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Some examples of composition operators and their approximation numbers on the Hardy space of the bi-disk

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Abstract. We give examples of composition operators \( C_\Phi \) on \( H^2(\mathbb{D}^2) \) showing that the condition \( \|\Phi\|_\infty = 1 \) is not sufficient for their approximation numbers \( a_n(C_\Phi) \) to satisfy \( \lim_{n \to \infty} [a_n(C_\Phi)]^{1/\sqrt{n}} = 1 \), contrary to the 1-dimensional case. We also give a situation where this implication holds. We make a link with the Monge-Ampère capacity of the image of \( \Phi \).

Key-words: approximation numbers; Bergman space; bidisk; composition operator; Green capacity; Hardy space; Monge-Ampère capacity; weighted composition operator.


1 Introduction and notation

1.1 Introduction

The purpose of this paper is to continue the study of composition operators on the polydisk initiated in [2], and in particular to examine to what extent one of the main results of [21] still holds.

Let \( H \) be a Hilbert space and \( T: H \to H \) a bounded operator. Recall that the approximation numbers of \( T \) are defined as:

\[
a_n(T) = \inf_{\text{rank} \ R < n} \|T - R\|, \quad n \geq 1,
\]

and we have:

\[
\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq \cdots
\]

The operator \( T \) is compact if and only if \( a_n(T) \xrightarrow{n \to \infty} 0. \)
For $d \geq 1$, we define:

$$
\begin{align*}
\beta_d^-(T) &= \liminf_{n \to \infty} \left[ a_n(T) \right]^{1/n} \\
\beta_d^+(T) &= \limsup_{n \to \infty} \left[ a_n(T) \right]^{1/n}
\end{align*}
$$

We have:

$$0 \leq \beta_d^-(T) \leq \beta_d^+(T) \leq 1,$$

and we simply write $\beta_d(T)$ in case of equality.

It may well happen in general (consider diagonal operators) that $\beta_d^-(T) = 0$ and $\beta_d^+(T) = 1$.

When $H = H^2(\mathbb{D})$ is the Hardy space on the open unit disk $\mathbb{D}$ of $\mathbb{C}$, and $T = C_\Phi$ is a composition operator, with $\Phi : \mathbb{D} \to \mathbb{D}$ a non-constant analytic function, we always have ([19]):

$$\beta_1^-(C_\Phi) > 0,$$

and one of the main results of [19] is the equivalence:

$$(1.1) \quad \beta_1^+(C_\Phi) < 1 \iff \|\Phi\|_\infty < 1.$$ 

An alternative proof was given in [21], as a consequence of a so-called “spectral radius formula”, which moreover shows that:

$$\beta_1^-(C_\Phi) = \beta_1^+(C_\Phi).$$

In [2], for $d \geq 2$, it is proved that, for a bounded symmetric domain $\Omega \subseteq \mathbb{C}^d$, if $\Phi : \Omega \to \Omega$ is analytic, such that $\Phi(\Omega)$ has a non-void interior, and the composition operator $C_\Phi : H^2(\Omega) \to H^2(\Omega)$ is compact, then:

$$\beta_d^-(C_\Phi) > 0.$$ 

On the other hand, if $\Omega$ is a product of balls, then:

$$\|\Phi\|_\infty < 1 \implies \beta_d^+(C_\Phi) < 1.$$ 

We do not know whether the converse holds and the purpose of this paper is to study some examples towards an answer.

The paper is organized as follows. Section 1 is this short introduction, as well as some notations and definitions on singular numbers of operators and Hardy spaces of the polydisk to follow. Section 2 contains preliminary results on weighted composition operators in one variable, which surprisingly play an important role in the study of non-weighted composition operators in two variables. Section 3 studies the case of symbols with “separated” variables. Our main one variable result extends in this case. Section 4 studies the “glued case” $\Phi(z_1, z_2) = (\phi(z_1), \phi(z_1))$ for which even boundedness is an issue. Here, the
Bergman space $B^2(\mathbb{D})$ enters the picture. Section 5 studies the case of “triangularly separated” variables. This section lets direct Hilbertian sums of weighted composition operators in one variable appear, and it contains our main result: an example of a symbol $\Phi$ satisfying $\|\Phi\|_\infty = 1$ and yet $\beta^+_2(C_\Phi) < 1$. The final Section 6 discusses the role of the Monge-Ampère pluricapacity, which is a multivariate extension of the Green capacity in the disk. Even though, as evidenced by our counterexample of Section 5, this capacity will not capture all the behavior of the parameter $\beta_m(C_\Phi)$, some partial results are obtained, relying on theorems of S. Nivoche and V. Zakharyuta.

1.2 Notation

We denote by $\mathbb{D}$ the open unit disk of the complex plane and by $\mathbb{T}$ its boundary, the 1-dimensional torus.

The Hardy space $H^2(\mathbb{D}^d)$ is the space of holomorphic functions $f : \mathbb{D}^d \to \mathbb{C}$ whose boundary values $f^*$ on $\mathbb{T}^d$ are square-integrable with respect to the Haar measure $m_d$ of $\mathbb{T}^d$, and normed with:

$$\|f\|_2^2 = \|f\|_{H^2(\mathbb{D}^d)}^2 = \int_{\mathbb{T}^d} |f^*(\xi_1, \ldots, \xi_d)|^2 \, dm_d(\xi_1, \ldots, \xi_d).$$

If $f(z_1, \ldots, z_d) = \sum_{\alpha_1, \ldots, \alpha_d \geq 0} a_{\alpha_1, \ldots, \alpha_d} z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, then:

$$\|f\|_2^2 = \sum_{\alpha_1, \ldots, \alpha_d \geq 0} |a_{\alpha_1, \ldots, \alpha_d}|^2.$$

We say that an analytic map $\Phi : \mathbb{D}^d \to \mathbb{D}^d$ is a symbol if its associated composition operator $C_\Phi : H^2(\mathbb{D}^d) \to H^2(\mathbb{D}^d)$, defined by $C_\Phi(f) = f \circ \Phi$, is bounded.

We say that $\Phi$ is truly $d$-dimensional if $\Phi(\mathbb{D}^d)$ has a non-void interior.

We will make use of two kinds of symbols defined on $\mathbb{D}$.

The lens map $\lambda_\theta : \mathbb{D} \to \mathbb{D}$ is defined, for $\theta \in (0, 1)$, by:

$$\lambda_\theta(z) = \frac{(1 + z)^\theta - (1 - z)^\theta}{(1 + z)^\theta + (1 - z)^\theta}$$

(see [26], p. 27, or [16], for more information), and corresponds to $u \mapsto u^\theta$ in the right half-plane.

The cusp map $\chi : \mathbb{D} \to \mathbb{D}$ was first defined in [15] and in a slightly different form in [20]; we actually use here the modified form introduced in [17], and then used in [18]. We first define:

$$\chi_\alpha(z) = \frac{\left(\frac{z - i}{iz - 1}\right)^{1/2} - i}{-i \left(\frac{z - i}{iz - 1}\right)^{1/2} + 1},$$
we note that $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2} - 1$. Then we set:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where:

$$a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)$$

is chosen in order that $\chi(0) = 0$. The image $\Omega$ of the (univalent) cusp map is formed by the intersection of the inside of the disk $D(1 - \frac{a}{2}, \frac{a}{2})$ and the outside of the two disks $D(1 + \frac{ia}{T}, \frac{a}{2})$ and $D(1 - \frac{ia}{T}, \frac{a}{2})$.

Besides the approximation numbers, we need other singular numbers for an operator $S: X \to Y$ between Banach spaces $X$ and $Y$.

The *Bernstein numbers* $b_n(S)$, $n \geq 1$, which are defined by:

$$(1.4) \quad b_n(S) = \sup_{E} \min_{x \in S_E} \|Sx\|,$$

where the supremum is taken over all $n$-dimensional subspaces of $X$ and $S_E$ is the unit sphere of $E$.

The *Gelfand numbers* $c_n(S)$, $n \geq 1$, which are defined by:

$$(1.5) \quad c_n(S) = \inf\{\|S_M\| : \text{codim } M < n\}.$$

The *Kolmogorov numbers* $d_n(S)$, $n \geq 1$, which are defined by:

$$(1.6) \quad d_n(S) = \inf_{\dim E < n} \left[ \sup_{x \in B_X} \text{dist} (Sx, E) \right].$$

Pietsch showed that all $s$-numbers on Hilbert spaces are equal (see [24], § 2, Corollary, or [25], Theorem 11.3.4); hence:

$$(1.7) \quad a_n(S) = b_n(S) = c_n(S) = d_n(S).$$

We denote $m$ the normalized Lebesgue measure on $T = \partial D$. If $\varphi: D \to D$, $m_\varphi$ is the pull-back measure on $\overline{D}$ defined by $m_\varphi(E) = m[\varphi^{-1}(E)]$, where $\varphi^*$ stands for the non-tangential boundary values of $\varphi$.

The notation $A \lesssim B$ means that $A \leq CB$ for some positive constant $C$ and we write $A \approx B$ if we have both $A \lesssim B$ and $B \lesssim A$. 

4
2 Preliminary results on weighted composition operators on $H^2(\mathbb{D})$

We see in this section that the presence of a “rapidly decaying” weight allows simpler estimates for the approximation numbers of a corresponding weighted composition operator. Such a study, but a bit different, is made in [14].

Let $\varphi: \mathbb{D} \to \mathbb{D}$ a non-constant analytic self-map in the disk algebra $A(\mathbb{D})$ such that, for some constant $C > 1$ and for all $z \in \mathbb{D}$:

\begin{equation}
\varphi(1) = 1, \quad |1 - \varphi(z)| \leq 1, \quad |1 - \varphi(z)| \leq C (1 - |\varphi(z)|)
\end{equation}

as well as $\varphi(z) \neq 1$ for $z \neq 1$. We can take for example $\varphi = \frac{1 + \lambda \theta}{2}$ where $\lambda \theta$ is the lens map with parameter $\theta$.

Let $w \in H^\infty$ and let $T$ be the weighted composition operator

$$T = M_{w \circ \varphi} C_\varphi : H^2 \to H^2.$$ 

Note that $M_{w \circ \varphi} C_\varphi = C_\varphi M_w$. We first show that:

**Theorem 2.1.** Let $T = M_{w \circ \varphi} C_\varphi : H^2 \to H^2$ be as above and let $B$ be a Blaschke product with length $< N$. Then, with the implied constant depending only on the number $C$ in (2.1) (and of $\varphi$):

$$a_N(T) \lesssim \sup_{|z-1| \leq 1, \ z \in \varphi(\mathbb{D})} |B(z)| |w(z)|.$$ 

**Proof.** The following preliminary observation (see also [16], p. 809), in which we denote by $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}$ the Carleson window with center $\xi \in \mathbb{T}$ and size $h$, and by $K_\varphi$ the support of the pull-back measure $m_\varphi$, will be useful.

\begin{equation}
\varphi \in S(\xi, h) \cap K_\varphi \implies u \in S(1, Ch) \cap K_\varphi.
\end{equation}

Indeed, if $|u - \xi| \leq h$ and $u \in K_\varphi$, (2.1) implies:

$$1 - |u| \leq |u - \xi| \leq h \quad \text{and} \quad |u - 1| \leq C(1 - |u|) \leq Ch.$$ 

Set $E = BH^2$. This is a subspace of codimension $< N$. If $f =Bg \in E$, with $\|g\| = \|f\|$ (isometric division by $B$ in $BH^2$), we have $Tf = (wBg) \circ \varphi$, whence:

$$\|T(f)\|^2 = \int_{\mathbb{D}} |B|^2 |w|^2 |g|^2 dm_\varphi,$$

implying $\|T(f)\|^2 \leq \|f\|^2 \|J\|^2$ where $J: H^2 \to L^2(\sigma)$ is the natural embedding and where

$$\sigma = |B|^2 |w|^2 dm_\varphi.$$ 

Now, Carleson’s embedding theorem for the measure $\sigma$ and (2.2) show that (the implied constants being absolute):

$$
\|J\|^2 \lesssim \sup_{0 < h < 1} \frac{1}{h} \int_{S(\xi, h) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi
$$

$$
\lesssim \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi
$$

$$
\lesssim \left( \sup_{|z-1| \leq 1, \ z \in \varphi(D)} |B(z)|^2 |w(z)|^2 \right) \left( \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} dm_\varphi \right)
$$

$$
\lesssim \sup_{|z-1| \leq 1, \ z \in \varphi(D)} |B(z)|^2 |w(z)|^2,
$$

since $m_\varphi$ is a Carleson measure for $H^2$ and where we used that, according to (2.1):

$$
K_\varphi \subseteq \varphi(D) \subseteq \{ z \in \mathbb{D}; \ |z-1| \leq 1 \}.
$$

This ends the proof of Theorem 2.1 with help of the equality of $a_N(T)$ with the Gelfand number $c_N(T)$ recalled in (1.7).

In order to specialize efficiently the general Theorem 2.1, we recall the following simple Lemma 2.3 of [16], where:

\begin{equation}
\rho(a, b) = \frac{|a - b|}{1 - ab}, \quad a, b \in \mathbb{D},
\end{equation}

is the pseudo-hyperbolic distance:

**Lemma 2.2** ([16]). Let $a, b \in \mathbb{D}$ such that $|a - b| \leq L \min(1 - |a|, 1 - |b|)$. Then:

$$
\rho(a, b) \leq \frac{L}{\sqrt{L^2 + 1}} =: \kappa < 1.
$$

We can now state:

**Theorem 2.3.** Assume that $\varphi$ is as in (2.1) and that the weight $w$ satisfies, for some parameters $0 < \theta \leq 1$ and $R > 0$:

$$
|w(z)| \leq \exp \left( - \frac{R}{|1 - z|^\theta} \right), \quad \forall z \in \mathbb{D} \text{ with } \Re z \geq 0.
$$

Then, the approximation numbers of $T = M_{w \circ \varphi} C_\varphi$ satisfy:

$$
a_{nm+1}(T) \lesssim \max \left[ \exp(-an), \exp(-R 2^n \theta) \right],
$$

for all integers $n, m \geq 1$, where $a = \log[\sqrt{16C^2 + 1}/(4C)] > 0$ and $C$ is as in (2.1).
Proof. Let \( p_l = 1 - 2^{-l} \), \( 0 \leq l < m \) and let \( B \) be the Blaschke product:

\[
B(z) = \prod_{0 \leq l < m} \left( \frac{z - p_l}{1 - p_l z} \right)^n.
\]

Let \( z \in K_\varphi \cap \mathbb{D} \) so that \( 0 < |z - 1| \leq 1 \). Let \( l \) be the non-negative integer such that \( 2^{-l-1} < |z - 1| \leq 2^{-l} \). We separate two cases:

**Case 1:** \( l \geq m \). Then, the weight does the job. Indeed, majorizing \(|B(z)|\) by 1 and using the assumption on \( w \), we get:

\[
|B(z)|^2 |w(z)|^2 \leq |w(z)|^2 \leq \exp \left( -\frac{2R}{|1 - z|} \right) \leq \exp(-2R 2^{l}) \leq \exp(-2R 2^{m}) .
\]

**Case 2:** \( l < m \). Then, the Blaschke product does the job. Indeed, majorize \(|w(z)|\) by 1 and estimate \(|B(z)|\) more accurately with help of Lemma 2.2; we observe that

\[
|z - p_l| \leq |z - 1| + 1 - p_l \leq 2 \times 2^{-l} = 2(1 - p_l) \leq 4C(1 - p_l)
\]

and then, since \( z \in K_\varphi \), we can write with \( C \geq 1 \) as in (2.1):

\[
1 - |z| \geq \frac{1}{C} |1 - z| \geq \frac{1}{2C} 2^{-l} \geq \frac{1}{4C} |z - p_l| ,
\]

so that the assumptions of Lemma 2.2 are verified with \( L = 4C \), giving:

\[
\rho(z, p_l) \leq \frac{4C}{\sqrt{16C^2 + 1}} = \exp(-a) < 1 .
\]

Hence, by definition, since \( l < m \):

\[
|B(z)| \leq |\rho(z, p_l)|^n \leq \exp(-an) .
\]

Putting both cases together, and observing that our Blaschke product has length \( nm < nm + 1 \), we get the result by applying Theorem 2.1 with \( N = nm + 1 \).

2.1 Some remarks

1. Twisting a composition operator by a weight may improve the compactness of this composition operator, or even may make this weighted composition operator compact though the non-weighted was not (see [8] or [14]). However, this is not possible for all symbols, as seen in the following proposition.

**Proposition 2.4.** Let \( w \in H^\infty \). If \( \varphi \) is inner, or more generally if \( |\varphi| = 1 \) on a subset of \( \mathbb{T} \) of positive measure, then \( M_w C_\varphi \) is never compact (unless \( w \equiv 0 \)).
Proof. Indeed, suppose \( T = M_w C_\varphi \) compact. Since \((z^n)_n\) converges weakly to 0 in \( H^2 \) and since \( T(z^n) = w \varphi^n \), we should have, since \( |\varphi| = 1 \) on \( E \), with \( m(E) > 0 \):

\[
\int_E |w|^2 \, dm = \int_E |w|^2 |\varphi|^{2n} \, dm \leq \int_T |w|^2 |\varphi|^{2n} \, dm = ||T(z^n)||^2 \rightarrow 0,
\]

but this would imply that \( w \) is null a.e. on \( E \) and hence \( w \equiv 0 \) (see [7], Theorem 2.2), which was excluded.

Note that É. Amar and A. Lederer proved in [1] that \( |\varphi| = 1 \) on a set of positive measure if and only if \( \varphi \) is an exposed point of the unit ball of \( H^\infty \); hence the following proposition can be viewed as the (almost) opposite case.

**Proposition 2.5.** Let \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) such that \( \|\varphi\|_\infty = 1 \). Assume that:

\[
\int_T \log(1 - |\varphi|) \, dm < -\infty
\]

(meaning that \( \varphi \) is not an extreme point of the unit ball of \( H^\infty \): see [7], Theorem 7.9). Then, if \( w \) is an outer function such that \( |w| = 1 - |\varphi| \), the weighted composition operator \( T = M_w C_\varphi \) is Hilbert-Schmidt.

Proof. We have:

\[
\sum_{n=0}^\infty \|T(z^n)\|^2 = \sum_{n=0}^\infty \int_T (1 - |\varphi|)^2 |\varphi|^{2n} \, dm = \int_T \frac{1 - |\varphi|}{1 + |\varphi|} \, dm < +\infty,
\]

and \( T \) is Hilbert-Schmidt, as claimed.

2. In [14], Theorem 2.5, it is proved that we always have, for some constants \( \delta, \rho > 0 \):

\[
(2.4) \quad a_n(M_w C_\varphi) \geq \delta \rho^n, \quad n = 1, 2, \ldots
\]

(if \( w \neq 0 \)). We give here an alternative proof, based on a result of Gunatillake ([9]), this result holding in a wider context.

**Theorem 2.6** (Gunatillake). Let \( T = M_w C_\varphi \) be a compact weighted composition operator on \( H^2 \) and assume that \( \varphi \) has a fixed point \( a \in \mathbb{D} \). Then the spectrum of \( T \) is the set:

\[
\sigma(T) = \{0, w(a), w(a) \varphi'(a), w(a) |\varphi'(a)|^2, \ldots, w(a) |\varphi'(a)|^n, \ldots\}
\]

Proof of (2.4). First observe that, in view of Proposition 2.4, \( \varphi \) cannot be an automorphism of \( \mathbb{D} \) so that the point \( a \) is the Denjoy-Wolff point of \( \varphi \) and is attractive. Theorem 2.6 is interesting only when \( w(a) \varphi'(a) \neq 0 \).
Now, we can give a new proof Theorem 2.5 of [14] as follows. Let \( a \in \mathbb{D} \) be such that \( w(a) \varphi'(a) \neq 0 \) (\( H(\mathbb{D}) \) is a division ring and \( \varphi' \neq 0, w \neq 0 \)). Let \( b = \varphi(a) \) and \( \tau \in \text{Aut} \mathbb{D} \) with \( \tau(b) = a \). We set:

\[
\psi = \tau \circ \varphi \quad \text{and} \quad S = M_w C_\psi = TC_\tau.
\]

This operator \( S \) is compact because \( T \) is.

Since \( \psi(a) = a \) and \( \psi'(a) = \tau'(b) \varphi'(a) \neq 0 \), Theorem 2.6 says that the non-zero eigenvalues of \( S \), arranged in non-increasing order, are the numbers \( \lambda_n = w(a) |\psi'(a)|^{n-1}, n \geq 1 \). As a consequence of Weyl’s inequalities, we know that:

\[
a_1(S) a_n(S) \geq |\lambda_{2n}|^2 \geq \delta \rho^n,
\]

with:

\[
\delta = |w(a)|^2 > 0 \quad \text{and} \quad \rho = |\psi'(a)|^4 > 0.
\]

To finish, it is enough to observe that \( a_n(S) \leq a_n(T) \|C_\tau\| \) by the ideal property of approximation numbers. \(\square\)

3 The splitted case

Theorem 3.1. Let \( \Phi = (\varphi, \psi) : \mathbb{D}^d \to \mathbb{D}^d \) be a truly \( d \)-dimensional symbol with \( \varphi : \mathbb{D} \to \mathbb{D} \) depending only on \( z_1 \) and \( \psi : \mathbb{D}^{d-1} \to \mathbb{D}^{d-1} \) only on \( z_2, \ldots, z_d \), i.e. \( \Phi(z_1, z_2, \ldots, z_d) = (\varphi(z_1), \psi(z_2, \ldots, z_d)) \). Then, whatever \( \psi \) behaves:

\[
\|\varphi\|_\infty = 1 \implies \beta_d(C_\Phi) = 1.
\]

Proof. The proof is based on the following simple lemma, certainly well-known.

Lemma 3.2. Let \( S : H_1 \to H_1 \) and \( T : H_2 \to H_2 \) be two compact linear operators, where \( H_1 \) and \( H_2 \) are Hilbert spaces. Let \( S \otimes T \) be their tensor product, acting on the tensor product \( H_1 \otimes H_2 \). Then:

\[
a_{mn}(S \otimes T) \geq a_m(S) a_n(T)
\]

for all positive integers \( m, n \).

We postpone the proof of the lemma and show how to conclude.

We can assume \( C_\Phi \) to be compact, so that \( C_\Phi \) is compact as well. Since \( \|\varphi\|_\infty = 1 \), we have, thanks to (1.1):

\[
a_m(C_\Phi) \geq e^{-m \varepsilon_m} \quad \text{with} \quad \varepsilon_m \to 0.\]

Replacing \( \varepsilon_m \) by \( \delta_m := \sup_{p \geq m} \varepsilon_p \), we can assume that \( (\varepsilon_m)_m \) is non-increasing. Moreover,

\[
m \varepsilon_m \to \infty
\]

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since $C_{\phi}$ is compact and hence $a_m(C_{\phi}) \to 0$. We next observe that, due to the separation of variables in the definition of $\phi$ and $\psi$, we can write:

\[ C_{\phi} = C_{\phi} \otimes C_{\psi}. \tag{3.1} \]

Indeed, write $z = (z_1, w)$ with $z_1 \in \mathbb{D}$ and $w \in \mathbb{D}^{d-1}$. If $f \in H^2(\mathbb{D})$ and $g \in H^2(\mathbb{D}^{d-1})$, we see that:

\[
C_{\phi}(f \otimes g)(z) = (f \otimes g)(\phi(z_1), \psi(w)) = f(\phi(z_1)) g(\psi(w))
= [C_{\phi} f(z_1)] [C_{\psi} g(w)] = (C_{\phi} f \otimes C_{\psi} g)(z).
\]

Since tensor products $f \otimes g$ generate $H^2(\mathbb{D}) = H^2(\mathbb{D}) \otimes H^2(\mathbb{D}^{d-1})$, this proves (3.1).

Let now $m$ be a large positive integer. Set $[\cdot]$ stands for the integer part:

\[ n_m = [m \varepsilon_m]^{d-1} \quad \text{and} \quad N_m = m n_m. \tag{3.2} \]

From what we know in dimension $d - 1$ (see [2], Theorem 3.1) and from the preceding, we can write (observe that $\psi$ has to be truly $(d-1)$-dimensional since $\Phi$ is truly $d$-dimensional):

\[
a_m(C_{\phi}) \geq \exp(-m \varepsilon_m) \quad \text{and} \quad a_n(C_{\psi}) \geq a \exp(-C n^{1/(d-1)}),
\]

for some positive constant $C$, which will be allowed to vary from one formula to another. Lemma 3.2 implies:

\[
a_{N_m}(C_{\phi}) \geq a \exp[-C (m \varepsilon_m + n_m^{1/(d-1)})].
\]

Since $n_m \lesssim (m \varepsilon_m)^{d-1}$, we get:

\[
a_{N_m}(C_{\phi}) \geq a \exp(-C m \varepsilon_m).
\]

Observe that $N_m = m n_m \sim m^d \varepsilon_m^{d-1}$ and so $N_m^{1/d} \sim m \varepsilon_m^{1-1/d}$. As a consequence:

\[
a_{N_m}(C_{\phi}) \geq a \exp(-C m \varepsilon_m) = a \exp \left[ -(C \varepsilon_m^{1/d}) (m \varepsilon_m^{1-1/d}) \right]
\geq a \exp(-\eta_m N_m^{1/d})
\]

with $\eta_m := C \varepsilon_m^{1/d}$.

Now, for $N > N_1$, let $m$ be the smallest integer satisfying $N_m \geq N$ (so that $N_m - 1 < N \leq N_m$), and set $\delta_N = \eta_m$. We have $\lim_{N \to \infty} \delta_N = 0$. Next, we note that $\lim_{m \to \infty} N_m/N_m - 1 = 1$, because $N_m \geq N_m - 1$ and:

\[
\frac{N_m}{N_m - 1} \leq \frac{m}{m - 1} \left( \frac{m \varepsilon_m + 1}{(m - 1) \varepsilon_m - 1} \right)^{d-1} \sim \left( \frac{\varepsilon_m}{\varepsilon_m - 1} \right)^{d-1} \leq 1.
\]

Finally, if $N$ is an arbitrary integer and $N_m - 1 < N \leq N_m$, we obtain:

\[
a_N(C_{\phi}) \geq a_{N_m}(C_{\phi}) \geq a \exp(-\eta_m N_m^{1/d}) \geq a \exp(-C \delta_N N^{1/d}),
\]

since we observed that $\lim_{m \to \infty} N_m/N_m - 1 = 1$.

This amounts to say that $\beta_d(C_{\phi}) = 1$. \qed
Proof of Lemma 3.2. It is rather formal. Start from the Schmidt decompositions of $S$ and $T$ respectively (recall that Hilbert spaces, the approximation numbers are equal to the singular ones):

$$S = \sum_{m=1}^{\infty} a_m(S) u_m \odot v_m, \quad T = \sum_{n=1}^{\infty} a_n(T) u'_n \odot v'_n,$$

where $(u_m), (v_m)$ are two orthonormal sequences of $H_1$, $(u'_n), (v'_n)$ two orthonormal sequences of $H_2$, and $u_m \odot v_m$ and $u'_n \odot v'_n$ denote the rank one operators defined by $(u_m \odot v_m)(x) = \langle x, v_m \rangle u_m, \ x \in H_1$, and $(u'_n \odot v'_n)(x) = \langle x, v'_n \rangle u'_n, \ x \in H_2$.

We clearly have:

$$(u_m \odot v_m) \otimes (u'_n \odot v'_n) = (u_m \otimes u'_n) \odot (v_m \otimes v'_n),$$

so that the Schmidt decomposition of $S \otimes T$ is (with SOT-convergence):

$$S \otimes T = \sum_{m,n \geq 1} a_m(S) a_n(T) (u_m \otimes u'_n) \odot (v_m \otimes v'_n),$$

since the two sequences $(u_m \otimes u'_n)_{m,n}$ and $(v_m \otimes v'_n)_{m,n}$ are orthonormal: for instance, we have by definition:

$$\langle u_{m_1} \otimes u'_{n_1}, u_{m_2} \otimes u'_{n_2} \rangle = \langle u_{m_1} , u_{m_2} \rangle \langle u'_{n_1} , u'_{n_2} \rangle.$$

This shows that the singular values of $S \otimes T$ are the non-increasing rearrangement of the positive numbers $a_m(S) a_n(T)$ and ends the proof of the lemma: the $mn$ numbers $a_k(S) a_l(T), \ 1 \leq k \leq m, \ 1 \leq l \leq n$ all satisfy $a_k(S) a_l(T) \geq a_m(S) a_n(T)$, so that $a_{mn}(S \otimes T) \geq a_m(S) a_n(T)$. \hfill \qed

4 The glued case

Here we consider symbols of the form:

$$(4.1) \quad \Phi(z_1, z_2) = (\phi(z_1), \phi(z_1)),$$

where $\phi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic map.

Note that such maps $\Phi$ are not truly 2-dimensional.

4.1 Preliminary

We begin by remarking the following fact.

Let $B^2(\mathbb{D})$ be the Bergman space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{B^2}^2 := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where $dA$ is the normalized area measure on $\mathbb{D}$. 

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Proposition 4.1. Assume that the composition operator $C_\phi$ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$. Then $C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$, defined by (4.1), is bounded.

Proof. If we write $f \in H^2(\mathbb{D}^2)$ as:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k,$$

we formally (or assuming that $f$ is a polynomial) have:

$$[C_\Phi f](z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\phi(z_2)]^k = \sum_{n=0}^\infty \left( \sum_{j+k=n} c_{j,k} \right) [\phi(z_1)]^n.$$

Hence, if we set $g(z) = \sum_{n=0}^\infty \left( \sum_{j+k=n} c_{j,k} \right) z^n$, we get:

$$[C_\Phi f](z_1, z_2) = [C_\Phi g](z_1),$$

so that, by integrating:

$$\|C_\Phi f\|_{H^2(\mathbb{D}^2)} = \|C_\Phi g\|_{H^2(\mathbb{D})}.$$ 

By hypothesis, there is a positive constant $M$ such that:

$$\|C_\Phi g\|_{H^2(\mathbb{D})} \leq M \|g\|_{B^2(\mathbb{D})}.$$ 

But, by the Cauchy-Schwarz inequality:

$$\|g\|_{B^2(\mathbb{D})}^2 = \sum_{n=0}^\infty \frac{1}{n+1} \left| \sum_{j+k=n} c_{j,k} \right|^2 \leq \sum_{n=0}^\infty \left( \sum_{j+k=n} |c_{j,k}|^2 \right) = \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2(\mathbb{D}^2)}^2,$$

and we obtain $\|C_\Phi f\|_{H^2(\mathbb{D}^2)} \leq M \|f\|_{H^2(\mathbb{D}^2)}$. \hfill \Box

4.2 Lens maps

Let $\lambda_\theta$ be a lens map of parameter $\theta$, $0 < \theta < 1$. We consider $\Phi_\theta : \mathbb{D}^2 \to \mathbb{D}^2$ defined by:

$$(4.2) \quad \Phi_\theta(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_2)).$$

We have the following result.

Theorem 4.2. The composition operator $C_{\Phi_\theta} : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is:

1) not bounded for $\theta > 1/2$;
2) bounded, but not compact for $\theta = 1/2$;
3) compact, and even Hilbert-Schmidt, for $0 < \theta < 1/2$. 

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Proof. The reproducing kernel of $H^2(\mathbb{D}^2)$ is, for $(a,b) \in \mathbb{D}^2$:

$$K_{a,b}(z_1, z_2) = \frac{1}{1 - a \bar{z}_1} \frac{1}{1 - b \bar{z}_2}, \quad (z_1, z_2) \in \mathbb{D}^2,$$

and:

$$\|K_{a,b}\|^2 = \frac{1}{(1 - |a|^2)(1 - |b|^2)};$$

1) If $C_{\Phi_{\theta}}$ were bounded, we should have, for some $M < \infty$:

$$\|C_{\Phi_{\theta}}(K_{a,b})\|_{H^2} \leq M \|K_{a,b}\|_{H^2}, \quad \text{for all } a,b \in \mathbb{D}.$$  

Since $C_{\Phi_{\theta}}(K_{a,b}) = K_{\lambda_{\theta}(a), \lambda_{\theta}(b)}$, we would get, with $b = 0$:

$$\left(\frac{1}{1 - |\lambda_{\theta}(a)|^2}\right)^2 \leq M^2 \frac{1}{1 - |a|^2};$$

but this is not possible for $\theta > 1/2$, since $1 - |\lambda_{\theta}(a)|^2 \approx 1 - |\lambda_{\theta}(a)| \sim (1 - a)^{\theta}$ when $a$ goes to 1, with $0 < a < 1$.

For 2) and 3), let us consider the pull-back measure $m_{\theta}$ of the normalized Lebesgue measure on $T = \partial \mathbb{D}$ by $\lambda_{\theta}$. It is easy to see that:

$$\sup_{\xi \in T} m_{\theta}(D(\xi, h) \cap \mathbb{D}) = m_{\theta}(D(1, h) \cap \mathbb{D}) \approx h^{1/\theta}.$$  

In particular, for $\theta \leq 1/2$, $m_{\theta}$ is a 2-Carleson measure, and hence (see [15], Theorem 2.1, for example) the canonical injection $j : B^2(\mathbb{D}) \rightarrow L^2(m_{\theta})$ is bounded, meaning that, for some positive constant $M < \infty$:

$$\int_{\mathbb{D}} |f(z)|^2 \, dm_{\theta}(z) \leq M^2 \|f\|^2_{H^2}.$$  

Since

$$\int_{\mathbb{D}} |f(z)|^2 \, dm_{\theta}(z) = \int_T |f[\lambda_{\theta}(u)]|^2 \, dm(u) = \|C_{\lambda_{\theta}}(f)\|^2_{H^2},$$

we get that $C_{\lambda_{\theta}}$ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$.

It follows from Proposition 4.1 that $C_{\Phi_{\theta}} : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ is bounded.

However, $C_{\Phi_{\lambda_{1/2}}}$ is not compact since $C_{\Phi_{\lambda_{1/2}}}(K_{a,b})/\|K_{a,b}\|$ does not converge to 0 as $a, b \rightarrow 1$, by the calculations made in 1).

For 3), let $e_{j,k}(z_1, z_2) = z_1^j z_2^k$, $j, k \geq 0$, be the canonical orthonormal basis of $H^2(\mathbb{D}^2)$; we have $|C_{\Phi_{\theta}}(e_{j,k})(z_1, z_2)| = |\lambda_{\theta}(z_1)|^{2+j+k}$. Hence:

$$\sum_{j,k \geq 0} \|C_{\Phi_{\theta}}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \leq \sum_{n=0}^{\infty} (2n+1) \int_T |\lambda_{\theta}|^{2n} \, dm \leq \int_T \frac{2}{(1 - |\lambda_{\theta}|^2)^2} \, dm.$$  

Since, by Lemma 4.3 below, $1 - |\lambda_{\theta}(e^{it})|^2 \geq |1 - e^{it}|^\theta \geq t^\theta$ for $|t| \leq \pi/2$, we get:

$$\sum_{j,k \geq 0} \|C_{\Phi_{\theta}}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \lesssim \int_0^{\pi/2} \frac{dt}{t^{2\theta}} < \infty,$$

since $\theta < 1/2$. Therefore $C_{\Phi_{\theta}}$ is Hilbert-Schmidt for $\theta < 1/2$.  

\qed
For sake of completeness, we recall the following elementary fact (see [26], p. 28, or also [16], Lemma 2.5).

**Lemma 4.3.** With \( \delta = \cos(\theta \pi / 2) \), we have, for \( |z| \leq 1 \) and \( \Re z \geq 0 \):

\[
1 - |\lambda_\theta(z)|^2 \geq \frac{\delta}{2} |1 - z|^\theta.
\]

**Proof.** We can write:

\[
\lambda_\theta(z) = \frac{1 - w}{1 + w} \quad \text{with} \quad w = \left(\frac{1 - z}{1 + z}\right)^\theta \quad \text{and} \quad |w| \leq 1.
\]

Then:

\[
\Re w \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta.
\]

Hence:

\[
1 - |\lambda_\theta(z)|^2 = \frac{4 \Re w}{|1 + w|^2} \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta,
\]

as announced \( \square \).

We now improve the result 3) of Theorem 4.2 by estimating the approximation numbers of \( C_{\Phi_\theta} \) and get that \( C_{\Phi_\theta} \) is in all Schatten classes of \( H^2(D^2) \) when \( \theta < 1/2 \).

**Theorem 4.4.** For \( 0 < \theta < 1/2 \), there exists \( b = b_\theta > 0 \) such that:

\[
a_n(C_{\Phi_\theta}) \lesssim e^{-b\sqrt{n}}.
\]

In particular \( \beta^+_1(C_{\Phi_\theta}) \leq e^{-b} < 1 \), though \( \|\Phi_\theta\|_\infty = 1 \), and even \( \Phi_\theta(T^2) \cap T^2 \neq \emptyset \).

**Proof.** Proposition 4.1 (and its proof) can be rephrased in the following way: if \( C_\phi \) maps boundedly \( B^2(D) \) into \( H^2(D) \), then, we have the following factorization:

\[
(4.6) \quad C_\phi : H^2(D^2) \overset{J}{\rightarrow} B^2(D^2) \overset{C_\phi}{\longrightarrow} H^2(D) \overset{I}{\rightarrow} H^2(D^2),
\]

where \( I : H^2(D) \rightarrow H^2(D^2) \) is the canonical injection given by \( (If)(z_1, z_2) = f(z_1) \) for \( f \in H^2(D) \), and \( J : H^2(D^2) \rightarrow B^2(D) \) is the contractive map defined by:

\[
(Jf)(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n,
\]

for \( f \in H^2(D^2) \) with \( f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k \).

In the proof of Theorem 4.2, we have seen that, for \( 0 < \theta \leq 1/2 \), the composition operator \( C_{\lambda_\theta} \) is bounded from \( B^2(D) \) into \( H^2(D) \); we get hence the factorization:

\[
C_{\Phi_\theta} : H^2(D^2) \overset{J}{\rightarrow} B^2(D^2) \overset{C_{\Phi_\theta}}{\longrightarrow} H^2(D) \overset{I}{\rightarrow} H^2(D^2),
\]

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Now, the lens maps have a semi-group property:

$$\lambda_{\theta_1 \theta_2} = \lambda_{\theta_1} \lambda_{\theta_2},$$

(4.7)

giving

$$C_{\lambda_{\theta_1 \theta_2}} = C_{\lambda_{\theta_1}} \circ C_{\lambda_{\theta_2}}.$$

For $0 < \theta < 1/2$, we therefore can write $C_{\lambda_\theta} = C_{\lambda_{2\theta}} \circ C_{\lambda_{1/2}}$ (note that $2\theta < 1$, so $C_{\lambda_{2\theta}} : H^2(D) \to H^2(D)$ is bounded), and we get:

$$C_{\Phi_\theta} = I C_{\lambda_{2\theta}} C_{\lambda_{1/2}} J.$$

Consequently:

$$a_n(C_{\Phi_\theta}) \leq \|I\| \|J\| \|C_{\lambda_{1/2}}\|_{B^2 \to H^2} a_n(C_{\lambda_{2\theta}}).$$

Now, we know ([16], Theorem 2.1) that $a_n(C_{\lambda_{2\theta}}) \lesssim e^{-b_0 \sqrt{n}}$, so we get that $a_n(C_{\Phi_\theta}) \lesssim e^{-b_0 \sqrt{n}}$.

Remark. In [2], we saw that for a truly 2-dimensional symbol $\Phi$, we have $\beta_2(C_{\Phi}) > 0$. Here the symbol $\Phi_\theta$ is not truly 2-dimensional, but we nevertheless have $\beta_2(C_{\Phi_\theta}) > 0$. In fact, let $E = \{ f \in H^2(D^2) ; \frac{\partial f}{\partial z_2} \equiv 0 \}; E$ is isometrically isomorphic to $H^2(D)$ and the restriction of $C_{\Phi_\theta}$ to $E$ behaves as the 1-dimensional composition operator $C_{\lambda_\theta} : H^2(D) \to H^2(D)$; hence ([19], Proposition 6.3):

$$e^{-b_0 \sqrt{n}} \lesssim a_n(C_{\lambda_\theta}) = a_n(C_{\Phi_\theta}|_E) \leq a_n(C_{\Phi_\theta}),$$

and $\beta_2(C_{\Phi_\theta}) \geq e^{-b_0} > 0$.

5 Triangularly separated variables

In this section, we consider symbols of the form:

$$\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2),$$

(5.1)

where $\phi, \psi : D \to D$ are non-constant analytic maps.

Such maps $\Phi$ are truly 2-dimensional.

More generally, if $h \in H^\infty$, with $h(0) = 0$ and $\|h\|_\infty \leq 1$, has its powers $h^k, k \geq 0$, orthogonal in $H^2$ (for convenience, we shall say that $h$ is a Rudin function), we can consider:

$$\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$$

(5.2)

For such $h$ we can take for example an inner function vanishing at the origin, but there are other such functions, as shown by C. Bishop:
\textbf{Theorem} (Bishop [4]). \textit{The function } $h$ \textit{is a Rudin function if and only if the pull-back measure } $\mu = \mu_h$ \textit{is radial and Jensen, i.e for every Borel set } $E$:\

$$\mu(e^{i\theta} E) = \mu(E) \quad \text{and} \quad \int \log(1/|z|) \, d\mu(z) < \infty.$$

Conversely, for every probability measure $\mu$ supported by $\overline{\mathbb{D}}$, which is radial and Jensen, there exists $h$ in the unit ball of $H^\infty$, with $h(0) = 0$, such that $\mu = \mu_h$.

If we take for $\mu$ the Lebesgue measure of $T$, we get an inner function. But, as remarked in [4], we can take for $\mu$ the Lebesgue measure on the union $T \cup (1/2)T$, normalized in order that $\mu(T) = \mu((1/2)T) = 1/2$. Then the corresponding $h$ is not inner since $|h| = 1/2$ on a subset of $T$ of positive measure. He also showed that $h(z)/z$ may be a non-constant outer function. Also, P. Bourdon ([6]) showed that the powers of $h$ are orthogonal if and only if its Nevanlinna counting function is almost everywhere constant on each circle centered on the origin.

### 5.1 General facts

We first observe that if $f \in H^2(\mathbb{D}^2)$ and:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k,$$

then we can write:

$$f(z_1, z_2) = \left( \sum_{k \geq 0} f_k(z_1) \right) z_2^k$$

with:

$$f_k(z_1) = \sum_{j \geq 0} c_{j,k} z_1^j,$$

and:

$$\|f\|^2_{H^2(\mathbb{D}^2)} = \sum_{j,k \geq 0} |c_{j,k}|^2 = \sum_{k \geq 0} \|f_k\|^2_{H^2(\mathbb{D})}.$$  

That means that we have an isometric isomorphism:

$$J : H^2(\mathbb{D}^2) \longrightarrow \bigoplus_{k=0}^{\infty} H^2(\mathbb{D}),$$

defined by $Jf = (f_k)_{k \geq 0}$.

Now, for symbols $\Phi$ as in (5.1), we have:

$$(C_{\Phi} f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\psi(z_1)]^k z_2^k,$$
so that \( JC \Phi J^{-1} \) appears as the operator \( \bigoplus_k M_{\psi^k} C_\phi \) on \( \bigoplus_k H^2(\mathbb{D}) \), where \( M_{\psi^k} \) is the multiplication operator by \( \psi^k \):

\[
[(M_{\psi^k} C_\phi) f_k](z_1) = \left[ \psi(z_1) \right]^k \left[ (f_k \circ \phi)(z_1) \right].
\]

When \( \Phi \) is as in (5.2), we have:

\[
(C \Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} \left[ \phi(z_1) \right]^j \left[ \psi(z_2) \right]^k [h(z_2)]^k,
\]

with:

\[
\|C \Phi f\|^2 \leq \sum_{k=0}^{\infty} \|T_k f_k\|^2
\]
and:

\[
T_k = M_{\psi^k} C_\phi;
\]

hence \( JC \Phi J^{-1} \) appears as pointwise dominated by the operator \( T = \bigoplus_k T_k \) on \( \bigoplus_k H^2(\mathbb{D}) \). This implies a factorization \( C \Phi = A T \) with \( \|A\| \leq 1 \), so that \( a_n(C \Phi) \leq a_n(T) \) for all \( n \geq 1 \).

We recall the following elementary fact.

**Lemma 5.1.** Let \( (H_k)_{k \geq 0} \) be a sequence of Hilbert spaces and \( T_k : H_k \to H_k \) be bounded operators. Let \( H = \bigoplus_k H_k \) and \( T : H \to H \) defined by \( Tx = (T_k x_k)_k \). Then:

1) \( T \) is bounded on \( H \) if and only if \( \sup_k \|T_k\| < \infty \);
2) \( T \) is compact on \( H \) if and only if each \( T_k \) is compact and \( \|T_k\| \to 0 \) as \( k \to \infty \).

Going back to the symbols of the form (5.1), we have \( \|M_{\psi^k}\| \leq \|\psi^k\|_{\infty} \leq 1 \), since \( \|\psi\|_{\infty} \leq 1 \); hence \( \|M_{\psi^k} C_\phi\| \leq \|C_\phi\| \) and the operator \( (M_{\psi^k} C_\phi)_k \) is bounded on \( \bigoplus_k H^2(\mathbb{D}) \). Therefore \( C \Phi \) is bounded on \( H^2(\mathbb{D}^2) \).

For approximation numbers, we have the following two facts.

**Lemma 5.2.** Let \( T_k : H_k \to H_k \) be bounded linear operators between Hilbert spaces \( H_k \), \( k \geq 0 \). Let \( H = \bigoplus_k H_k \) and \( T = (T_k)_k : H \to H \), assumed to be compact. Then, for every \( n_1, \ldots, n_K \geq 1 \), and \( 0 \leq m_1 < \cdots < m_K, K \geq 1 \), we have:

\[
a_N(T) \geq \inf_{1 \leq k \leq K} a_{n_k}(T_{m_k}),
\]

where \( N = n_1 + \cdots + n_K \).

**Proof.** We use the Bernstein numbers \( b_n \) (see (1.4)), which are equal to the approximation numbers (see (1.7)).

For \( k = 1, \ldots, K \), there is an \( n_k \)-dimensional subspace \( E_k \) of \( H_{m_k} \) such that:

\[
b_{n_k}(T_{m_k}) \leq \|T_{m_k} x\|, \quad \text{for all } x \in S_{E_k}. \]
Then \( E = \bigoplus_{k=1}^{K} E_k \) is an \( N \)-dimensional subspace of \( H \) and for every \( x = (x_1, x_2, \ldots) \in E \), we have:

\[
\|Tx\|^2 = \sum_{k \leq K} \|T_{mk}x_{mk}\|^2 \geq \sum_{k \leq K} [b_{nk}(T_{mk})]^2 \|x_{mk}\|^2
\]

\[
\geq \inf_{k \leq K} [b_{nk}(T_{mk})]^2 \sum_{k \leq K} \|x_{mk}\|^2 = \inf_{k \leq K} [b_{nk}(T_{mk})]^2 \|x\|^2 ;
\]

hence \( b_N(T) \geq \inf_{k \leq K} b_{nk}(T_{mk}) \), and we get the announced result. \( \square \)

**Lemma 5.3.** Let \( T = \bigoplus_{k=0}^{\infty} T_k \) acting on a Hilbertian sum \( H = \bigoplus_{k=0}^{\infty} H_k \). Let \( n_0, \ldots, n_K \) be positive integers and \( N = n_0 + \cdots + n_K - K \). Then, the approximation numbers of \( T \) satisfy:

\[
a_N(T) \leq \max \left( \max_{0 \leq k \leq K} a_{nk}(T_k), \sup_{k > K} \|T_k\| \right).
\]

**Proof.** Denote by \( S \) the right-hand side of (5.4). Let \( R_k, 0 \leq k \leq K \) be operators on \( H_k \) of respective rank \( < n_k \) such that \( \|T_k - R_k\| = a_{nk}(T_k) \) and let \( R = \bigoplus_{k=0}^{K} R_k \). Then \( R \) is an operator of rank \( \leq n_0 + \cdots + n_K - K - 1 < N \). If \( f = \sum_{k=0}^{\infty} f_k \in H \), we see that:

\[
\|Tf - Rf\|^2 = \sum_{k=0}^{K} \|T_k f_k - R_k f_k\|^2 + \sum_{k > K} \|T_k f_k\|^2
\]

\[
\leq \sum_{k=0}^{K} a_{nk}(T_k)^2 \|f_k\|^2 + \sum_{k > K} \|T_k f_k\|^2 \leq S^2 \sum_{k=0}^{\infty} \|f_k\|^2 = S^2 \|f\|^2 ,
\]

hence the result. \( \square \)

We give now two corollaries of Lemma 5.3.

**Example 1.** We first use lens maps. We get:

**Theorem 5.4.** Let \( \lambda_0 \) the lens map of parameter \( \theta \) and let \( \psi : D \to D \) such that \( \|
\psi\|_{\infty} := c < 1 \) and \( h \) a Rudin function. We consider:

\[
\Phi(z_1, z_2) = (\lambda_0(z_1), \psi(z_1) h(z_2)).
\]

Then, for some positive constant \( \beta \), we have, for all \( N \geq 1 \):

\[
a_N(C_{\Phi}) \lesssim e^{-\beta N^{1/3}}.
\]

**Proof.** Let \( T_k = M_{\psi^k} C_{\lambda_0} \). We have \( \|T_k\| \leq c^k \), so \( \sup_{k > K} \|T_k\| \leq c^K \). On the other hand, we have \( a_n(T_k) \leq c^k a_n(C_{\lambda_0}) \leq a_n(C_{\lambda_0}) \lesssim e^{-\beta \sqrt{n}} \) ([16], Theorem 2.1). Taking \( n_0 = n_1 = \cdots = n_K = K^2 \) in Lemma 5.3, we get:

\[
\max_{0 \leq k \leq K} a_{nk}(T_k) \lesssim e^{-\beta K}.
\]

Since \( n_0 + \cdots + n_K - K \approx K^2 \), we obtain \( a_K \lesssim e^{-\beta K} \), which gives the claimed result, by taking \( \beta = \max(\beta_0, \log(1/c)) \). \( \square \)
Example 2. We consider the cusp map $\chi$. We have:

**Theorem 5.5.** Let $\chi$ be the cusp map, $h$ a Rudin function, and $\psi$ in the unit ball of $H^\infty$, with $\|\psi\|_\infty := c < 1$. Let:

$$\Phi(z_1, z_2) = (\chi(z_1), \psi(z_1) h(z_2)).$$

Then, for positive constant $\beta$, we have, for all $N \geq 1$:

$$a_N(C_\Phi) \lesssim e^{-\beta \sqrt{N}/\log N}.\]$$

**Proof.** Let $T_k = M_\psi C_\chi$. As above, we have $\sup_{k>1} \|T_k\| \leq c^K$. For the cusp map, we have $a_n(C_\chi) \lesssim e^{-\alpha n/\log n}$ ([20], Theorem 4.3); hence $a_n(T_k) \lesssim e^{-\alpha n/\log n}$. We take $n_0 = n_1 = \cdots = n_K = K \lceil \log K \rceil$ (where $\lceil \log K \rceil$ is the integer part of $\log K$). Since $n_0 + \cdots + n_K \approx K^2 \lceil \log K \rceil$, we get, for another $\alpha > 0$:

$$a_{K^2 \lceil \log K \rceil}(C_\Phi) \lesssim e^{-\alpha K},$$

which reads: $a_N(C_\Phi) \lesssim e^{-\beta \sqrt{N}/\log N}$, as claimed. \qed

### 5.2 Lower bounds

In this subsection, we give lower bounds for approximation numbers of composition operators on $H^2$ of the bidisk, attached to a symbol $\Phi$ of the previous form $\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$ where $h$ is a Rudin function. The sharpness of those estimates will be discussed in the next subsection. We first need some lemmas in dimension one.

**Lemma 5.6.** Let $u, v: \mathbb{D} \to \mathbb{D}$ be two non-constant analytic self-maps and $T = M_v C_u: H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the associated weighted composition operator. For $0 < r < 1$, we set $A = u(r \mathbb{D})$ and $\Gamma = \exp \left( -\frac{1}{\text{Cap}(A)} \right)$. Then, for $0 < \delta \leq \inf_{|z|=r} |v(z)|$, we have:

$$a_n(T) \gtrsim \sqrt{1 - r \delta} \Gamma^n.$$

In this lemma, $\text{Cap}(A)$ denotes the Green capacity of the compact subset $A \subseteq \mathbb{D}$ (see [21], § 2.3 for the definition).

For the proof, we need the following result ([27], Theorem 7, p. 353).

**Theorem 5.7 (Widom).** Let $A$ be a compact subset of $\mathbb{D}$ and $C(A)$ be the space of continuous functions on $A$ with its natural norm. Set:

$$\tilde{d}_n(A) = \inf E \sup_{f \in B_{H^\infty}} \text{dist}(f, E),$$

where $E$ runs over all $(n-1)$-dimensional subspaces of $C(A)$ and $\text{dist}(f, E) = \inf_{h \in E} \|f - h\|_{C(A)}$. Then

$$\tilde{d}_n(A) \geq \alpha e^{-n/\text{Cap}(A)}$$

for some positive constant $\alpha$. 

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Proof of Lemma 5.6. We apply Theorem 5.7 to the compact set \( A = u(r \overline{D}) \).

Let \( E \) be an \((n - 1)\)-dimensional subspace of \( H^2 = H^2(\mathbb{D}) \); it can be viewed as a subspace of \( \mathcal{C}(A) \), so, by Theorem 5.7, there exists \( f \in H^\infty \subseteq H^2 \) with \( \|f\|_2 \leq \|f\|_\infty \leq 1 \) such that:

\[
\|f - h\|_{\mathcal{C}(A)} \geq \alpha \Gamma^n, \quad \forall h \in E.
\]

Then:

\[
\|v(f \circ u - h \circ u)\|_{\mathcal{C}(rT)} \geq \delta \|f - h\|_{\mathcal{C}(rT)} = \delta \|f - h\|_{\mathcal{C}(A)} \geq \alpha \delta \Gamma^n.
\]

But:

\[
\|v(f \circ u - h \circ u)\|_{\mathcal{C}(rT)} \leq \frac{1}{\sqrt{1 - r^2}} \|v(f \circ u - h \circ u)\|_{H^2};
\]

Hence:

\[
\|Tf - Th\|_{H^2} \geq \alpha \sqrt{1 - r^2} \delta \Gamma^n \geq \alpha \sqrt{1 - r} \delta \Gamma^n.
\]

Since \( h \) is an arbitrary function of \( E \), we get (\( B_{H^2} \) being the unit ball of \( H^2 \)):

\[
\inf_{\dim E < n} \left[ \sup_{f \in B_{H^2}} \text{dist}(Tf, T(E)) \right] \geq \alpha \sqrt{1 - r} \delta \Gamma^n.
\]

But the left-hand side is equal to the Kolmogorov number \( d_n(T) \) of \( T \) (see [21], Lemma 3.12), and, as recalled in (1.7), in Hilbert spaces, the Kolmogorov numbers are equal to the approximation numbers; hence we obtain:

(5.8) \[
a_n(T) \geq \alpha \sqrt{1 - r} \delta \Gamma^n, \quad n = 1, 2, \ldots,
\]
as announced.

The next lemma shows that some Blaschke products are far away from 0 on some circles centered at 0.

We consider a strongly interpolating sequence \((z_j)_{j \geq 1}\) of \( \mathbb{D} \) in the sense that, if \( \varepsilon_j := 1 - |z_j| \), then:

(5.9) \[
\varepsilon_{j+1} \leq \sigma \varepsilon_j
\]
and so \( \varepsilon_j \leq \sigma^{j-1} \varepsilon_1 \), where \( 0 < \sigma < 1 \) is fixed. Equivalently, the sequence \((|z_j|)_{j \geq 1}\) is interpolating. We consider the corresponding interpolating Blaschke product:

(5.10) \[
B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j}z}.
\]

The following lemma is probably well-known, but we could find no satisfactory reference (see yet [10] for related estimates) and provide a simple proof.
Lemma 5.8. Let \((z_j)_{j \geq 1}\) be a strongly interpolating sequence as in (5.9) and \(B\) the associated Blaschke product (5.10).

Then there exists a sequence \(r_l := 1 - \rho_l\) such that:

\[
C_1 \sigma^l \leq \rho_l \leq C_2 \sigma^l,
\]

where \(C_1, C_2\) are positive constants, and for which:

\[
|z| = r_l \quad \implies \quad |B(z)| \geq \delta,
\]

where \(\delta > 0\) does not depend on \(l\).

Proof. Let us denote by \(p_l, 1 \leq p_l \leq l\), the biggest integer such that \(\varepsilon_{p_l} \geq \sigma \varepsilon_1 - 1 \varepsilon_1\).

We separate two cases.

Case 1: \(\varepsilon_{p_l} \geq 2 \sigma \varepsilon_1\).

Then, we choose \(\rho_l = \alpha \sigma \varepsilon_1\) with \(\alpha\) fixed, \(1 < \alpha < 2\). Since \(\rho(\xi, \zeta) \geq \rho(|\xi|, |\zeta|)\) for all \(\xi, \zeta \in \mathbb{D}\) (recall that \(\rho\) is the pseudo-hyperbolic distance on \(\mathbb{D}\)), we have the following lower bound for \(|z| = r_l\):

\[
|B(z)| = \prod_{j=1}^{\infty} \rho(z, z_j) \geq \prod_{j=1}^{\infty} \rho(r_l, |z_j|) = \prod_{j \leq p_l} \rho(r_l, |z_j|) \times \prod_{j > p_l} \rho(r_l, |z_j|) := P_1 \times P_2,
\]

and we estimate \(P_1\) and \(P_2\) separately.

We first observe that \(\frac{\rho_l}{\varepsilon_{p_l}} \leq \frac{\alpha \sigma \varepsilon_1}{2 \sigma \varepsilon_1} \leq \frac{\alpha}{2}\), and then:

\[
\frac{\rho_l}{\varepsilon_{p_l}} = \frac{\rho_l}{\varepsilon_{p_l}} \leq \frac{\alpha}{2} \sigma^{p_l - j}.
\]

The inequality \(\rho(1 - u, 1 - v) \geq \frac{|u - v|}{(u + v)}\) for \(0 < u, v \leq 1\) now gives us:

\[
\rho(r_l, |z_j|) \geq \frac{\varepsilon_j - \rho_l}{\varepsilon_j + \rho_l} = \frac{1 - \rho_l / \varepsilon_j}{1 + \rho_l / \varepsilon_j} \geq \frac{1 - (\alpha/2) \sigma^{p_l - j}}{1 + (\alpha/2) \sigma^{p_l - j}}, \quad \text{for } j \leq p_l,
\]

and:

\[
P_1 \geq \prod_{k=0}^{\infty} \left(1 - \frac{\alpha/2 \sigma^k}{1 + \alpha/2 \sigma^k}\right).
\]

Similarly:

\[
\frac{\varepsilon_{p_l+1}}{p_l} \leq \frac{\sigma^{p_l-1} \varepsilon_1}{\alpha \sigma^{p_l-1} \varepsilon_1} \leq \frac{1}{\alpha}
\]

and:

\[
\frac{\varepsilon_j}{p_l} \leq \frac{1}{\alpha} \sigma^{j - p_l - 1} \quad \text{for } j > p_l;
\]

so that:

\[
\rho(r_l, |z_j|) \geq \frac{\rho_l - \varepsilon_j}{\rho_l + \varepsilon_j} = \frac{1 - \varepsilon_j / p_l}{1 + \varepsilon_j / p_l} \geq \frac{1 - \alpha^{-1} \sigma^{j - p_l - 1}}{1 + \alpha^{-1} \sigma^{j - p_l - 1}}, \quad \text{for } j > p_l,
\]
and
\[ P_2 \geq \prod_{k=0}^{\infty} \left( \frac{1 - \alpha^{-1}\sigma^k}{1 + \alpha^{-1}\sigma^k} \right). \]

Finally, the condition of lower and upper bound for \( \rho_l \) is fulfilled by construction.

**Case 2:** \( \varepsilon_{pl} \leq 2\sigma^{l-1}\varepsilon_1 \).

Then, we choose \( \rho_l = a\varepsilon_{pl} \) with \( \sigma < a < 1 \) fixed. Computations exactly similar to those of Case 1 give us:

\[ |B(z)| \geq \prod_{k=0}^{\infty} \left( \frac{1 - a\sigma^k}{1 + a\sigma^k} \right) \times \prod_{k=0}^{\infty} \left( \frac{1 - \alpha^{-1}\sigma^k}{1 + \alpha^{-1}\sigma^k} \right) =: \delta > 0, \quad \text{for } |z| = r_l. \]

Moreover, in this case:

\[ a\sigma^{l-1}\varepsilon_1 \leq \rho_l \leq 2 a\sigma^{l-1}\varepsilon_1, \]

and the proof is ended. \( \square \)

Now, we have the following estimation.

**Theorem 5.9.** Let \( \phi, \psi : \mathbb{D} \to \mathbb{D} \) be two non-constant analytic self-maps and \( \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)) \), where \( h \) is inner.

Let \( (r_l)_{l \geq 1} \) be an increasing sequence of positive numbers with limit 1 such that:

\[ \inf_{|z| = r_l} |\psi(z)| \geq \delta_l > 0, \]

with \( \delta_l \leq e^{-1/\text{Cap}(A_l)} \), where \( A_l = \phi(r_l\mathbb{D}) \).

Then the approximation numbers \( a_N(C_{\Phi}), N \geq 1, \) of the composition operator \( C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \) satisfy:

\[ a_N(C_{\Phi}) \gtrsim \sup_{l \geq 1} \left[ \sqrt{1 - r_l} \exp \left( -8 \sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)} \right) \right], \]

where:

\[ \Gamma_l = e^{-1/\text{Cap}(A_l)}. \]

**Proof.** Since \( h \) is inner, the sequence \( (h^k)_{k \geq 0} \) is orthonormal in \( H^2 \) and hence \( a_n(C_{\Phi}) = a_n(T) \) for all \( n \geq 1, \) where \( T = \bigoplus_{k=0}^{\infty} T_k \) and \( T_k = M_{\psi_k} C_\phi. \) Then Lemma 5.6 gives:

\[ a_n(T_k) \gtrsim \sqrt{1 - r_l} \delta_l^k \Gamma_l^n \]

for all \( n \geq 1 \) and all \( k \geq 0. \)

Let now:

\[ p_l = \left[ \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)} \right], \]
where \([.\.\.)\) stands for the integer part, and:

\[
(5.22) \quad n_k = p_k k, \quad \text{for} \quad k = 1, \ldots, K.
\]

By Lemma 5.2, applied with \(m_k = k\) (i.e. to \(H_1, \ldots, H_K\)), we have, if \(N = n_1 + \cdots + n_K\):

\[
a_N(T) \geq \inf_{1 \leq k \leq K} \alpha \sqrt{1 - r_l} \delta_l^k \Gamma^k_l = \alpha \sqrt{1 - r_l} \delta_l^K \Gamma^m_l.
\]

But, since \(p_l \leq \log(1/\delta_l)/\log(1/\Gamma_l)\):

\[
\delta_l^K \Gamma^m_l = \exp\left[-(K \log(1/\delta_l) + p_l K \log(1/\Gamma_l))\right] \geq \exp[-2K \log(1/\delta_l)].
\]

Since:

\[
N = p_l K(K + 1)/2 \geq p_l K^2/4 \geq K^2 \log(1/\delta_l)/16 \log(1/\Gamma_l),
\]

we get:

\[
\delta_l^K \Gamma^m_l \geq \exp\left[-8 \sqrt{N \log(1/\delta_l)/\log(1/\Gamma_l)}\right],
\]

and the result ensues. \(\square\)

**Example 1.** We take \(\phi = \lambda_\theta\), a lens map, and \(\psi = B\), a Blaschke product associated to a strongly regular sequence, as defined in (5.10); then we get:

**Theorem 5.10.** Let \(\Phi: \mathbb{D}^2 \to \mathbb{D}^2\) be defined by:

\[
\Phi(z_1, z_2) = (\lambda_\theta(z_1), cB(z_1)h(z_2)),
\]

where \(B\) is a Blaschke product as in (5.10), \(0 < c < 1\), and \(h\) is an arbitrary inner function, we have, for some positive constant \(b\), for all \(N \geq 1\):

\[
(5.23) \quad a_N(C_\Phi) \gtrsim e^{-b N^{1/3}} = \exp(-b \sqrt{N}/N^{1/6}).
\]

In particular \(\beta_2(C_\Phi) = \beta_2^2(C_\Phi) = 1\).

**Remark.** We saw in Theorem 5.4 that this is the exact size, since we have:

\[
a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}.
\]

**Proof.** By Lemma 5.8, there is a sequence of numbers \(r_l \approx \sigma_l\) such that \(|B(z)| \geq \delta\) for \(|z| = r_l\), where \(\delta\) is a positive constant (depending on \(\sigma\)). Since \(\lambda_\theta(0) = 0\), we have:

\[
\text{diam}_\rho(A_l) \geq \lambda_\theta(r_l) \gtrsim 1 - (1 - r_l)^\theta;
\]

hence, by [21], Theorem 3.13, we have:

\[
\text{Cap}(A_l) \gtrsim \log \frac{1}{1 - r_l} \gtrsim l,
\]

or, equivalently: \(\Gamma_l \geq e^{-b/l}\), some some \(b > 0\). Then (5.18) gives, for all \(l \geq 1\) (with another \(b\)):

\[
a_N(C_\Phi) \gtrsim \exp\left[-b \left(l + \sqrt{N}/l\right)\right].
\]

Taking \(l = N^{1/3}\), we get the result. \(\square\)
Example 2. By taking the cusp instead of a lens map, we obtain a better result, close to the extremal one.

Theorem 5.11. Let \( \Phi(z_1, z_2) = (\chi(z_1), cB(z_1)h(z_2)) \), where \( \chi \) is the cusp map, \( B \) a Blaschke product as in (5.10), \( 0 < c < 1 \), and \( h \) an arbitrary inner function. Then, for all \( N \geq 1 \):

\[
a_N(C_{\Phi}) \gtrsim e^{-b \sqrt{N}/\log N}.
\]

In particular \( \beta_2(C_{\Phi}) = 1 \).

Remark. We saw in Theorem 5.5 that this is the exact size, since we have:

\[
a_N(C_{\phi}) \lesssim e^{-\beta_1 \sqrt{N}/\log N}.
\]

Proof. The proof is the same as that of Proposition 5.10, except that, for the cusp map, we have (note that \( \chi(0) = 0 \)):

\[
diam(\rho(A_l)) \geq \chi(r_l).
\]

But when \( r \) goes to 1:

\[
1 - \chi(r) \sim \frac{\pi (\sqrt{2} - 1)}{2} \frac{1}{\log (1/(1-r))}
\]

(see [20], Lemma 4.2). Hence, by [21], Theorem 3.13, again, we have:

\[
\text{Cap}(A_l) \gtrsim \log \left( \log \left( 1/(1-r_l) \right) \right),
\]

so \( \Gamma_l \geq e^{-b/\log l} \). Then, (5.18) gives (with another \( b \)):

\[
a_N(C_{\Phi}) \gtrsim \exp \left[ -b \left( l + \frac{\sqrt{N}}{\log l} \right) \right].
\]

In taking \( l = \sqrt{N}/\log N \), we get the announced result. \( \square \)

5.3 Upper bounds

All previous results point in the direction that, if \( \|\Phi\|_{\infty} = 1 \), then however small \( a_n(C_{\Phi}) \) is, it will always be larger than \( \alpha e^{-\beta_1 \sqrt{n}} \) with \( \varepsilon_n \to 0^+ \), as this is the case in dimension one (with \( n \) instead of \( \sqrt{n} \)). But Theorem 5.12 to follow shows that we cannot hope, in full generality, to get the same result in dimension \( d \geq 2 \), and that other phenomena await to be understood. Here is our main result. It shows that, even for a truly 2-dimensional symbol \( \Phi \), we can have \( \|\Phi\|_{\infty} = 1 \) and nevertheless \( \beta_2^+(C_{\Phi}) < 1 \), in contrast to the 1-dimensional case where (1.1) holds.

Theorem 5.12. There exist a map \( \Phi: \mathbb{D}^2 \to \mathbb{D}^2 \) such that:

1) the composition operator \( C_{\Phi}: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \) is bounded and compact;

2) we have \( \|\Phi\|_{\infty} = 1 \) and \( \Phi \) is truly 2-dimensional, so that \( \beta_2^-(C_{\Phi}) > 0 \);

3) the singular numbers satisfy \( a_n(C_{\Phi}) \leq \alpha e^{-\beta \sqrt{n}} \) for some positive constants \( \alpha, \beta \); in particular \( \beta_2^+(C_{\Phi}) < 1 \).
Proof. Let $0 < \theta < 1$ be fixed, and $\lambda_\theta$ be the corresponding lens map. We set:

\[
\begin{align*}
\phi &= \frac{1 + \lambda_\theta}{2} \\
 w(z) &= \exp \left[ - \left( \frac{1 + z}{1 - z} \right)^\theta \right] \\
\psi &= w \circ \phi.
\end{align*}
\]

Note that $\|\phi\|_\infty = 1$.

Setting $\delta = \cos(\theta \pi / 2) > 0$, we have for $z \in \mathbb{D}$:

\[
|1 - \phi(z)| = \frac{1}{2} |1 - \lambda_\theta(z)| = \left| \frac{(1 - z)^\theta}{(1 - z)^\theta + (1 + z)^\theta} \right| \leq \frac{|1 - z|^\theta}{\delta}.
\]

Indeed, the argument $\alpha$ of $(1 \pm z)^\theta$ satisfies $|\alpha| \leq \theta \pi / 2$ for $z \in \mathbb{D}$, and we get:

\[
|(1 - z)^\theta + (1 + z)^\theta| \geq \Re [(1 - z)^\theta + (1 + z)^\theta] \geq \delta (|1 + z|^\theta + |1 - z|^\theta) \geq \delta.
\]

We also see that $\phi(\mathbb{D})$ touches the boundary $\partial \mathbb{D}$ only at $1$ in a non-tangential way, meaning that for some constant $C > 1$:

\[
1 - |\phi(z)| \geq \frac{1}{C} |1 - \phi(z)|, \quad \forall z \in \mathbb{D}.
\]

Now, we have the following two inequalities:

\[
\begin{align*}
\Re z \geq 0 & \implies |w(z)| \leq \exp \left( - \frac{\delta}{|1 - z|^\theta} \right) \\
 z \in \mathbb{D} & \implies |\psi(z)| \leq \exp \left( - \frac{\delta^2}{|1 - z|^{\theta^2}} \right).
\end{align*}
\]

Indeed, with $S(z) = \left( \frac{1 + z}{1 - z} \right)^\theta$, we have $\Re S(z) \geq \delta |S(z)| \geq \delta |1 - z|^{-\theta}$ when $\Re z \geq 0$, giving (5.25), and (5.24) and (5.25) imply, since $\Re \phi(z) \geq 0$:

\[
|\psi(z)| = |w(\phi(z))| \leq \exp \left( - \frac{\delta}{|1 - \phi(z)|^\theta} \right) \leq \exp \left( - \frac{\delta^2}{|1 - z|^{\theta^2}} \right).
\]

We now set:

\[
\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) \circ h(z_2)),
\]

with $h$ a Rudin function.

Observe that $\phi \in A(\mathbb{D})$ and $\psi = w \circ \phi \in A(\mathbb{D})$ as well ($w \in A(\mathbb{D})$ with $w(1) = 0$; this is due to the presence of the parameter $\theta < 1$). Hence if we take for $h$ a finite Blaschke product, the two components of $\Phi$ are in the bidisk algebra $A(\mathbb{D}^2)$.
We have $\|\psi\|_\infty := \rho < 1$. In fact, for $\Re u \geq 0$, we have:

$$\left|\frac{1+u}{1-u}\right| \geq 2^{-\theta}|1+u|^{\theta} \geq 2^{-\theta}(1+\Re u)^{\theta} \geq 2^{-\theta},$$

hence:

$$\Re \left[\left(\frac{1+u}{1-u}\right)^{\theta}\right] \geq \left(\frac{\cos \theta \pi}{2}\right) \left|\frac{1+u}{1-u}\right|^{\theta} \geq \left(\cos \frac{\theta \pi}{2}\right) 2^{-\theta} = \delta 2^{-\theta},$$

and $\|w \circ \phi\|_\infty \leq e^{2-\theta} \delta$.

Now, 1) follows from the orthogonal model presented in Section 5.1, because $\|\psi\|_\infty < 1$.

The assertion 2) follows from [2], Theorem 3.1, since $\|\phi\|_\infty = 1$.

We now prove 3).

As observed, $\Phi$ can be viewed as a direct sum $T = \bigoplus_{k=0}^{\infty} T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^{\infty} H_k$, where $T_k$ acts on a copy $H_k$ of $H^2(\mathbb{D})$ with:

$$T_k = M_{\psi_k} C_{\phi}.$$ 

We fix the positive integer $n$. The rest of the proof will consist of three lemmas.

**Lemma 5.13.** We have $\|T_k\| \leq 2 \rho^{-k} \leq 2 \rho^{-n}$ for $k > n$.

**Proof.** Indeed, since $\phi(0) = 1/2$, we know that $\|C_{\phi}\| \leq \sqrt{\frac{1+\phi(0)}{1-\phi(0)}} = \sqrt{3} \leq 2$, so that $\|T_k\| \leq \|\psi_k\|_\infty \|C_{\phi}\| \leq \rho^{-k} \times 2$. \hfill \Box

**Lemma 5.14.** Set $b = a/\delta^2$ where $a > 0$ is given by $e^{-a} = 4C/\sqrt{16C^2+1}$ and $C$ is as in (2.1). Let $m_k$ be the smallest integer such that $k \delta^2 2^{m_k} \theta^2 \geq an$; namely:

$$m_k = \left\lceil \frac{\log(b n/k)}{\theta^2 \log 2} \right\rceil + 1,$$

where $\lceil . \rceil$ stands for the integer part. Then, with $a' = \min(\log 2, a)$:

$$a_{nm_{k+1}}(T_k) \leq e^{-a' n}.$$ 

**Proof.** This follows from Theorem 2.3 applied with $w = \psi^k$, $R = k \delta^2$ and $\theta$ changed into $\theta^2$. This is possible thanks to (5.26) and to Lemma 5.13. Moreover we have adjusted $m_k$ so as to make the two terms in Theorem 2.3 of the same order. \hfill \Box

**Lemma 5.15.** The dimension $d := \sum_{k=0}^{n} m_k$ satisfies, for some positive constant $\alpha$:

$$d \leq \alpha n^2.$$
Proof. Indeed, it is well-known that:
\[\sum_{k=1}^{n} \log k = n \log n - n + O(\log n),\]
and, in view of (5.28), we have
\[m_k \leq \alpha_\theta' \log(bn/k) \leq \alpha_\theta''(\log n - \log k);\]
and we get
\[d \leq \alpha_\theta'' n^2 + O(n \log n) \leq \alpha_\theta n^2.\]

Alternatively, we could have used a Riemann sum for the function \(\log(1/x)\) on \((0,1].\)

Finally, putting things together and using as well Proposition 5.3 with
\[K = n\] and \(n_k = nm_k + 1\) so that \((\sum_{k=0}^{n} n_k) - n = (\sum_{k=0}^{n} m_k) + 1 = d + 1,\) we get
ignoring once more multiplicative constants:
\[a_{n+}(T) \lesssim a_d(T) \leq \alpha e^{-\beta n}\]
with positive constants \(\alpha, \beta.\) This ends the proof of Theorem 5.12. \(\-boxed{}\)

6 Monge-Ampère capacity and applications

6.1 Definition

Let \(K\) be a compact subset of \(\mathbb{D}^m\) (in this section, for notational reasons, we denote the dimension by \(m\) instead of \(d\)). The Monge-Ampère capacity of \(K\) has been defined by Bedford and Taylor ([3]; see also [13], § 5 or [11], Chapter 1) as:
\[\text{Cap}_m(K) = \sup \left\{ \int_K (dd^c u)^m ; u \in PSH \text{ and } 0 \leq u \leq 1 \right\},\]
where \(PSH\) is the set of plurisubharmonic functions on \(\mathbb{D}^m, dd^c = 2i\partial\bar{\partial},\) and \((dd^c)^m = dd^c \wedge \cdots \wedge dd^c\) \((m\) times). When \(u \in PSH \cap C^2(\mathbb{D}^m),\) we have:
\[(dd^c u)^m = 4^m m! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV(z),\]
where \(dV(z) = (i/2)^m dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m\) is the usual volume in \(\mathbb{C}^m.\)

A more convenient formula (because \(\mathbb{D}^m\) is bounded and hyperconvex: see [11], p. 80, for the definition) is:
\[\text{Cap}_m(K) = \int_K (dd^c u_K^*)^m,\]
where \(u_K^\ast\) is called the extremal function of \(K\) and is the upper semi-continuous regularization of:
\[u_K = \sup \{ v \in PSH ; v \leq 0 \text{ and } v \leq -1 \text{ on } K \},\]
but we will not need that.

As in [28], we set:

\begin{equation}
\tau_m(K) = \frac{1}{(2\pi)^m} \text{Cap}_m(K).
\end{equation}

For \( m = 1 \), \( \tau(K) := \tau_1(K) \) is equal to the Green capacity \( \text{Cap}(K) \) of \( K \) with respect to \( \mathbb{D} \), with the definition used in [21] (see [13], Theorem 8.1, where a factor \( 2\pi \) is introduced).

We further set:

\begin{equation}
\Gamma_m(K) = \exp \left[ -m! \tau_m(K) \right]^{1/m}.
\end{equation}

We proved in [21] that, for \( m = 1 \), and \( \varphi: \mathbb{D} \to r\mathbb{D} \), with \( 0 < r < 1 \), we have:

\begin{equation}
\beta_1(C\varphi) = \Gamma_1(\varphi(\mathbb{D})).
\end{equation}

The goal of this section is to see that Theorem 5.12 shows that this no longer holds for \( m = 2 \).

**6.2 A seminal example**

In one variable, our initial motivation had been the simple-minded example \( \varphi(z) = rz, 0 < r < 1 \), for which \( C\varphi(z^n) = r^n z^n \), implying \( a_n(C\varphi) = r^{n-1} \) and \( \beta_1(C\varphi) = r \). If \( K = \varphi(\mathbb{D}) = \overline{D}(0, r) \), we have \( \text{Cap}(K) = \frac{1}{\log(1/r)} \) and \( \Gamma_1(K) = r \), so that \( \beta_1(C\varphi) = \Gamma_1(K) \).

Let us examine the multivariate example (where \( 0 < r_j < 1 \)):

\( \Phi(z_1, z_2, \ldots, z_m) = (r_1z_1, r_2z_2, \ldots, r_mz_m) \).

If \( K = \Phi(\mathbb{D}^m) \), we have \( K = \prod_{k=1}^m \overline{D}(0, r_k) \), and hence ([5], Theorem 3):

\begin{equation}
\tau_m(K) = \prod_{k=1}^m \frac{1}{\log(1/r_k)}.
\end{equation}

On the other hand, \( C\Phi(z_1^{n_1}z_2^{n_2} \cdots z_m^{n_m}) = r_1^{n_1}r_2^{n_2} \cdots r_m^{n_m} \) \( \overline{D}(0, r_1^{n_1}r_2^{n_2} \cdots r_m^{n_m}) \) so that the sequence \( (a_n)_n \) of approximation numbers of \( C\Phi \) is the non-increasing rearrangement of the numbers \( r_1^{n_1}r_2^{n_2} \cdots r_m^{n_m} \). It is convenient to state the following simple lemma.

**Lemma 6.1.** Let \( \lambda_1, \ldots, \lambda_m \) be positive numbers. Let \( N_A \) be the number of \( m \)-tuples of non-negative integers \( (n_1, \ldots, n_m) \) such that \( \sum_{k=1}^m \lambda_k n_k \leq A \). Then, as \( A \to \infty \):

\[ N_A \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}. \]
Indeed, just apply Karamata’s tauberian theorem (see [12] p. 30) to the generalized Dirichlet series:

\[ S(\varepsilon) := \prod_{k=1}^{m} \frac{1}{1 - e^{-\lambda_k \varepsilon}} = \sum_{n_1, \ldots, n_m \geq 0} e^{-(\sum_{k=1}^{m} \lambda_k n_k) \varepsilon} ; \]

we have \( S(\varepsilon) \sim \frac{\varepsilon^{-m}}{(\lambda_1 \cdots \lambda_m) m!} \) as \( \varepsilon \to 0^+ \).

Let now \( N \) be a positive integer and \( \varepsilon = a_N \). Setting \( \lambda_k = \log(1/r_k) \) and \( A = \log(1/\varepsilon) \), we see that \( N \) is the number of \( m \)-tuples \( (n_1, \ldots, n_m) \) of non-negative integers such that \( r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \geq \varepsilon \), i.e. such that \( \sum_{k=1}^{m} \lambda_k n_k \leq A \). This number \( N \) is hence nothing but the number \( N_A \) of the previous lemma, so that:

\[ N \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!} . \]

Inverting this formula, we get:

\[ a_N(C\phi) = \exp \left[ - (1 + o(1)) \left( m!(\lambda_1 \lambda_2 \cdots \lambda_m) N \right)^{1/m} \right] \]

and:

\[ \beta_m(C\phi) = \exp \left[ - (m! \lambda_1 \lambda_2 \cdots \lambda_m)^{1/m} \right] = \Gamma_m(K) , \]

in view of (6.2) and (6.4).

On the view of the simple-minded previous example, the extension of the spectral radius formula (6.3) to the multivariate case holds, and it is tempting to conjecture that this is a general phenomenon as in dimension one, all the more as the following extension of Widom’s theorem was proved by Zakharyuta, based on the solution by S. Nivoche of Zakharyuta’s conjecture ([23]); see also [28], Theorem 5.4. A compact subset \( K \) of \( \mathbb{D}^m \) is said to be regular if its extremal function \( u_K^* \) is continuous on \( \mathbb{D}^m \).

**Theorem 6.2** ([28], Theorem 5.6). Let \( K \) be a regular compact subset of \( \mathbb{D}^m \) and \( J : H^\infty(\mathbb{D}^m) \to C(K) \) the canonical injection; then the Kolmogorov numbers \( d_n(J) \) satisfy:

\[ \lim_{n \to \infty} \left[ d_n(J) \right]^{1/n^{1/m}} = \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right] . \]

Note that the right side is nothing but \( \Gamma_m(K) \).

We will see consequences of this result in a forthcoming paper ([22]).

### 6.3 Upper bound

For the upper bound, the situation behaves better, as stated in the following theorem.
Theorem 6.3 ([28], Proposition 6.1). Let $K$ be a compact subset of $\mathbb{D}^m$ with non-void interior. Then:

$$
\limsup_{n \to \infty} \left[ d_n(J) \right]^{1/n^1/m} \leq \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right].
$$

Note that $(K, \mathbb{D}^m)$ is a condenser since $K$ has non-void interior. We deduce the following upper bound.

Theorem 6.4. Let $\Phi$ be an analytic self-map of $\mathbb{D}^m$ with $\|\Phi\|_\infty = \rho < 1$, thus inducing a compact composition operator on $H^2(\mathbb{D}^m)$. Then we have:

$$
\beta_m^* (C_\Phi) \leq \Gamma_m (\overline{\Phi(\mathbb{D}^m)}).
$$

Proof. This proof provides in particular a simplification of that given in [21] in dimension $m = 1$.

Changing $n$ into $n^m$, Theorem 6.3 means that for every $\varepsilon > 0$, there exists an $(n^m - 1)$-dimensional subspace $V$ of $C(K)$ such that, for any $g \in H^\infty(\mathbb{D}^m)$, there exists $h \in V$ such that:

$$
\|g - h\|_{C(K)} \leq C_\varepsilon (1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n \|g\|_{\infty}.
$$

Let $l$ be an integer to be adjusted later, and $f(z) = \sum_\alpha b_\alpha z^\alpha \in B_{H^2}$, as well as $g(z) = \sum_{|\alpha| \leq l} b_\alpha z^\alpha$. We first note that (with $M_m$ depending only on $m$ and $\rho$, and since the number of $\alpha$’s such that $|\alpha| \leq p$ is $O(p^m)$):

$$
\sum_{|\alpha| \leq l} \rho^{2|\alpha|} \leq M_m \sum_{p \geq l} p^m \rho^{2p} \leq M_m l^m \frac{\rho^{2l}}{(1 - \rho^{2})^{m+1}}.
$$

We next observe that, by the Cauchy-Schwarz and Parseval inequalities:

$$
\|g\|_{\infty} \leq M_m l^{m/2},
$$

and

$$
|f(z) - g(z)| \leq M_m l^{m/2} \frac{|z|^{l}}{(1 - |z|_{\infty}^{2})(m+1)^{1/2}}, \quad \forall z \in \mathbb{D}^m.
$$

where $|z|_{\infty} := \max_{j \leq m} |z_j|$ if $z = (z_1, \ldots, z_m)$. The subspace $F$ formed by functions $v \circ \Phi$, for $v \in V$, can be viewed as a subspace of $L^\infty(\mathbb{T}^m) \subseteq L^2(\mathbb{T}^m)$ with respect to the Haar measure of $\mathbb{T}^m$, the distinguished boundary of $\mathbb{D}^m$ (indeed, we can write $(v \circ \Phi)^* = v \circ \Phi^*$, where $\Phi^*$ denotes the almost everywhere existing radial limits of $\Phi(rz)$, which belong to $K$). Let finally $E = P(F) \subseteq H^2(\mathbb{D}^m)$ where $P : L^2(\mathbb{T}^m) \to H^2(\mathbb{T}^m) = H^2(\mathbb{D}^m)$ is the orthogonal projection. This is a subspace of $H^2$ with dimension $< n^m$. Set temporarily $\eta = C_\varepsilon (1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n$. It follows from (6.7) and (6.8) that, for some $h \in V$:

$$
\|g - h\|_{C(K)} \leq \eta \|g\|_{\infty} \leq \eta M_m l^{m/2}
$$
and hence:
\[ \|g \circ \Phi - h \circ \Phi\|_2 \leq \|g \circ \Phi - h \circ \Phi\|_\infty \leq \eta M_m t^{m/2}, \]

implying by orthogonal projection:
\[ \text{dist} (C_\Phi g, E) \leq \|g \circ \Phi - P(h \circ \Phi)\|_2 \leq \eta M_m t^{m/2}. \]

Now, since \( C_\Phi f(z) - C_\Phi g(z) = f(\Phi(z)) - g(\Phi(z)) \), (6.9) gives:
\[ \|C_\Phi f - C_\Phi g\|_2 \leq \|C_\Phi f - C_\Phi g\|_\infty \leq M_m t^{m/2} \frac{\rho^l}{(1 - \rho^2)(m+1)/2} \]

and hence:
\[ \text{dist} (C_\Phi f, E) \leq M_m t^{m/2} \left( \frac{\rho^l}{(1 - \rho^2)(m+1)/2} + C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n \right). \]

It ensues, since \( a_N(C_\Phi) = d_N(C_\Phi) \), that:
\[ \left[ a_n^{\infty}(C_\Phi) \right]^{1/n} \leq (M_m t^{m/2})^{1/n} \left[ \frac{\rho^l/2}{(1 - \rho^2)(m+1)/2n} + C_\varepsilon^{1/n} (1 + \varepsilon)^n \Gamma_m(K) \right]. \]

Taking now for \( l \) the integer part of \( n \log n \), and passing to the upper limit as \( n \to \infty \), we obtain (since \( l/n \to \infty \) and \( (\log l)/n \to 0 \)):
\[ \beta_m^+(C_\Phi) \leq (1 + \varepsilon) \Gamma_m(K), \]

and Theorem 6.4 follows.

\[ \Box \]

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