 < \delta \leq \inf_{|z|=r} |v(z)|$, we have:*

$$a_n(T) \gtrsim \sqrt{1-r}\,\delta\,\Gamma^n\,. \tag{5.6}$$

In this lemma, $\mathrm{Cap}\,(A)$ denotes the Green capacity of the compact subset $A \subseteq \mathbb{D}$ (see [21], § 2.3 for the definition).

For the proof, we need the following result ([27], Theorem 7, p. 353).

Theorem 5.7 (Widom)**.** *Let A be a compact subset of $\mathbb{D}$ and $\mathcal{C}(A)$ be the space of continuous functions on A with its natural norm. Set:*

$$\tilde{d}_n(A) = \inf_E \Big[\sup_{f \in B_{H^\infty}} \mathrm{dist}\,(f, E)\Big]\,,$$

where E runs over all $(n-1)$-dimensional subspaces of $\mathcal{C}(A)$ and $\mathrm{dist}\,(f, E) = \inf_{h \in E} \|f - h\|_{\mathcal{C}(A)}$. Then

$$\tilde{d}_n(A) \geq \alpha\, \mathrm{e}^{-n/\mathrm{Cap}\,(A)} \tag{5.7}$$

for some positive constant α.

Proof of Lemma 5.6. We apply Theorem 5.7 to the compact set $A = u(r\,\overline{\mathbb{D}})$.

Let E be an $(n-1)$-dimensional subspace of $H^2 = H^2(\mathbb{D})$; it can be viewed as a subspace of $\mathcal{C}(A)$, so, by Theorem 5.7, there exists $f \in H^\infty \subseteq H^2$ with $\|f\|_2 \leq \|f\|_\infty \leq 1$ such that:

$$\|f - h\|_{\mathcal{C}(A)} \geq \alpha\,\Gamma^n\,, \quad \forall h \in E\,.$$

Then:

$$\|v\,(f \circ u - h \circ u)\|_{\mathcal{C}(r\mathbb{T})} \geq \delta\,\|(f - h) \circ u\|_{\mathcal{C}(r\mathbb{T})} = \delta\,\|f - h\|_{\mathcal{C}(A)} \geq \alpha\,\delta\,\Gamma^n\,.$$

But:

$$\|v\,(f \circ u - h \circ u)\|_{\mathcal{C}(r\mathbb{T})} \leq \frac{1}{\sqrt{1 - r^2}}\,\|v\,(f \circ u - h \circ u)\|_{H^2}\,;$$

Hence:

$$\|Tf - Th\|_{H^2} \geq \alpha\,\sqrt{1 - r^2}\,\delta\,\Gamma^n \geq \alpha\,\sqrt{1 - r}\,\delta\,\Gamma^n\,.$$

Since h is an arbitrary function of E, we get (B_{H^2} being the unit ball of H^2):

$$\inf_{\dim E < n}\left[\sup_{f \in B_{H^2}} \operatorname{dist}\big(Tf, T(E)\big)\right] \geq \alpha\,\sqrt{1 - r}\,\delta\,\Gamma^n\,.$$

But the left-hand side is equal to the Kolmogorov number $d_n(T)$ of T (see [21], Lemma 3.12), and, as recalled in (1.7), in Hilbert spaces, the Kolmogorov numbers are equal to the approximation numbers; hence we obtain:

$$a_n(T) \geq \alpha\,\sqrt{1 - r}\,\delta\,\Gamma^n\,, \quad n = 1, 2, \ldots\,, \tag{5.8}$$

as announced. □

The next lemma shows that some Blaschke products are far away from 0 on some circles centered at 0.

We consider a *strongly interpolating sequence* $(z_j)_{j\geq 1}$ of $\mathbb{D}$ in the sense that, if $\varepsilon_j := 1 - |z_j|$, then:

$$\varepsilon_{j+1} \leq \sigma\,\varepsilon_j \tag{5.9}$$

and so $\varepsilon_j \leq \sigma^{j-1}\varepsilon_1$, where $0 < \sigma < 1$ is fixed. Equivalently, the sequence $(|z_j|)_{j\geq 1}$ is interpolating. We consider the corresponding interpolating Blaschke product:

$$B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j}\,\frac{z_j - z}{1 - \overline{z_j}z}\,. \tag{5.10}$$

The following lemma is probably well-known, but we could find no satisfactory reference (see yet [10] for related estimates) and provide a simple proof.

Lemma 5.8. *Let $(z_j)_{j\geq 1}$ be a strongly interpolating sequence as in* (5.9) *and B the associated Blaschke product* (5.10).

Then there exists a sequence $r_l := 1-\rho_l$ such that:

$$C_1\,\sigma^l \leq \rho_l \leq C_2\,\sigma^l\,, \tag{5.11}$$

where C_1, C_2 are positive constants, and for which:

$$|z| = r_l \quad \Longrightarrow \quad |B(z)| \geq \delta\,, \tag{5.12}$$

where $\delta > 0$ does not depend on l.

Proof. Let us denote by p_l, $1 \leq p_l \leq l$, the biggest integer such that $\varepsilon_{p_l} \geq \sigma^{l-1}\varepsilon_1$.

We separate two cases.

Case 1: $\varepsilon_{p_l} \geq 2\,\sigma^{l-1}\varepsilon_1$.

Then, we choose $\rho_l = \alpha\,\sigma^{l-1}\varepsilon_1$ with α fixed, $1<\alpha<2$. Since $\rho(\xi,\zeta) \geq \rho(|\xi|,|\zeta|)$ for all $\xi,\zeta\in\mathbb{D}$ (recall that ρ is the pseudo-hyperbolic distance on $\mathbb{D}$), we have the following lower bound for $|z| = r_l$:

$$|B(z)| = \prod_{j=1}^{\infty}\rho(z,z_j) \geq \prod_{j=1}^{\infty}\rho(r_l,|z_j|) = \prod_{j\leq p_l}\rho(r_l,|z_j|)\times\prod_{j>p_l}\rho(r_l,|z_j|) := P_1\times P_2\,,$$

and we estimate P_1 and P_2 separately.

We first observe that $\dfrac{\rho_l}{\varepsilon_{p_l}} \leq \dfrac{\alpha\,\sigma^{l-1}\varepsilon_1}{2\,\sigma^{l-1}\varepsilon_1} \leq \dfrac{\alpha}{2}$, and then:

$$\frac{\rho_l}{\varepsilon_j} = \frac{\rho_l}{\varepsilon_{p_l}}\frac{\varepsilon_{p_l}}{\varepsilon_j} \leq \frac{\alpha}{2}\,\sigma^{p_l-j}.$$

The inequality $\rho(1-u,1-v) \geq \frac{|u-v|}{(u+v)}$ for $0<u,v\leq 1$ now gives us:

$$\rho(r_l,|z_j|) \geq \frac{\varepsilon_j-\rho_l}{\varepsilon_j+\rho_l} = \frac{1-\rho_l/\varepsilon_j}{1+\rho_l/\varepsilon_j} \geq \frac{1-(\alpha/2)\,\sigma^{p_l-j}}{1+(\alpha/2)\,\sigma^{p_l-j}}\,, \quad \text{for } j\leq p_l\,, \tag{5.13}$$

and:

$$P_1 \geq \prod_{k=0}^{\infty}\left(\frac{1-(\alpha/2)\,\sigma^k}{1+(\alpha/2)\,\sigma^k}\right). \tag{5.14}$$

Similarly:

$$\frac{\varepsilon_{p_l+1}}{\rho_l} \leq \frac{\sigma^{l-1}\varepsilon_1}{\alpha\,\sigma^{l-1}\varepsilon_1} \leq \frac{1}{\alpha}$$

and:

$$\frac{\varepsilon_j}{\rho_l} \leq \frac{1}{\alpha}\,\sigma^{j-p_l-1} \quad \text{for } j>p_l\,;$$

so that:

$$\rho(r_l,|z_j|) \geq \frac{\rho_l-\varepsilon_j}{\rho_l+\varepsilon_j} = \frac{1-\varepsilon_j/\rho_l}{1+\varepsilon_j/\rho_l} \geq \frac{1-\alpha^{-1}\sigma^{j-p_l-1}}{1+\alpha^{-1}\sigma^{j-p_l-1}}\,, \quad \text{for } j>p_l\,, \tag{5.15}$$

and

$$P_2 \geq \prod_{k=0}^{\infty} \left(\frac{1-\alpha^{-1}\sigma^k}{1+\alpha^{-1}\sigma^k} \right). \tag{5.16}$$

Finally, the condition of lower and upper bound for ρ_l is fulfilled by construction.

Case 2: $\varepsilon_{p_l} \leq 2\,\sigma^{l-1}\varepsilon_1$.

Then, we choose $\rho_l = a\,\varepsilon_{p_l}$ with $\sigma < a < 1$ fixed. Computations exactly similar to those of Case 1 give us:

$$|B(z)| \geq \prod_{k=0}^{\infty} \left(\frac{1-a\,\sigma^k}{1+a\,\sigma^k} \right) \times \prod_{k=0}^{\infty} \left(\frac{1-a^{-1}\sigma^k}{1+a^{-1}\sigma^k} \right) =: \delta > 0\,, \quad \text{for } |z| = r_l\,. \tag{5.17}$$

Moreover, in this case:

$$a\,\sigma^{l-1}\varepsilon_1 \leq \rho_l \leq 2\,a\,\sigma^{l-1}\varepsilon_1\,,$$

and the proof is ended. □

Now, we have the following estimation.

Theorem 5.9. *Let $\phi, \psi\colon \mathbb{D} \to \mathbb{D}$ be two non-constant analytic self-maps and $\Phi(z_1, z_2) = \big(\phi(z_1), \psi(z_1)\,h(z_2)\big)$, where h is inner.*

Let $(r_l)_{l\geq 1}$ be an increasing sequence of positive numbers with limit 1 such that:

$$\inf_{|z|=r_l} |\psi(z)| \geq \delta_l > 0\,,$$

with $\delta_l \leq \mathrm{e}^{-1/\mathrm{Cap}\,(A_l)}$, where $A_l = \phi\big(r_l\overline{\mathbb{D}}\big)$.

Then the approximation numbers $a_N(C_\Phi)$, $N \geq 1$, of the composition operator $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ satisfy:

$$a_N(C_\Phi) \gtrsim \sup_{l\geq 1} \Big[\sqrt{1-r_l}\, \exp\big(-8\sqrt{N}\,\sqrt{\log(1/\delta_l)}\,\sqrt{\log(1/\Gamma_l}\ \big) \Big]\,, \tag{5.18}$$

where:

$$\Gamma_l = \mathrm{e}^{-1/\mathrm{Cap}\,(A_l)}\,. \tag{5.19}$$

Proof. Since h is inner, the sequence $(h^k)_{k\geq 0}$ is orthonormal in H^2 and hence $a_n(C_\Phi) = a_n(T)$ for all $n \geq 1$, where $T = \bigoplus_{k=0}^{\infty} T_k$ and $T_k = M_{\psi^k} C_\phi$. Then Lemma 5.6 gives:

$$a_n(T_k) \gtrsim \sqrt{1-r_l}\,\delta_l^k \Gamma_l^n \tag{5.20}$$

for all $n \geq 1$ and all $k \geq 0$.

Let now:

$$p_l = \left[\frac{\log(1/\delta_l)}{\log(1/\Gamma_l)} \right], \tag{5.21}$$

where $[\,.\,]$ stands for the integer part, and:

$$n_k = p_l k\,, \qquad \text{for} \quad k = 1, \ldots, K\,. \tag{5.22}$$

By Lemma 5.2, applied with $m_k = k$ (i.e. to $H_1, \ldots, H_K$), we have, if $N = n_1 + \cdots + n_K$:

$$a_N(T) \geq \inf_{1 \leq k \leq K} \alpha \sqrt{1 - r_l}\, \delta_l^k\, \Gamma_l^n = \alpha \sqrt{1 - r_l}\, \delta_l^K\, \Gamma_l^{n_K}\,.$$

But, since $p_l \leq \log(1/\delta_l)/\log(1/\Gamma_l)$:

$$\delta_l^K\, \Gamma_l^{n_K} = \exp\big[- \big(K \log(1/\delta_l) + p_l K \log(1/\Gamma_l)\big)\big] \geq \exp[-2K \log(1/\delta_l)]\,.$$

Since:

$$N = p_l \frac{K(K+1)}{2} \geq p_l \frac{K^2}{4} \geq \frac{K^2}{16}\, \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)}\,,$$

we get:

$$\delta_l^K\, \Gamma_l^{n_K} \geq \exp\big[- 8 \sqrt{N}\, \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)}\big]\,,$$

and the result ensues. □

Example 1. We take $\phi = \lambda_\theta$, a lens map, and $\psi = B$, a Blaschke product associated to a strongly regular sequence, as defined in (5.10); then we get:

Theorem 5.10. *Let $\Phi \colon \mathbb{D}^2 \to \mathbb{D}^2$ be defined by:*

$$\Phi(z_1, z_2) = \big(\lambda_\theta(z_1), c\, B(z_1)\, h(z_2)\big)\,,$$

where B is a Blaschke product as in (5.10), $0 < c < 1$, and h is an arbitrary inner function, we have, for some positive constant b, for all $N \geq 1$:

$$a_N(C_\Phi) \gtrsim \exp(-b\, N^{1/3}) = \exp(-b \sqrt{N}/N^{1/6})\,. \tag{5.23}$$

In particular $\beta_2(C_\Phi) = \beta_2^\pm(C_\Phi) = 1$.

Remark. We saw in Theorem 5.4 that this is the exact size, since we have: $a_N(C_\Phi) \lesssim \mathrm{e}^{-\beta\, N^{1/3}}$.

Proof. By Lemma 5.8, there is a sequence of numbers $r_l \approx \sigma^l$ such that $|B(z)| \geq \delta$ for $|z| = r_l$, where δ is a positive constant (depending on σ). Since $\lambda_\theta(0) = 0$, we have:

$$\mathrm{diam}_\rho(A_l) \geq \lambda_\theta(r_l) \gtrsim 1 - (1 - r_l)^\theta\,;$$

hence, by [21], Theorem 3.13, we have:

$$\mathrm{Cap}\,(A_l) \gtrsim \log \frac{1}{1 - r_l} \gtrsim l\,,$$

or, equivalently: $\Gamma_l \geq \mathrm{e}^{-b/l}$, some some $b > 0$. Then (5.18) gives, for all $l \geq 1$ (with another b):

$$a_N(C_\Phi) \gtrsim \exp\left[- b\bigg(l + \frac{\sqrt{N}}{\sqrt{l}}\bigg)\right].$$

Taking $l = N^{1/3}$, we get the result. □

Example 2. By taking the cusp instead of a lens map, we obtain a better result, close to the extremal one.

Theorem 5.11. *Let* $\Phi(z_1, z_2) = \big(\chi(z_1), c\, B(z_1)\, h(z_2)\big)$, *where* χ *is the cusp map,* B *a Blaschke product as in* (5.10), $0 < c < 1$, *and* h *an arbitrary inner function. Then, for all* $N \geq 1$*:*

$$a_N(C_\Phi) \gtrsim \mathrm{e}^{-b\,\sqrt{N}/\sqrt{\log N}}\,.$$

In particular $\beta_2(C_\Phi) = 1$.

Remark. We saw in Theorem 5.5 that this is the exact size, since we have: $a_N(C_\phi) \lesssim \mathrm{e}^{-\beta\sqrt{N/\log N}}$.

Proof. The proof is the same as that of Proposition 5.10, except that, for the cusp map, we have (note that $\chi(0) = 0$):

$$\mathrm{diam}_\rho(A_l) \geq \chi(r_l)\,.$$

But when r goes to 1:

$$1 - \chi(r) \sim \frac{\pi\,(\sqrt{2} - 1)}{2}\, \frac{1}{\log\big(1/(1 - r)\big)}$$

(see [20], Lemma 4.2). Hence, by [21], Theorem 3.13, again, we have:

$$\mathrm{Cap}\,(A_l) \gtrsim \log\big(\log\big(1/(1 - r_l)\big)\big)\,,$$

so $\Gamma_l \geq \mathrm{e}^{-b/\log l}$. Then, (5.18) gives (with another b):

$$a_N(C_\Phi) \gtrsim \exp\bigg[-b\bigg(l + \frac{\sqrt{N}}{\sqrt{\log l}}\bigg)\bigg]\,.$$

In taking $l = \sqrt{N/\log N}$, we get the announced result. □

5.3 Upper bounds

All previous results point in the direction that, if $\|\Phi\|_\infty = 1$, then however small $a_n(C_\Phi)$ is, it will always be larger than $\alpha\,\mathrm{e}^{-\beta\varepsilon_n\sqrt{n}}$ with $\varepsilon_n \to 0^+$, as this is the case in dimension one (with n instead of $\sqrt{n}$). But Theorem 5.12 to follow shows that we cannot hope, in full generality, to get the same result in dimension $d \geq 2$, and that other phenomena await to be understood. Here is our main result. It shows that, even for a truly 2-dimensional symbol Φ, we can have $\|\Phi\|_\infty = 1$ and nevertheless $\beta_2^+(C_\Phi) < 1$, in contrast to the 1-dimensional case where (1.1) holds.

Theorem 5.12. *There exist a map* $\Phi\colon \mathbb{D}^2 \to \mathbb{D}^2$ *such that:*

1) *the composition operator* $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ *is bounded and compact;*

2) *we have* $\|\Phi\|_\infty = 1$ *and* Φ *is truly 2-dimensional, so that* $\beta_2^-(C_\Phi) > 0$*;*

3) *the singular numbers satisfy* $a_n(C_\Phi) \leq \alpha\,\mathrm{e}^{-\beta\,\sqrt{n}}$ *for some positive constants* α, β*; in particular* $\beta_2^+(C_\Phi) < 1$.

Proof. Let $0 < \theta < 1$ be fixed, and λ_θ be the corresponding lens map. We set:

$$\begin{cases} \phi & = & \dfrac{1+\lambda_\theta}{2} \\ w(z) & = & \exp\Big[-\Big(\dfrac{1+z}{1-z}\Big)^\theta\Big] \\ \psi & = & w\circ\phi\,. \end{cases}$$

Note that $\|\phi\|_\infty = 1$.

Setting $\delta = \cos(\theta\pi/2) > 0$, we have for $z \in \mathbb{D}$:

$$|1-\phi(z)| = \frac{1}{2}\,|1-\lambda_\theta(z)| = \left|\frac{(1-z)^\theta}{(1-z)^\theta + (1+z)^\theta}\right| \leq \frac{|1-z|^\theta}{\delta}\,. \tag{5.24}$$

Indeed, the argument α of $(1\pm z)^\theta$ satisfies $|\alpha| \leq \theta\pi/2$ for $z \in \mathbb{D}$, and we get:

$$|(1-z)^\theta + (1+z)^\theta| \geq \mathfrak{Re}\,[(1-z)^\theta + (1+z)^\theta] \geq \delta(|1+z|^\theta + |1-z|^\theta) \geq \delta\,.$$

We also see that $\phi(\mathbb{D})$ touches the boundary $\partial\mathbb{D}$ only at 1 in a non-tangential way, meaning that for some constant $C > 1$:

$$1 - |\phi(z)| \geq \frac{1}{C}\,|1-\phi(z)|\,, \quad \forall z \in \mathbb{D}\,.$$

Now, we have the following two inequalities:

$$\mathfrak{Re}\,z \geq 0 \quad\Longrightarrow\quad |w(z)| \leq \exp\Big(-\frac{\delta}{|1-z|^\theta}\Big) \tag{5.25}$$

$$z \in \mathbb{D} \quad\Longrightarrow\quad |\psi(z)| \leq \exp\Big(-\frac{\delta^2}{|1-z|^{\theta^2}}\Big)\,. \tag{5.26}$$

Indeed, with $S(z) = \big(\frac{1+z}{1-z}\big)^\theta$, we have $\mathfrak{Re}\,S(z) \geq \delta\,|S(z)| \geq \delta|1-z|^{-\theta}$ when $\mathfrak{Re}\,z \geq 0$, giving (5.25), and (5.24) and (5.25) imply, since $\mathfrak{Re}\,\phi(z) \geq 0$:

$$|\psi(z)| = |w\big(\phi(z)\big)| \leq \exp\Big(-\frac{\delta}{|1-\phi(z)|^\theta}\Big) \leq \exp\Big(-\frac{\delta^2}{|1-z|^{\theta^2}}\Big)\,.$$

We now set:

$$\Phi(z_1, z_2) = \big(\phi(z_1), \psi(z_1)\,h(z_2)\big)\,, \tag{5.27}$$

with h a Rudin function.

Observe that $\phi \in A(\mathbb{D})$ and $\psi = w\circ\phi \in A(\mathbb{D})$ as well ($w \in A(\mathbb{D})$ with $w(1) = 0$; this is due to the presence of the parameter $\theta < 1$). hence if we take for h a finite Blaschke product, the two components of Φ are in the bidisk algebra $A(\mathbb{D}^2)$.

We have $\|\psi\|_\infty := \rho < 1$. In fact, for $\Re\mathrm{e}\, u \geq 0$, we have:

$$\left|\frac{1+u}{1-u}\right| \geq 2^{-\theta}|1+u|^\theta \geq 2^{-\theta}(1+\Re\mathrm{e}\, u)^\theta \geq 2^{-\theta}\,,$$

hence:

$$\Re\mathrm{e}\left[\left(\frac{1+u}{1-u}\right)^\theta\right] \geq \left(\cos\frac{\theta\pi}{2}\right)\left|\frac{1+u}{1-u}\right|^\theta \geq \left(\cos\frac{\theta\pi}{2}\right)2^{-\theta} = \delta\, 2^{-\theta}\,,$$

and $\|w\circ\phi\|_\infty \leq \mathrm{e}^{2^{-\theta}\delta}$.

Now, 1) follows from the orthogonal model presented in Section 5.1, because $\|\psi\|_\infty < 1$.

The assertion 2) follows from [2], Theorem 3.1, since $\|\phi\|_\infty = 1$.

We now prove 3).

As observed, C_Φ can be viewed as a direct sum $T = \bigoplus_{k=0}^\infty T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^\infty H_k$, where T_k acts on a copy H_k of $H^2(\mathbb{D})$ with:

$$T_k = M_{\psi^k} C_\phi\,.$$

We fix the positive integer n. The rest of the proof will consist of three lemmas.

Lemma 5.13. *We have $\|T_k\| \leq 2\,\rho^{-k} \leq 2\,\rho^{-n}$ for $k > n$.*

Proof. Indeed, since $\phi(0) = 1/2$, we know that $\|C_\phi\| \leq \sqrt{\frac{1+\phi(0)}{1-\phi(0)}} = \sqrt{3} \leq 2$, so that $\|T_k\| \leq \|\psi^k\|_\infty \|C_\phi\| \leq \rho^{-k}\times 2$. □

Lemma 5.14. *Set $b = a/\delta^2$ where $a > 0$ is given by $\mathrm{e}^{-a} = 4C/\sqrt{16C^2+1}$ and C is as in (2.1). Let m_k be the smallest integer such that $k\,\delta^2 2^{m_k\theta^2} \geq an$; namely:*

$$m_k = \left[\frac{\log(b\,n/k)}{\theta^2\log 2}\right] + 1\,, \tag{5.28}$$

where $[\,.\,]$ stands for the integer part. Then, with $a' = \min(\log 2, a)$:

$$a_{nm_k+1}(T_k) \lesssim \mathrm{e}^{-a'n}\,.$$

Proof. This follows from Theorem 2.3 applied with $w = \psi^k$, $R = k\,\delta^2$ and θ changed into θ^2. This is possible thanks to (5.26) and to Lemma 5.13. Moreover we have adjusted m_k so as to make the two terms in Theorem 2.3 of the same order. □

Lemma 5.15. *The dimension $d := \sum_{k=0}^n n\,m_k$ satisfies, for some positive constant α:*

$$d \leq \alpha\, n^2\,.$$

Proof. Indeed, it is well-known that:

$$\sum_{k=1}^{n} \log k = n \log n - n + \mathrm{O}\,(\log n)\,,$$

and, in view of (5.28), we have $m_k \leq \alpha'_\theta \log(b\, n/k) \leq \alpha''_\theta (\log n - \log k)$; hence:

$$\sum_{k=1}^{n} m_k \leq \alpha''_\theta \big[n \log n - \big(n \log n - n + \mathrm{O}\,(\log n) \big) \big] = \alpha''_\theta\, n + \mathrm{O}\,(\log n)\,,$$

and we get $d \leq \alpha''_\theta\, n^2 + \mathrm{O}\,(n \log n) \leq \alpha_\theta\, n^2$.

Alternatively, we could have used a Riemann sum for the function $\log(1/x)$ on $(0, 1]$. □

Finally, putting things together and using as well Proposition 5.3 with $K = n$ and $n_k = n m_k + 1$ so that $(\sum_{k=0}^{n} n_k) - n = (\sum_{k=0}^{n} n\, m_k) + 1 = d + 1$, we get ignoring once more multiplicative constants:

$$a_{n^2}(T) \lesssim a_d(T) \leq \alpha\, \mathrm{e}^{-\beta n}$$

with positive constants α, β. This ends the proof of Theorem 5.12. □

6 Monge-Ampère capacity and applications

6.1 Definition

Let K be a compact subset of $\mathbb{D}^m$ (in this section, for notational reasons, we denote the dimension by m instead of d). The Monge-Ampère capacity of K has been defined by Bedford and Taylor ([3]; see also [13], § 5 or [11], Chapter 1) as:

$$\mathrm{Cap}_m\,(K) = \sup \left\{ \int_K (dd^c u)^m\,;\ u \in PSH \text{ and } 0 \leq u \leq 1 \right\},$$

where PSH is the set of plurisubharmonic functions on $\mathbb{D}^m$, $dd^c = 2i\partial\bar{\partial}$, and $(dd^c)^m = dd^c \wedge \cdots \wedge dd^c$ (m times). When $u \in PSH \cap \mathcal{C}^2(\mathbb{D}^m)$, we have:

$$(dd^c u)^m = 4^m m! \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV(z)\,,$$

where $dV(z) = (i/2)^m dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m$ is the usual volume in $\mathbb{C}^m$. A more convenient formula (because $\mathbb{D}^m$ is bounded and hyperconvex: see [11], p. 80, for the definition) is:

$$\mathrm{Cap}_m\,(K) = \int_K (dd^c u_K^*)^m\,,$$

where u_K^* is called the *extremal function of* K and is the upper semi-continuous regularization of:

$$u_K = \sup\{ v \in PSH\,;\ v \leq 0 \text{ and } v \leq -1 \text{ on } K \}\,,$$

but we will not need that.

As in [28], we set:

$$\tau_m(K) = \frac{1}{(2\pi)^m}\,\mathrm{Cap}_m\,(K)\,. \tag{6.1}$$

For $m = 1$, $\tau(K) := \tau_1(K)$ is equal to the Green capacity $\mathrm{Cap}\,(K)$ of K with respect to $\mathbb{D}$, with the definition used in [21] (see [13], Theorem 8.1, where a factor 2π is introduced).

We further set:

$$\Gamma_m(K) = \exp\left[-\left(\frac{m!}{\tau_m(K)}\right)^{1/m}\right]. \tag{6.2}$$

We proved in [21] that, for $m = 1$, and $\varphi\colon \mathbb{D} \to r\mathbb{D}$, with $0 < r < 1$, we have:

$$\beta_1(C_\varphi) = \Gamma_1\big(\overline{\varphi(\mathbb{D})}\big)\,. \tag{6.3}$$

The goal of this section is to see that Theorem 5.12 shows that this no longer holds for $m = 2$.

6.2 A seminal example

In one variable, our initial motivation had been the simple-minded example $\varphi(z) = rz$, $0 < r < 1$, for which $C_\varphi(z^n) = r^n z^n$, implying $a_n(C_\varphi) = r^{n-1}$ and $\beta_1(C_\varphi) = r$. If $K = \overline{\varphi(\mathbb{D})} = \overline{D}(0, r)$, we have $\mathrm{Cap}\,(K) = \frac{1}{\log 1/r}$ and $\Gamma_1(K) = r$, so that $\beta_1(C_\varphi) = \Gamma_1(K)$. Let us examine the multivariate example (where $0 < r_j < 1$):

$$\Phi(z_1, z_2, \ldots, z_m) = (r_1 z_1, r_2 z_2, \ldots, r_m z_m).$$

If $K = \overline{\Phi(\mathbb{D}^m)}$, we have $K = \prod_{k=1}^m \overline{D}(0, r_k)$, and hence ([5], Theorem 3):

$$\tau_m(K) = \prod_{k=1}^m \frac{1}{\log(1/r_k)}\,\cdot \tag{6.4}$$

On the other hand, $C_\Phi(z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}) = r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m}\, z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}$ so that the sequence $(a_n)_n$ of approximation numbers of C_Φ is the non-increasing rearrangement of the numbers $r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m}$. It is convenient to state the following simple lemma.

Lemma 6.1. *Let $\lambda_1, \ldots, \lambda_m$ be positive numbers. Let N_A be the number of m-tuples of non-negative integers $(n_1, \ldots, n_m)$ such that $\sum_{k=1}^m \lambda_k n_k \leq A$. Then, as $A \to \infty$:*

$$N_A \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m)\, m!}\,\cdot$$

Indeed, just apply Karamata's tauberian theorem (see [12] p. 30) to the generalized Dirichlet series:

$$S(\varepsilon) := \prod_{k=1}^{m} \frac{1}{1 - \mathrm{e}^{-\lambda_k \varepsilon}} \quad = \sum_{n_1, \ldots, n_m \geq 0} \mathrm{e}^{-(\sum_{k=1}^{m} \lambda_k n_k)\, \varepsilon} \, ;$$

we have $S(\varepsilon) \sim \frac{\varepsilon^{-m}}{(\lambda_1 \cdots \lambda_m)}$ as $\varepsilon \to 0^+$.

Let now N be a positive integer and $\varepsilon = a_N$. Setting $\lambda_k = \log(1/r_k)$ and $A = \log(1/\varepsilon)$, we see that N is the number of m-tuples $(n_1, \ldots, n_m)$ of non-negative integers such that $r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \geq \varepsilon$, i.e. such that $\sum_{k=1}^{m} \lambda_k n_k \leq A$. This number N is hence nothing but the number N_A of the previous lemma, so that:

$$N \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m)\, m!} \, \cdot$$

Inverting this formula, we get:

$$a_N(C_\Phi) = \exp\big[- (1 + \mathrm{o}\,(1))\, (m!(\lambda_1 \lambda_2 \cdots \lambda_m)\, N)^{1/m} \big]$$

and:

$$\beta_m(C_\Phi) = \exp\big[- (m! \lambda_1 \lambda_2 \cdots \lambda_m)^{1/m} \big] = \Gamma_m(K) \, ,$$

in view of (6.2) and (6.4).

On the view of the simple-minded previous example, the extension of the spectral radius formula (6.3) to the multivariate case holds, and it is tempting to conjecture that this is a general phenomenon as in dimension one, all the more as the following extension of Widom's theorem was proved by Zakharyuta, based on the solution by S. Nivoche of Zakharyuta's conjecture ([23]); see also [28], Theorem 5.4. A compact subset K of $\mathbb{D}^m$ is said to be *regular* if its extremal function u_K^* is continuous on $\mathbb{D}^m$.

Theorem 6.2 ([28], Theorem 5.6)**.** *Let K be a regular compact subset of $\mathbb{D}^m$ and $J \colon H^\infty(\mathbb{D}^m) \to \mathcal{C}(K)$ the canonical injection; then the Kolmogorov numbers $d_n(J)$ satisfy:*

$$\lim_{n \to \infty} \big[d_n(J) \big]^{1/n^{1/m}} = \exp\left[- \left(\frac{m!}{\tau_m(K)} \right)^{1/m} \right] \cdot \tag{6.5}$$

Note that the right side is nothing but $\Gamma_m(K)$.

We will see consequences of this result in a forthcoming paper ([22]).

6.3 Upper bound

For the upper bound, the situation behaves better, as stated in the following theorem.

Theorem 6.3 ([28], Proposition 6.1)**.** *Let K be a compact subset of $\mathbb{D}^m$ with non-void interior. Then:*

$$\limsup_{n\to\infty} \big[d_n(J)\big]^{1/n^{1/m}} \leq \exp\bigg[-\Big(\frac{m!}{\tau_m(K)}\Big)^{1/m}\bigg]. \tag{6.6}$$

Note that $(K, \mathbb{D}^m)$ is a condenser since K has non-void interior. We deduce the following upper bound.

Theorem 6.4. *Let Φ be an analytic self-map of $\mathbb{D}^m$ with $\|\Phi\|_\infty = \rho < 1$, thus inducing a compact composition operator on $H^2(\mathbb{D}^m)$. Then we have:*

$$\beta_m^+(C_\Phi) \leq \Gamma_m\big(\overline{\Phi(\mathbb{D}^m)}\big).$$

Proof. This proof provides in particular a simplification of that given in [21] in dimension $m = 1$.

Changing n into n^m, Theorem 6.3 means that for every $\varepsilon > 0$, there exists an $(n^m - 1)$-dimensional subspace V of $\mathcal{C}(K)$ such that, for any $g \in H^\infty(\mathbb{D}^m)$, there exists $h \in V$ such that:

$$\|g - h\|_{\mathcal{C}(K)} \leq C_\varepsilon (1+\varepsilon)^n \big[\Gamma_m(K)\big]^n \|g\|_\infty. \tag{6.7}$$

Let l be an integer to be adjusted later, and $f(z) = \sum_\alpha b_\alpha z^\alpha \in B_{H^2}$, as well as $g(z) = \sum_{|\alpha|\leq l} b_\alpha z^\alpha$. We first note that (with M_m depending only on m and ρ, and since the number of α's such that $|\alpha| \leq p$ is $\mathrm{O}\,(p^m)$):

$$\sum_{|\alpha|>l} \rho^{2|\alpha|} \leq M_m \sum_{p>l} p^m \rho^{2p} \leq M_m l^m \,\frac{\rho^{2l}}{(1-\rho^2)^{m+1}}.$$

We next observe that, by the Cauchy-Schwarz and Parseval inequalities:

$$\|g\|_\infty \leq M_m\, l^{m/2}, \tag{6.8}$$

and

$$|f(z) - g(z)| \leq M_m\, l^{m/2} \frac{|z|_\infty^l}{(1-|z|_\infty^2)^{(m+1)/2}}, \qquad \forall z \in \mathbb{D}^m. \tag{6.9}$$

where $|z|_\infty := \max_{j\leq m} |z_j|$ if $z = (z_1, \ldots, z_m)$.

The subspace F formed by functions $v \circ \Phi$, for $v \in V$, can be viewed as a subspace of $L^\infty(\mathbb{T}^m) \subseteq L^2(\mathbb{T}^m)$ with respect to the Haar measure of $\mathbb{T}^m$, the distinguished boundary of $\mathbb{D}^m$ (indeed, we can write $(v\circ\Phi)^* = v \circ \Phi^*$, where Φ^* denotes the almost everywhere existing radial limits of $\Phi(rz)$, which belong to K). Let finally $E = P(F) \subseteq H^2(\mathbb{D}^m)$ where $P\colon L^2(\mathbb{T}^m) \to H^2(\mathbb{T}^m) = H^2(\mathbb{D}^m)$ is the orthogonal projection. This is a subspace of H^2 with dimension $< n^m$. Set temporarily $\eta = C_\varepsilon(1+\varepsilon)^n \big[\Gamma_m(K)\big]^n$. It follows from (6.7) and (6.8) that, for some $h \in V$:

$$\|g - h\|_{\mathcal{C}(K)} \leq \eta\, \|g\|_\infty \leq \eta\, M_m\, l^{m/2}$$

and hence:

$$\|g\circ\Phi - h\circ\Phi\|_2 \leq \|g\circ\Phi - h\circ\Phi\|_\infty \leq \eta\, M_m\, l^{m/2}\,,$$

implying by orthogonal projection:

$$\operatorname{dist}\,(C_\Phi g, E) \leq \|g\circ\Phi - P(h\circ\Phi)\|_2 \leq \eta\, M_m\, l^{m/2}\,.$$

Now, since $C_\Phi f(z) - C_\Phi g(z) = f\big(\Phi(z)\big) - g\big(\Phi(z)\big)$, (6.9) gives:

$$\|C_\Phi f - C_\Phi g\|_2 \leq \|C_\Phi f - C_\Phi g\|_\infty \leq M_m\, l^{m/2}\, \frac{\rho^l}{(1-\rho^2)^{(m+1)/2}}$$

and hence:

$$\operatorname{dist}\,(C_\Phi f, E) \leq M_m\, l^{m/2} \left(\frac{\rho^l}{(1-\rho^2)^{(m+1)/2}} + C_\varepsilon (1+\varepsilon)^n \big[\Gamma_m(K)\big]^n \right).$$

It ensues, since $a_N(C_\Phi) = d_N(C_\Phi)$, that:

$$\big[a_{n^m}(C_\Phi)\big]^{1/n} \leq (M_m\, l^{m/2})^{1/n} \left[\frac{\rho^{l/n}}{(1-\rho^2)^{(m+1)/2n}} + C_\varepsilon^{1/n} (1+\varepsilon)\, \Gamma_m(K) \right].$$

Taking now for l the integer part of $n\log n$, and passing to the upper limit as $n\to\infty$, we obtain (since $l/n\to\infty$ and $(\log l)/n \to 0$):

$$\beta_m^+(C_\Phi) \leq (1+\varepsilon)\, \Gamma_m(K)\,,$$

and Theorem 6.4 follows. □

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Daniel Li
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec
Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Apartado de Correos 1160 41 080 SEVILLA, SPAIN
piazza@us.es