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Some examples of composition operators and their approximation numbers on the Hardy space of the bi-disk

Daniel Li, Hervé Queffélec, L. Rodríguez-Piazza

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Abstract. We give examples of composition operators $C_\Phi$ on $H^2(D^2)$ showing that the condition $\|\Phi\|_\infty = 1$ is not sufficient for their approximation numbers $a_n(C_\Phi)$ to satisfy $\lim_{n \to \infty} [a_n(C_\Phi)]^{1/\sqrt{n}} = 1$, contrary to the 1-dimensional case. We also give a situation where this implication holds. We make a link with the Monge-Ampère capacity of the image of $\Phi$.

Key-words: approximation numbers; Bergman space; bidisk; composition operator; Green capacity; Hardy space; Monge-Ampère capacity; weighted composition operator.


1 Introduction and notation

1.1 Introduction

The purpose of this paper is to continue the study of composition operators on the polydisk initiated in [2], and in particular to examine to what extent one of the main results of [21] still holds.

Let $H$ be a Hilbert space and $T : H \to H$ a bounded operator. Recall that the approximation numbers of $T$ are defined as:

$$a_n(T) = \inf_{\text{rank } R < n} \| T - R \|, \quad n \geq 1,$$

and we have:

$$\| T \| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq \cdots$$

The operator $T$ is compact if and only if $a_n(T) \xrightarrow{n \to \infty} 0$. 

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For \( d \geq 1 \), we define:
\[
\begin{align*}
\beta_d^-(T) &= \liminf_{n \to \infty} \left[ a_{n^d}(T) \right]^{1/n} \\
\beta_d^+(T) &= \limsup_{n \to \infty} \left[ a_{n^d}(T) \right]^{1/n}
\end{align*}
\]
We have:
\[ 0 \leq \beta_d^-(T) \leq \beta_d^+(T) \leq 1, \]
and we simply write \( \beta_d(T) \) in case of equality.

It may well happen in general (consider diagonal operators) that \( \beta_d^-(T) = 0 \) and \( \beta_d^+(T) = 1 \).

When \( H = H^2(\mathbb{D}) \) is the Hardy space on the open unit disk \( \mathbb{D} \) of \( \mathbb{C} \), and \( T = C_\Phi \) is a composition operator, with \( \Phi : \mathbb{D} \to \mathbb{D} \) a non-constant analytic function, we always have ([19]):
\[
\beta_1^-(C_\Phi) > 0,
\]
and one of the main results of [19] is the equivalence:
\[
(1.1) \quad \beta_1^+(C_\Phi) < 1 \iff \|\Phi\|_\infty < 1.
\]
An alternative proof was given in [21], as a consequence of a so-called “spectral radius formula”, which moreover shows that:
\[
\beta_1^-(C_\Phi) = \beta_1^+(C_\Phi).
\]

In [2], for \( d \geq 2 \), it is proved that, for a bounded symmetric domain \( \Omega \subseteq \mathbb{C}^d \), if \( \Phi : \Omega \to \Omega \) is analytic, such that \( \Phi(\Omega) \) has a non-void interior, and the composition operator \( C_\Phi : H^2(\Omega) \to H^2(\Omega) \) is compact, then:
\[
\beta_d^-(C_\Phi) > 0.
\]
On the other hand, if \( \Omega \) is a product of balls, then:
\[
\|\Phi\|_\infty < 1 \quad \Rightarrow \quad \beta_d^+(C_\Phi) < 1.
\]
We do not know whether the converse holds and the purpose of this paper is to study some examples towards an answer.

The paper is organized as follows. Section 1 is this short introduction, as well as some notations and definitions on singular numbers of operators and Hardy spaces of the polydisk to follow. Section 2 contains preliminary results on weighted composition operators in one variable, which surprisingly play an important role in the study of non-weighted composition operators in two variables. Section 3 studies the case of symbols with “separated” variables. Our main one variable result extends in this case. Section 4 studies the “glued case” \( \Phi(z_1, z_2) = (\phi(z_1), \phi(z_1)) \) for which even boundedness is an issue. Here, the
Bergman space $B^2(D)$ enters the picture. Section 5 studies the case of “triangularly separated” variables. This section lets direct Hilbertian sums of weighted composition operators in one variable appear, and it contains our main result: an example of a symbol $\Phi$ satisfying $\|\Phi\|_{\infty} = 1$ and yet $\beta_D(C\Phi) < 1$. The final Section 6 discusses the role of the Monge-Ampère pluricapacity, which is a multivariate extension of the Green capacity in the disk. Even though, as evidenced by our counterexample of Section 5, this capacity will not capture all the behavior of the parameter $\beta_m(C\Phi)$, some partial results are obtained, relying on theorems of S. Nivoche and V. Zakharyuta.

### 1.2 Notation

We say that an analytic map $\Phi: D^d \rightarrow D^d$ is a symbol if its associated composition operator $C\Phi: H^2(D^d) \rightarrow H^2(D^d)$, defined by $C\Phi(f) = f \circ \Phi$, is bounded.

We say that $\Phi$ is truly $d$-dimensional if $\Phi(D^d)$ has a non-void interior.

We will make use of two kinds of symbols defined on $D$.

The lens map $\lambda_\theta: D \rightarrow D$ is defined, for $\theta \in (0, 1)$, by:

$$
\lambda_\theta(z) = \frac{(1 + z)^\theta - (1 - z)^\theta}{(1 + z)^\theta + (1 - z)^\theta}
$$

(see [25], p. 27, or [16], for more information), and corresponds to $u \mapsto u^\theta$ in the right half-plane.

The cusp map $\chi: D \rightarrow D$ was first defined in [15] and in a slightly different form in [20]; we actually use here the modified form introduced in [17], and then used in [18]. We first define:

$$
\chi_0(z) = \frac{(z - i)^{1/2} - i}{-i \left(\frac{z - i}{i z - 1}\right)^{1/2} + 1};
$$

we note that $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2} - 1$.

Then we set:

$$
\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},
$$

and finally:

$$
\chi(z) = 1 - \chi_3(z),
$$

where:

$$
a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)
$$

is chosen in order that $\chi(0) = 0$. The image $\Omega$ of the (univalent) cusp map is formed by the intersection of the inside of the disk $D(1 - \frac{2}{\pi} \frac{a}{2})$ and the outside of the two disks $D(1 + \frac{ia}{\sqrt{2}})$ and $D(1 - \frac{ia}{\sqrt{2}})$. 


Besides the approximation numbers, we need other singular numbers for an operator $S: X \to Y$ between Banach spaces $X$ and $Y$.

The **Bernstein numbers** $b_n(S)$, $n \geq 1$, which are defined by:

\begin{equation}
(1.4) \quad b_n(S) = \sup_{E} \min_{x \in S_E} \|Sx\|,
\end{equation}

where the supremum is taken over all $n$-dimensional subspaces of $X$ and $S_E$ is the unit sphere of $E$.

The **Gelfand numbers** $c_n(S)$, $n \geq 1$, which are defined by:

\begin{equation}
(1.5) \quad c_n(S) = \inf \{\|S|_M\; ; \; \text{codim } M < n\}.
\end{equation}

The **Kolmogorov numbers** $d_n(S)$, $n \geq 1$, which are defined by:

\begin{equation}
(1.6) \quad d_n(S) = \inf_{\dim E < n} \left[ \sup_{x \in B_X} \text{dist} (Sx, E) \right].
\end{equation}

Pietsch showed that all $s$-numbers on Hilbert spaces are equal (see [23], § 2, Corollary, or [24], Theorem 11.3.4); hence:

\begin{equation}
(1.7) \quad a_n(S) = b_n(S) = c_n(S) = d_n(S).
\end{equation}

We denote $m$ the normalized Lebesgue measure on $T = \partial \mathbb{D}$. If $\varphi: \mathbb{D} \to \mathbb{D}$, $m_\varphi$ is the pull-back measure on $\mathbb{D}$ defined by $m_\varphi(E) = m[\varphi^{-1}(E)]$, where $\varphi^*$ stands for the non-tangential boundary values of $\varphi$.

The notation $A \lesssim B$ means that $A \leq CB$ for some positive constant $C$ and we write $A \approx B$ if we have both $A \lesssim B$ and $B \lesssim A$.

## 2 Preliminary results on weighted composition operators on $H^2(\mathbb{D})$

We see in this section that the presence of a “rapidly decaying” weight allows simpler estimates for the approximation numbers of a corresponding weighted composition operator. Such a study, but a bit different, is made in [14].

Let $\varphi: \mathbb{D} \to \mathbb{D}$ a non-constant analytic self-map in the disk algebra $A(\mathbb{D})$ such that, for some constant $C > 1$ and for all $z \in \mathbb{D}$:

\begin{equation}
(2.1) \quad \varphi(1) = 1, \quad |1 - \varphi(z)| \leq 1, \quad |1 - \varphi(z)| \leq C (1 - |\varphi(z)|)
\end{equation}

as well as $\varphi(z) \neq 1$ for $z \neq 1$. We can take for example $\varphi = \frac{1 + \lambda \theta}{2}$ where $\lambda_\theta$ is the lens map with parameter $\theta$.

Let $w \in H^\infty$ and let $T$ be the weighted composition operator

\[ T = M_{w \circ \varphi} C_\varphi : H^2 \to H^2. \]

Note that $M_{w \circ \varphi} C_\varphi = C_\varphi M_w$. We first show that:
**Theorem 2.1.** Let \( T = M_w \circ \varphi C \varphi : H^2 \to H^2 \) be as above and let \( B \) be a Blaschke product with length \(< N\). Then, with the implied constant depending only on the number \( C \) in (2.1) (and of \( \varphi \)):

\[
\| T(f) \|_2 \lesssim \sup_{|z-1| \leq 1, z \in \varphi(D)} |B(z)| |w(z)|.
\]

**Proof.** The following preliminary observation (see also [16], p. 809), in which we denote by \( S(\xi, h) = \{ z \in \mathbb{D}; |z - \xi| \leq h \} \) the Carleson window with center \( \xi \in \mathbb{T} \) and size \( h \), and by \( K_\varphi \) the support of the pull-back measure \( m_\varphi \), will be useful.

(2.2) \( u \in S(\xi, h) \cap K_\varphi \implies u \in S(1, Ch) \cap K_\varphi \).

Indeed, if \( |u - \xi| \leq h \) and \( u \in K_\varphi \), (2.1) implies:

\[
1 - |u| \leq |u - \xi| \leq h \quad \text{and} \quad |u - 1| \leq C(1 - |u|) \leq Ch.
\]

Set \( E = BH^2 \). This is a subspace of codimension \(< N\). If \( f = Bg \in E \), with \( \|g\| = \|f\| \) (isometric division by \( B \) in \( BH^2 \)), we have \( Tf = (wBg) \circ \varphi \), whence:

\[
\| T(f) \|_2 = \int_{\mathbb{D}} |B|^2 |w|^2 |g|^2 dm_\varphi,
\]

implying \( \| T(f) \|_2 \leq \|f\|^2 \|J\|^2 \) where \( J : H^2 \to L^2(\sigma) \) is the natural embedding and where

\[
\sigma = |B|^2 |w|^2 dm_\varphi.
\]

Now, Carleson’s embedding theorem for the measure \( \sigma \) and (2.2) show that (the implied constants being absolute):

\[
\|J\|^2 \lesssim \sup_{\xi \in \mathbb{T}, 0 < h < 1} \frac{1}{h} \int_{S(\xi, h) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi
\]

\[
\lesssim \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi
\]

\[
\lesssim \left( \sup_{|z-1| \leq 1, z \in \varphi(D)} |B(z)|^2 |w(z)|^2 \right) \left( \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} dm_\varphi \right)
\]

\[
\lesssim \sup_{|z-1| \leq 1, z \in \varphi(\mathbb{D})} |B(z)|^2 |w(z)|^2,
\]

since \( m_\varphi \) is a Carleson measure for \( H^2 \) and where we used that, according to (2.1):

\[
K_\varphi \subseteq \varphi(\overline{\mathbb{D}}) \subseteq \{ z \in \mathbb{D}; |z - 1| \leq 1 \}.
\]

This ends the proof of Theorem 2.1 with help of the equality of \( a_N(T) \) with the Gelfand number \( c_N(T) \) recalled in (1.7). \( \square \)
In order to specialize efficiently the general Theorem 2.1, we recall the following simple Lemma 2.3 of [16], where:

\[
\rho(a, b) = \left| \frac{a - b}{1 - ab} \right|, \quad a, b \in \mathbb{D},
\]

is the pseudo-hyperbolic distance:

**Lemma 2.2 ([16])**. Let \( a, b \in \mathbb{D} \) such that \( |a - b| \leq L \min(1 - |a|, 1 - |b|) \). Then:

\[
\rho(a, b) \leq \frac{L}{\sqrt{L^2 + 1}} =: \kappa < 1.
\]

We can now state:

**Theorem 2.3.** Assume that \( \varphi \) is as in (2.1) and that the weight \( w \) satisfies, for some parameters \( 0 < \theta \leq 1 \) and \( R > 0 \):

\[
|w(z)| \leq \exp\left(-\frac{R}{|1 - z|^\theta}\right), \quad \forall z \in \mathbb{D}.
\]

Then, the approximation numbers of \( T = M_w \circ \varphi C \) satisfy:

\[
a_{nm+1}(T) \lesssim \max\left[ \exp(-an), \exp(-R 2^m \theta) \right],
\]

for all integers \( n, m \geq 1 \), where \( a = \log[\sqrt{16C^2 + 1}/(4C)] > 0 \) and \( C \) is as in (2.1).

**Proof.** Let \( p_l = 1 - 2^{-l}, 0 \leq l < m \) and let \( B \) be the Blaschke product:

\[
B(z) = \prod_{0 \leq l < m} \left( \frac{z - p_l}{1 - p_l z} \right)^n.
\]

Let \( z \in K_{\varphi} \cap \mathbb{D} \) so that \( 0 < |z - 1| \leq 1 \). Let \( l \) be the non-negative integer such that \( 2^{-l-1} < |z - 1| \leq 2^{-l} \). We separate two cases:

**Case 1: \( l \geq m \).** Then, the weight does the job. Indeed, majorizing \(|B(z)|\) by 1 and using the assumption on \( w \), we get:

\[
|B(z)|^2|w(z)|^2 \leq |w(z)|^2 \leq \exp\left(-\frac{2R}{|1 - z|^\theta}\right) \leq \exp(-2R 2^l \theta) \leq \exp(-2R 2^m \theta).
\]

**Case 2: \( l < m \).** Then, the Blaschke product does the job. Indeed, majorize \(|w(z)|\) by 1 and estimate \(|B(z)|\) more accurately with help of Lemma 2.2; we observe that

\[
|z - p_l| \leq |z - 1| + 1 - p_l \leq 2 \times 2^{-l} = 2(1 - p_l) \leq 4C(1 - p_l)
\]

and then, since \( z \in K_{\varphi} \), we can write with \( C \geq 1 \) as in (2.1):

\[
1 - |z| \geq \frac{1}{C} |1 - z| \geq \frac{1}{2C} 2^{-l} \geq \frac{1}{4C} |z - p_l|,
\]
so that the assumptions of Lemma 2.2 are verified with $L = 4C$, giving:

$$\rho(z, p_l) \leq \frac{4C}{\sqrt{16C^2 + 1}} = \exp(-a) < 1.$$  

Hence, by definition, since $l < m$:

$$|B(z)| \leq [\rho(z, p_l)]^n \leq \exp(-an).$$

Putting both cases together, and observing that our Blaschke product has length $nm < nm + 1$, we get the result by applying Theorem 2.1 with $N = nm + 1$.

2.1 Some remarks

1. Twisting a composition operator by a weight may improve the compactness of this composition operator, or even may make this weighted composition operator compact though the non-weighted was not (see [8] or [14]). However, this is not possible for all symbols, as seen in the following proposition.

**Proposition 2.4.** Let $w \in H^\infty$. If $\varphi$ is inner, or more generally if $|\varphi| = 1$ on a subset of $\mathbb{T}$ of positive measure, then $M_w C_\varphi$ is never compact (unless $w \equiv 0$).

**Proof.** Indeed, suppose $T = M_w C_\varphi$ compact. Since $(z^n)_{n=0}^\infty$ converges weakly to 0 in $H^2$ and since $T(z^n) = w \varphi^n$, we should have, since $|\varphi| = 1$ on $E$, with $m(E) > 0$:

$$\int_E |w|^2 \, dm = \int_E |w|^2 |\varphi|^{2n} \, dm \leq \int_T |w|^2 |\varphi|^{2n} \, dm = \|T(z^n)\|_2^2 \xrightarrow{n \to \infty} 0,$$

but this would imply that $w$ is null a.e. on $E$ and hence $w \equiv 0$ (see [7], Theorem 2.2), which was excluded. □

Note that É. Amar and A. Lederer proved in [1] that $|\varphi| = 1$ on a set of positive measure if and only if $\varphi$ is an exposed point of of the unit ball of $H^\infty$; hence the following proposition can be viewed as the (almost) opposite case.

**Proposition 2.5.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\|\varphi\|_\infty = 1$. Assume that:

$$\int_\mathbb{T} \log(1 - |\varphi|) \, dm > -\infty$$

(meaning that $\varphi$ is not an extreme point of the unit ball of $H^\infty$: see [7], Theorem 7.9). Then, if $w$ is an outer function such that $|w| = 1 - |\varphi|$, the weighted composition operator $T = M_w C_\varphi$ is Hilbert-Schmidt.

**Proof.** We have:

$$\sum_{n=0}^\infty \|T(z^n)\|^2 = \sum_{n=0}^\infty \int_\mathbb{T} (1 - |\varphi|)^2 |\varphi|^{2n} \, dm = \int_\mathbb{T} \frac{1 - |\varphi|}{1 + |\varphi|} \, dm < +\infty,$$

and $T$ is Hilbert-Schmidt, as claimed. □
2. In [14], Theorem 2.5, it is proved that we always have, for some constants \( \delta, \rho > 0 \):
\[
(2.4) \quad a_n(M_w C_\varphi) \geq \delta \rho^n, \quad n = 1, 2, \ldots
\]
(if \( w \neq 0 \)). We give here an alternative proof, based on a result of Gunatillake ([9]), this result holding in a wider context.

**Theorem 2.6** (Gunatillake). Let \( T = M_w C_\varphi \) be a compact weighted composition operator on \( H^2 \) and assume that \( \varphi \) has a fixed point \( a \in \mathbb{D} \). Then the spectrum of \( T \) is the set:
\[
\sigma(T) = \{0, w(a), w(a) \varphi'(a), w(a) [\varphi'(a)]^2, \ldots, w(a) [\varphi'(a)]^n, \ldots \}
\]

**Proof of (2.4).** First observe that, in view of Proposition 2.4, \( \varphi \) cannot be an automorphism of \( \mathbb{D} \) so that the point \( a \) is the Denjoy-Wolff point of \( \varphi \) and is attractive. Theorem 2.6 is interesting only when \( w(a) \varphi'(a) \neq 0 \).

Now, we can give a new proof Theorem 2.5 of [14] as follows. Let \( a \in \mathbb{D} \) be such that \( w(a) \varphi'(a) \neq 0 \) (\( H(\mathbb{D}) \) is a division ring and \( \varphi' \neq 0 \), \( w \neq 0 \)). Let \( b = \varphi(a) \) and \( \tau \in \text{Aut} \mathbb{D} \) with \( \tau(b) = a \). We set:
\[
\psi = \tau \circ \varphi \quad \text{and} \quad S = M_w C_\psi = TC_\tau.
\]
This operator \( S \) is compact because \( T \) is.

Since \( \psi(a) = a \) and \( \psi'(a) = \tau'(b) \varphi'(a) \neq 0 \), Theorem 2.6 says that the non-zero eigenvalues of \( S \), arranged in non-increasing order, are the numbers \( \lambda_n = w(a) [\psi'(a)]^{n-1}, n \geq 1 \). As a consequence of Weyl's inequalities, we know that:
\[
a_1(S) a_n(S) \geq |
\lambda_{2n}^2 \geq \delta \rho^n,
\]
with:
\[
\delta = |w(a)|^2 > 0 \quad \text{and} \quad \rho = |\psi'(a)|^2 > 0.
\]
To finish, it is enough to observe that \( a_n(S) \leq a_n(T) \|C_\tau\| \) by the ideal property of approximation numbers. \( \square \)

3 The splitted case

**Theorem 3.1.** Let \( \Phi = (\phi, \psi) : \mathbb{D}^d \to \mathbb{D}^d \) be a truly \( d \)-dimensional symbol with \( \phi : \mathbb{D} \to \mathbb{D} \) depending only on \( z_1 \) and \( \psi : \mathbb{D}^{d-1} \to \mathbb{D}^{d-1} \) only on \( z_2, \ldots, z_d \), i.e. \( \Phi(z_1, z_2, \ldots, z_d) = (\phi(z_1), \psi(z_2, \ldots, z_d)) \). Then, whatever \( \psi \) behaves:
\[
\|\phi\|_\infty = 1 \quad \implies \quad \beta_d(C_\Phi) = 1.
\]

**Proof.** The proof is based on the following simple lemma, certainly well-known.

**Lemma 3.2.** Let \( S : H_1 \to H_1 \) and \( T : H_2 \to H_2 \) be two compact linear operators, where \( H_1 \) and \( H_2 \) are Hilbert spaces. Let \( S \otimes T \) be their tensor product, acting on the tensor product \( H_1 \otimes H_2 \). Then:
\[
a_{mn}(S \otimes T) \geq a_m(S) a_n(T)
\]
for all positive integers \( m, n \).
We postpone the proof of the lemma and show how to conclude.

We can assume $C\Phi$ to be compact, so that $C\phi$ is compact as well. Since $\|\phi\|_\infty = 1$, we have, thanks to (1.1):

\[ a_m(C\phi) \geq e^{-m \varepsilon_m} \quad \text{with} \quad \varepsilon_m \to 0. \]

Replacing $\varepsilon_m$ by $\delta_m := \sup_{p \geq m} \varepsilon_p$, we can assume that $(\varepsilon_m)_m$ is non-increasing.

Moreover, $m \varepsilon_m \to \infty$ since $C\phi$ is compact and hence $a_m(C\phi) \to 0$. We next observe that, due to the separation of variables in the definition of $\phi$ and $\psi$, we can write:

\[ C\Phi = C\phi \otimes C\psi. \]

Indeed, write $z = (z_1, w)$ with $z_1 \in \mathbb{D}$ and $w \in \mathbb{D}^{d-1}$. If $f \in H^2(\mathbb{D})$ and $g \in H^2(\mathbb{D}^{d-1})$, we see that:

\[
C\Phi(f \otimes g)(z) = (f \otimes g)(\phi(z_1), \psi(w)) = f(\phi(z_1))g(\psi(w))
\]

\[
= [C\phi f(z_1)] [C\psi g(w)] = (C\phi \otimes C\psi g)(z).
\]

Since tensor products $f \otimes g$ generate $H^2(\mathbb{D}) = H^2(\mathbb{D}) \otimes H^2(\mathbb{D}^{d-1})$, this proves (3.1).

Let now $m$ be a large positive integer. Set (\[.] stands for the integer part):

\[ n_m = \lfloor m \varepsilon_m \rfloor^{d-1} \quad \text{and} \quad N_m = m n_m. \]

From what we know in dimension $d-1$ (see [2], Theorem 3.1) and from the preceding, we can write (observe that $\psi$ has to be truly $(d-1)$-dimensional since $\Phi$ is truly $d$-dimensional):

\[ a_m(C\phi) \geq \exp(-m \varepsilon_m) \quad \text{and} \quad a_n(C\psi) \geq a \exp(-C n^{1/(d-1)}), \]

for some positive constant $C$, which will be allowed to vary from one formula to another. Lemma 3.2 implies:

\[ a_{N_m}(C\Phi) \geq a \exp[-C (m \varepsilon_m + n_m^{1/(d-1)})]. \]

Since $n_m \lesssim (m \varepsilon_m)^{d-1}$, we get:

\[ a_{N_m}(C\Phi) \geq a \exp(-C m \varepsilon_m). \]

Observe that $N_m = m n_m \sim m^d \varepsilon_m^{d-1}$ and so $N_m^{1/d} \sim m \varepsilon_m^{1-1/d}$. As a consequence:

\[ a_{N_m}(C\Phi) \geq a \exp(-C m \varepsilon_m) = a \exp\left[-(C \varepsilon_m^{1/d}) (m \varepsilon_m^{1-1/d})\right] \geq a \exp(-\eta_m N_m^{1/d}) \]
with \( \eta_m := C \varepsilon_m^{1/d} \).

Now, for \( N > N_1 \), let \( m \) be the smallest integer satisfying \( N_m \geq N \) (so that \( N_{m-1} < N \leq N_m \)), and set \( \delta_N = \eta_m \). We have \( \lim_{N \to \infty} \delta_N = 0 \). Next, we note that \( \lim_{m \to \infty} N_m/N_{m-1} = 1 \), because \( N_m \geq N_{m-1} \) and:

\[
\frac{N_m}{N_{m-1}} \leq \frac{m}{m-1} \left( \frac{m \varepsilon_m + 1}{(m-1) \varepsilon_m} \right)^{d-1} \sim \left( \frac{\varepsilon_m}{\varepsilon_{m-1}} \right)^{d-1} \leq 1.
\]

Finally, if \( N \) is an arbitrary integer and \( N_{m-1} < N \leq N_m \), we obtain:

\[
a_N(C_\Phi) \geq a_{N_m}(C_\Phi) \geq a \exp(-\eta_m N_m^{1/d}) \geq a \exp(-C \delta_N N^{1/d}),
\]

since we observed that \( \lim_{m \to \infty} N_m/N_{m-1} = 1 \).

This amounts to say that \( \beta_d(C_\Phi) = 1 \).

**Proof of Lemma 3.2.** It is rather formal. Start from the Schmidt decompositions of \( S \) and \( T \) respectively (recall that Hilbert spaces, the approximation numbers are equal to the singular ones):

\[
S = \sum_{m=1}^{\infty} a_m(S) u_m \otimes v_m, \quad T = \sum_{n=1}^{\infty} a_n(T) u'_n \otimes v'_n,
\]

where \((u_m), (v_m)\) are two orthonormal sequences of \( H_1 \), \((u'_n), (v'_n)\) two orthonormal sequences of \( H_2 \), and \( u_m \otimes v_m \) and \( u'_n \otimes v'_n \) denote the rank one operators defined by \((u_m \otimes v_m)(x) = \langle x, v_m \rangle u_m, x \in H_1\), and \((u'_n \otimes v'_n)(x) = \langle x, v'_n \rangle u'_n, x \in H_2\).

We clearly have:

\[
(u_m \otimes v_m) \otimes (u'_n \otimes v'_n) = (u_m \otimes u'_n) \otimes (v_m \otimes v'_n),
\]

so that the Schmidt decomposition of \( S \otimes T \) is (with SOT-convergence):

\[
S \otimes T = \sum_{m,n \geq 1} a_m(S) a_n(T) (u_m \otimes u'_n) \otimes (v_m \otimes v'_n),
\]

since the two sequences \((u_m \otimes u'_n)_{m,n}\) and \((v_m \otimes v'_n)_{m,n}\) are orthonormal: for instance, we have by definition:

\[
\langle u_{m_1} \otimes u'_{n_1}, u_{m_2} \otimes u'_{n_2} \rangle = \langle u_{m_1}, u_{m_2} \rangle \langle u'_{n_1}, u'_{n_2} \rangle.
\]

This shows that the singular values of \( S \otimes T \) are the non-increasing rearrangement of the positive numbers \( a_m(S) a_n(T) \) and ends the proof of the lemma: the \( mn \) numbers \( a_k(S) a_l(T) \), for \( 1 \leq k \leq m, 1 \leq l \leq n \) all satisfy \( a_k(S) a_l(T) \geq a_m(S) a_n(T) \), so that \( a_{mn}(S \otimes T) \geq a_m(S) a_n(T) \).
4 The glued case

Here we consider symbols of the form:

\[ \Phi(z_1, z_2) = (\phi(z_1), \phi(z_1)), \]

where \( \phi: \mathbb{D} \to \mathbb{D} \) is a non-constant analytic map.

Note that such maps \( \Phi \) are not truly 2-dimensional.

4.1 Preliminary

We begin by remarking the following fact.

Let \( B^2(\mathbb{D}) \) be the Bergman space of all analytic functions \( f: \mathbb{D} \to \mathbb{C} \) such that:

\[ \|f\|_{B^2}^2 := \int_{\mathbb{D}} |f(z)|^2 \ dA(z) < \infty, \]

where \( dA \) is the normalized area measure on \( \mathbb{D} \).

**Proposition 4.1.** Assume that the composition operator \( C_\phi \) maps boundedly \( B^2(\mathbb{D}) \) into \( H^2(\mathbb{D}) \). Then \( C_\Phi: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \), defined by (4.1), is bounded.

**Proof.** If we write \( f \in H^2(\mathbb{D}^2) \) as:

\[ f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k, \quad \text{with} \quad \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2}^2, \]

we formally (or assuming that \( f \) is a polynomial) have:

\[ \Phi(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} \phi(z_1)^j \phi(z_1)^k = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) \phi(z_1)^n. \]

Hence, if we set \( g(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n \), we get:

\[ \|C_\Phi(f)(z_1, z_2) = [C_\phi(g)](z_1), \]

so that, by integrating:

\[ \|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} = \|C_\phi(g)\|_{H^2(\mathbb{D})}. \]

By hypothesis, there is a positive constant \( M \) such that:

\[ \|C_\phi(g)\|_{H^2(\mathbb{D})} \leq M \|g\|_{B^2(\mathbb{D})}. \]

But, by the Cauchy-Schwarz inequality:

\[ \|g\|_{B^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \sum_{j+k=n} c_{j,k} \right)^2 \]

\[ \leq \sum_{n=0}^{\infty} \left( \sum_{j+k=n} |c_{j,k}|^2 \right) = \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2(\mathbb{D}^2)}^2, \]

and we obtain \( \|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} \leq M \|f\|_{H^2(\mathbb{D}^2)}. \)
4.2 Lens maps

Let \( \lambda_\theta \) be a lens map of parameter \( \theta, 0 < \theta < 1 \). We consider \( \Phi_\theta: \mathbb{D}^2 \to \mathbb{D}^2 \) defined by:

\[
\Phi_\theta(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_1)).
\]

We have the following result.

**Theorem 4.2.** The composition operator \( C_{\Phi_\theta}: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \) is:

1) not bounded for \( \theta > 1/2 \);
2) bounded, but not compact for \( \theta = 1/2 \);
3) compact, and even Hilbert-Schmidt, for \( 0 < \theta < 1/2 \).

**Proof.** The reproducing kernel of \( H^2(\mathbb{D}^2) \) is, for \((a, b) \in \mathbb{D}^2\):

\[
K_{a,b}(z_1, z_2) = \frac{1}{1 - \overline{az}_1} \frac{1}{1 - bz_2}, \quad (z_1, z_2) \in \mathbb{D}^2,
\]

and:

\[
\|K_{a,b}\|^2 = \frac{1}{(1 - |a|^2)(1 - |b|^2)}.
\]

1) If \( C_{\Phi_\theta} \) were bounded, we should have, for some \( M < \infty \):

\[
\|C_{\Phi_\theta}(K_{a,b})\|_{H^2} \leq M \|K_{a,b}\|_{H^2}, \quad \text{for all } a, b \in \mathbb{D}.
\]

Since \( C_{\Phi_\theta}(K_{a,b}) = K_{\Phi_\theta(a), \Phi_\theta(b)} = K_{\lambda_\theta(a), \lambda_\theta(b)} \), we would get, with \( b = 0 \):

\[
\left( \frac{1}{1 - |\lambda_\theta(a)|^2} \right)^2 \leq M^2 \frac{1}{1 - |a|^2}.
\]

but this is not possible for \( \theta > 1/2 \), since \( 1 - |\lambda_\theta(a)|^2 \approx 1 - |\lambda_\theta(a)| \sim (1 - a)^\theta \) when \( a \) goes to 1, with \( 0 < a < 1 \).

For 2) and 3), let us consider the pull-back measure \( m_\theta \) of the normalized Lebesgue measure on \( T = \partial \mathbb{D} \) by \( \lambda_\theta \). It is easy to see that:

\[
\sup_{\xi \in T} m_\theta[D(\xi, h) \cap \mathbb{D}] = m_\theta[D(1, h) \cap \mathbb{D}] \approx h^{1/\theta}.
\]

In particular, for \( \theta \leq 1/2 \), \( m_\theta \) is a 2-Carleson measure, and hence (see [15], Theorem 2.1, for example) the canonical injection \( j: B^2(\mathbb{D}) \to L^2(m_\theta) \) is bounded, meaning that, for some positive constant \( M < \infty \):

\[
\int_{\mathbb{D}} |f(z)|^2 dm_\theta(z) \leq M^2 \|f\|_{B^2}^2.
\]

Since

\[
\int_{\mathbb{D}} |f(z)|^2 dm_\theta(z) = \int_T |f(\lambda_\theta(u))|^2 dm(u) = \|C_{\lambda_\theta}(f)\|_{H^2}^2,
\]

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we get that $C_\lambda$ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$.

It follows from Proposition 4.1 that $C_{\Phi}: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is bounded.

However, $C_{\Phi_{1/2}}$ is not compact since $C_{\Phi_{1/2}}(K_{a,b})/\|K_{a,b}\|$ does not converge to 0 as $a, b \to 1$, by the calculations made in 1).

For 3), let $e_{j,k}(z_1, z_2) = z_1^{j+1/2} z_2^{j+1}$, $j, k \geq 0$, be the canonical orthonormal basis of $H^2(\mathbb{D}^2)$; we have $[C_{\Phi}(e_{j,k})](z_1, z_2) = [\lambda_\theta(z_1)]^{j+k}$. Hence:

$$
\sum_{j, k \geq 0} \|C_{\Phi}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \leq \sum_{n=0}^{\infty} (2n+1) \int_{\mathbb{T}} |\lambda_\theta|^{2n} \, dm \leq \int_{\mathbb{T}} \frac{2}{(1-|\lambda_\theta|^2)^2} \, dm.
$$

Since, by Lemma 4.3 below, $1 - |\lambda_\theta(e^{it})|^2 \gtrsim |1 - e^{it}|^\theta$ for $|t| \leq \pi/2$, we get:

$$
\sum_{j, k \geq 0} \|C_{\Phi}(e_{j,k})\|^2_{H^2(\mathbb{D}^2)} \lesssim \int_{0}^{\pi/2} \frac{dt}{t^{\theta/2}} < \infty,
$$

since $\theta < 1/2$. Therefore $C_{\Phi}$ is Hilbert-Schmidt for $\theta < 1/2$.

For sake of completeness, we recall the following elementary fact (see [25], p. 28, or also [16], Lemma 2.5).

**Lemma 4.3.** With $\delta = \cos(\theta \pi/2)$, we have, for $|z| \leq 1$ and $\Re z \geq 0$:

$$
1 - |\lambda_\theta(z)|^2 \geq \frac{\delta}{2} |1 - z|^\theta.
$$

**Proof.** We can write:

$$
\lambda_\theta(z) = \frac{1-w}{1+w} \quad \text{with} \quad w = \left(\frac{1-z}{1+z}\right)^\theta \quad \text{and} \quad |w| \leq 1.
$$

Then:

$$
\Re w \geq \Re |w| \geq \frac{\delta}{2} |1-z|^\theta.
$$

Hence:

$$
1 - |\lambda_\theta(z)|^2 = \frac{4 \Re w}{1+|w|^2} \geq \frac{\delta}{2} |1-z|^\theta,
$$

as announced.

We now improve the result 3) of Theorem 4.2 by estimating the approximation numbers of $C_{\Phi}$ and get that $C_{\Phi}$ is in all Schatten classes of $H^2(\mathbb{D}^2)$ when $\theta < 1/2$.

**Theorem 4.4.** For $0 < \theta < 1/2$, there exists $b = b_\theta > 0$ such that:

$$
a_n(C_{\Phi}) \lesssim e^{-b\sqrt{n}}.
$$

In particular $\beta_2^+ (C_{\Phi}) \leq e^{-b} < 1$, though $\|\Phi_\theta\|_\infty = 1$, and even $\Phi_\theta(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$. 

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Proof. Proposition 4.1 (and its proof) can be rephrased in the following way: if $C_\phi$ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$, then, we have the following factorization:

$$C_\phi: H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^2),$$

where $I: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^2)$ is the canonical injection given by $(I f)(z_1, z_2) = f(z_1)$ for $f \in H^2(\mathbb{D})$, and $J: H^2(\mathbb{D}^2) \rightarrow B^2(\mathbb{D})$ is the contractive map defined by:

$$(J f)(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n,$$

for $f \in H^2(\mathbb{D}^2)$ with $f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k$.

In the proof of Theorem 4.2, we have seen that, for $0 < \theta \leq 1/2$, the composition operator $C_{\lambda_\theta}$ is bounded from $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$; we get hence the factorization:

$$C_{\Phi_\theta}: H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_{\lambda_\theta}} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^2),$$

Now, the lens maps have a semi-group property:

$$\lambda_{\theta_1 \theta_2} = \lambda_{\theta_1} \lambda_{\theta_2},$$

giving $C_{\lambda_{\theta_1 \theta_2}} = C_{\lambda_{\theta_1}} \circ C_{\lambda_{\theta_2}}$.

For $0 < \theta < 1/2$, we therefore can write $C_{\lambda_{\theta}} = C_{\lambda_{2\theta}} \circ C_{\lambda_{1/2}}$ (note that $2\theta < 1$, so $C_{\lambda_{2\theta}}: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ is bounded), and we get:

$$C_{\Phi_\theta} = I C_{\lambda_{2\theta}} C_{\lambda_{1/2}} J.$$

Consequently:

$$a_n(C_{\Phi_\theta}) \leq \|I\| \|J\| \|C_{\lambda_{1/2}}\|_{B^2 \rightarrow H^2} a_n(C_{\lambda_{2\theta}}).$$

Now, we know ([16], Theorem 2.1) that $a_n(C_{\lambda_{2\theta}}) \lesssim e^{-b\sqrt{n}}$, so we get that $a_n(C_{\Phi_\theta}) \lesssim e^{-b\sqrt{n}}$. \qed

5 Triangularly separated variables

In this section, we consider symbols of the form:

$$\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2),$$

where $\phi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ are non-constant analytic maps.

Such maps $\Phi$ are truly 2-dimensional.
More generally, if \( h \in H^\infty \), with \( h(0) = 0 \) and \( \| h \|_\infty \leq 1 \), has its powers \( h^k, k \geq 0 \), orthogonal in \( H^2 \) (for convenience, we shall say that \( h \) is a Rudin function), we can consider:

\[
\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))
\]

For such \( h \) we can take for example an inner function vanishing at the origin, but there are other such functions, as shown by C. Bishop:

**Theorem** (Bishop \cite{4}). The function \( h \) is a Rudin function if and only if the pull-back measure \( \mu = \mu_h \) is radial and Jensen, i.e for every Borel set \( E \):

\[
\mu(e^{i\theta} E) = \mu(E) \quad \text{and} \quad \int_D \log(1/|z|) \, d\mu(z) < \infty.
\]

Conversely, for every probability measure \( \mu \) supported by \( \overline{D} \), which is radial and Jensen, there exists \( h \) in the unit ball of \( H^\infty \), with \( h(0) = 0 \), such that \( \mu = \mu_h \).

If we take for \( \mu \) the Lebesgue measure of \( \mathbb{T} \), we get an inner function. But, as remarked in \cite{4}, we can take for \( \mu \) the Lebesgue measure on the union \( \mathbb{T} \cup (1/2)^\mathbb{T} \), normalized in order that \( \mu(T) = \mu((1/2)\mathbb{T}) = 1/2 \). Then the corresponding \( h \) is not inner since \( |h| = 1/2 \) on a subset of \( \mathbb{T} \) of positive measure. He also showed that \( h(z)/z \) may be a non-constant outer function. Also, P. Bourdon (\cite{6}) showed that the powers of \( h \) are orthogonal if and only if its Nevanlinna counting function is almost everywhere constant on each circle centered on the origin.

### 5.1 General facts

We first observe that if \( f \in H^2(\mathbb{D}^2) \) and:

\[
f(z_1, z_2) = \sum_{j, k \geq 0} c_{j,k} z_1^j z_2^k,
\]

then we can write:

\[
f(z_1, z_2) = \left( \sum_{k \geq 0} f_k(z_1) \right) z_2^k
\]

with:

\[
f_k(z_1) = \sum_{j \geq 0} c_{j,k} z_1^j,
\]

and:

\[
\|f\|_{H^2(\mathbb{D}^2)}^2 = \sum_{j, k \geq 0} |c_{j,k}|^2 = \sum_{k \geq 0} \|f_k\|_{H^2(\mathbb{D})}^2.
\]

That means that we have an isometric isomorphism:

\[
J : H^2(\mathbb{D}^2) \rightarrow \bigoplus_{k=0}^{\infty} H^2(\mathbb{D})
\]

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defined by \( Jf = (f_k)_{k \geq 0} \).

Now, for symbols \( \Phi \) as in (5.1), we have:

\[
(C\Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\psi(z_1)]^k z_2^k,
\]

so that \( JC\Phi J^{-1} \) appears as the operator \( \bigoplus_k M_{\psi^k} C\phi \) on \( \bigoplus_k H^2(\mathbb{D}) \), where \( M_{\psi^k} \) is the multiplication operator by \( \psi^k \):

\[
[(M_{\psi^k} C\phi) f_k](z_1) = [\psi(z_1)]^k [(f_k \circ \phi)(z_1)].
\]

When \( \Phi \) is as in (5.2), we have:

\[
(C\Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\psi(z_1)]^k [h(z_2)]^k,
\]

with:

\[
\|C\Phi f\|^2 \leq \sum_{k=0}^\infty \|T_k f_k\|^2
\]

and:

\[
T_k = M_{\psi^k} C\phi;
\]

hence \( JC\Phi J^{-1} \) appears as pointwise dominated by the operator \( T = \bigoplus_k T_k \) on \( \bigoplus_k H^2(\mathbb{D}) \). This implies a factorization \( C\Phi = AT \) with \( \|A\| \leq 1 \), so that \( a_n(C\Phi) \leq a_n(T) \) for all \( n \geq 1 \).

We recall the following elementary fact.

**Lemma 5.1.** Let \( (H_k)_{k \geq 0} \) be a sequence of Hilbert spaces and \( T_k : H_k \to H_k \) be bounded operators. Let \( H = \bigoplus_{k=0}^\infty H_k \) and \( T : H \to H \) defined by \( Tx = (T_k x_k)_k \). Then:

1) \( T \) is bounded on \( H \) if and only if \( \sup_{k} \|T_k\| < \infty \);

2) \( T \) is compact on \( H \) if and only if each \( T_k \) is compact and \( \|T_k\| \to 0 \) as \( k \to \infty \).

Going back to the symbols of the form (5.1), we have \( \|M_{\psi^k}\| \leq \|\psi^k\|_{\infty} \leq 1 \), since \( \|\psi\|_{\infty} \leq 1 \); hence \( \|M_{\psi^k} C\phi\| \leq \|C\phi\| \) and the operator \( (M_{\psi^k} C\phi)_k \) is bounded on \( \bigoplus_k H^2(\mathbb{D}) \). Therefore \( C\phi \) is bounded on \( H^2(\mathbb{D}) \).

For approximation numbers, we have the following two facts.

**Lemma 5.2.** Let \( T_k : H_k \to H_k \) be bounded linear operators between Hilbert spaces \( H_k, k \geq 0 \). Let \( H = \bigoplus_k H_k \) and \( T = (T_k)_k : H \to H \), assumed to be compact. Then, for every \( n_1, \ldots, n_K \geq 1 \), and \( 0 \leq m_1 < \cdots < m_K, K \geq 1 \), we have:

\[
a_N(T) \geq \inf_{1 \leq k \leq K} a_{n_k}(T_{m_k}),
\]

where \( N = n_1 + \cdots + n_K \).
Proof. We use the Bernstein numbers $b_n$ (see (1.4)), which are equal to the approximation numbers (see (1.7)). For $k = 1, \ldots, K$, there is an $n_k$-dimensional subspace $E_k$ of $H_{n_k}$ such that:

\[ b_{n_k}(T_{n_k}) \leq \|T_{n_k}x\|, \quad \text{for all } x \in S_{E_k}. \]

Then $E = \bigoplus_{k=1}^{K} E_k$ is an $N$-dimensional subspace of $H$ and for every $x = (x_1, x_2, \ldots) \in E$, we have:

\[ \|Tx\|^2 = \sum_{k \leq K} \|T_{n_k}x_{n_k}\|^2 \geq \sum_{k \leq K} [b_{n_k}(T_{n_k})]^2 \|x_{n_k}\|^2 \]

\[ \geq \inf_{k \leq K} [b_{n_k}(T_{n_k})]^2 \sum_{k \leq K} \|x_{n_k}\|^2 = \inf_{k \leq K} [b_{n_k}(T_{n_k})]^2 \|x\|^2; \]

hence $b_N(T) = \inf_{k \leq K} b_{n_k}(T_{n_k})$, and we get the announced result. \hfill \box

Lemma 5.3. Let $T = \bigoplus_{k=0}^{\infty} T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^{\infty} H_k$. Let $n_0, \ldots, n_K$ be positive integers and $N = n_0 + \cdots + n_K - K$. Then, the approximation numbers of $T$ satisfy:

\[ a_N(T) \leq \max \left( \max_{0 \leq k \leq K} a_{n_k}(T_k), \sup_{k > K} \|T_k\| \right). \]  

(5.4)

Proof. Denote by $S$ the right-hand side of (5.4). Let $R_k$, $0 \leq k \leq K$ be operators on $H_k$ of respective rank $\leq n_k$ such that $\|T_k - R_k\| = a_{n_k}(T_k)$ and let $R = \bigoplus_{k=0}^{K} R_k$ be an operator of rank $\leq n_0 + \cdots + n_K - K - 1 < N$. If $f = \sum_{k=0}^{\infty} f_k \in H$, we see that:

\[ \|Tf - Rf\|^2 = \sum_{k=0}^{K} \|T_k f_k - R_k f_k\|^2 + \sum_{k > K} ||T_k f_k||^2 \]

\[ \leq \sum_{k=0}^{K} a_{n_k}(T_k)^2 \|f_k\|^2 + \sum_{k > K} ||T_k f_k||^2 \leq S^2 \sum_{k=0}^{\infty} \|f_k\|^2 = S^2 \|f\|^2, \]

hence the result. \hfill \box

We give now two corollaries of Lemma 5.3.

Example 1. We first use lens maps. We get:

Theorem 5.4. Let $\lambda_0$ the lens map of parameter $\theta$ and let $\psi : \mathbb{D} \to \mathbb{D}$ such that $\|\psi\|_\infty := c < 1$ and $h$ a Rudin function. We consider:

\[ \Phi(z_1, z_2) = (\lambda_0(z_1), \psi(z_1) h(z_2)). \]

Then, for some positive constant $\beta$, we have, for all $N \geq 1$:

\[ a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}. \]  

(5.5)
Proof. Let \( T_k = M_{\varphi_k} C_{\lambda_k} \). We have \( \|T_k\| \leq e^k \), so \( \sup_{k \geq K} \|T_k\| \leq e^K \). On the other hand, we have \( a_n(T_k) \leq e^k a_n(C_{\lambda_k}) \leq a_n(C_{\lambda_n}) \leq e^{-\beta_0 \sqrt{n}} \) ([16], Theorem 2.1). Taking \( n_0 = n_1 = \cdots = n_K = K^2 \) in Lemma 5.3, we get:
\[
\max_{0 \leq k \leq K} a_n(T_k) \lesssim e^{-\beta_0 K}.
\]
Since \( n_0 + \cdots + n_K - K \approx K^2 \), we obtain \( a_{K^2} \lesssim e^{-\beta_0 K} \), which gives the claimed result, by taking \( \beta = \max(\beta_0, \log(1/c)) \).

**Example 2.** We consider the cusp map \( \chi \). We have:

**Theorem 5.5.** Let \( \chi \) be the cusp map, \( h \) a Rudin function, and \( \psi \) in the unit ball of \( H^\infty \), with \( \|\psi\|_\infty := c < 1 \). Let:
\[
\Phi(z_1, z_2) = \left( \chi(z_1), \psi(z_1) h(z_2) \right).
\]
Then, for positive constant \( \beta \), we have, for all \( N \geq 1 \):
\[
a_N(C_{\Phi}) \lesssim e^{-\beta \sqrt{N/\log N}}.
\]

**Proof.** Let \( T_k = M_{\varphi_k} C_{\lambda_k} \). As above, we have \( \sup_{k \geq K} \|T_k\| \leq e^K \). For the cusp map, we have \( a_n(C_{\lambda_n}) \lesssim e^{-\alpha n/\log n} \) ([20], Theorem 4.3); hence \( a_n(T_k) \lesssim e^{-\alpha n/\log n} \). We take \( n_0 = n_1 = \cdots = n_K = K[\log K] \) (where \( [\log K] \) is the integer part of \( \log K \)). Since \( n_0 + \cdots + n_K \approx K^2[\log K] \), we get, for another \( \alpha > 0 \):
\[
a_{K^2[\log K]}(C_{\Phi}) \lesssim e^{-\alpha K},
\]
which reads: \( a_N(C_{\Phi}) \lesssim e^{-\beta \sqrt{N/\log N}} \), as claimed.

### 5.2 Lower bounds

In this subsection, we give lower bounds for approximation numbers of composition operators on \( H^2 \) of the bidisk, attached to a symbol \( \Phi \) of the previous form \( \Phi(z_1, z_2) = (\varphi(z_1), \psi(z_1) h(z_2)) \) where \( h \) is a Rudin function. The sharpness of those estimates will be discussed in the next subsection. We first need some lemmas in dimension one.

**Lemma 5.6.** Let \( u, v : \mathbb{D} \to \mathbb{D} \) be two non-constant analytic self-maps and \( T = M_u C_u : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \) be the associated weighted composition operator. For \( 0 < r < 1 \), we set \( A = u(r \mathbb{D}) \) and \( \Gamma = \exp\left( -1/\operatorname{Cap}(A) \right) \). Then, for \( 0 < \delta \leq \inf |z| = r |v(z)| \), we have:
\[
a_n(T) \gtrsim \sqrt{1 - r \delta} \Gamma^n.
\]

In this lemma, \( \operatorname{Cap}(A) \) denotes the Green capacity of the compact subset \( A \subseteq \mathbb{D} \) (see [21], § 2.3 for the definition).

For the proof, we need the following result ([26], Theorem 7, p. 353).
**Theorem 5.7** (Widom). Let $A$ be a compact subset of $\mathbb{D}$ and $\mathcal{C}(A)$ be the space of continuous functions on $A$ with its natural norm. Set:

$$
\tilde{d}_n(A) = \inf_E \left[ \sup_{f \in B_{H^\infty}} \text{dist}(f, E) \right],
$$

where $E$ runs over all $(n - 1)$-dimensional subspaces of $\mathcal{C}(A)$ and $\text{dist}(f, E) = \inf_{h \in E} \|f - h\|_{\mathcal{C}(A)}$. Then

(5.7) $$
\tilde{d}_n(A) \geq \alpha e^{-n/\text{Cap}(A)}
$$

for some positive constant $\alpha$.

**Proof of Lemma 5.6.** We apply Theorem 5.7 to the compact set $A = u(r \mathbb{D})$.

Let $E$ be an $(n - 1)$-dimensional subspace of $H^2 = H^2(\mathbb{D})$; it can be viewed as a subspace of $\mathcal{C}(A)$, so, by Theorem 5.7, there exists $f \in H^\infty \subseteq H^2$ with $\|f\|_2 \leq \|f\|_\infty \leq 1$ such that:

$$
\|f - h\|_{\mathcal{C}(A)} \geq \alpha \Gamma_n,
$$

$\forall h \in E$.

Then:

$$
\|v(f \circ u - h \circ u)\|_{\mathcal{C}(rT)} \geq \delta \|f - h\|_{\mathcal{C}(rT)} = \delta \|f - h\|_{\mathcal{C}(A)} \geq \alpha \delta \Gamma_n.
$$

But:

$$
\|v(f \circ u - h \circ u)\|_{\mathcal{C}(rT)} \leq \frac{1}{\sqrt{1 - r^2}} \|v(f \circ u - h \circ u)\|_{H^2};
$$

Hence:

$$
\|Tf - Th\|_{H^2} \geq \alpha \sqrt{1 - r^2} \delta \Gamma_n \geq \alpha \sqrt{1 - r} \delta \Gamma_n.
$$

Since $h$ is an arbitrary function of $E$, we get ($B_{H^2}$ being the unit ball of $H^2$):

$$
\inf_{\dim E < n} \left[ \sup_{f \in B_{H^2}} \text{dist}(Tf, T(E)) \right] \geq \alpha \sqrt{1 - r} \delta \Gamma_n.
$$

But the left-hand side is equal to the Kolmogorov number $d_n(T)$ of $T$ (see [21], Lemma 3.12), and, as recalled in (1.7), in Hilbert spaces, the Kolmogorov numbers are equal to the approximation numbers; hence we obtain:

(5.8) $$
a_n(T) \geq \alpha \sqrt{1 - r} \delta \Gamma_n, \quad n = 1, 2, \ldots,
$$

as announced.

The next lemma shows that some Blaschke products are far away from 0 on some circles centered at 0.

We consider a strongly interpolating sequence $(z_j)_{j \geq 1}$ of $\mathbb{D}$ in the sense that, if $\varepsilon_j := 1 - |z_j|$, then:

(5.9) $$
\varepsilon_{j+1} \leq \sigma \varepsilon_j
$$
and so $\varepsilon_j \leq \sigma^{j-1}\varepsilon_1$, where $0 < \sigma < 1$ is fixed. Equivalently, the sequence $(|z_j|)_{j \geq 1}$ is interpolating. We consider the corresponding interpolating Blaschke product:

$$B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{\varepsilon_j} \frac{z_j - z}{1 - z_j \bar{z}}. \quad \text{(5.10)}$$

The following lemma is probably well-known, but we could find no satisfactory reference (see yet [10] for related estimates) and provide a simple proof.

**Lemma 5.8.** Let $(z_j)_{j \geq 1}$ be a strongly interpolating sequence as in (5.9) and $B$ the associated Blaschke product (5.10).

Then there exists a sequence $r_l := 1 - \rho_l$ such that:

$$C_1 \sigma^l \leq \rho_l \leq C_2 \sigma^l, \quad \text{where } C_1, C_2 \text{ are positive constants, and for which:}$$

$$|z| = r_l \implies |B(z)| \geq \delta,$$

where $\delta > 0$ does not depend on $l$.

**Proof.** Let us denote by $p_l$, $1 \leq p_l \leq l$, the biggest integer such that $\varepsilon_{p_l} \geq \sigma^{l-1}\varepsilon_1$.

We separate two cases.

**Case 1:** $\varepsilon_{p_l} \geq 2\sigma^{l-1}\varepsilon_1$.

Then, we choose $\rho_l = \alpha \sigma^{l-1}\varepsilon_1$ with $\alpha$ fixed, $1 < \alpha < 2$. Since $\rho(\xi, \zeta) \geq \rho(|\xi|, |\zeta|)$ for all $\xi, \zeta \in \mathbb{D}$ (recall that $\rho$ is the pseudo-hyperbolic distance on $\mathbb{D}$), we have the following lower bound for $|z| = r_l$:

$$|B(z)| = \prod_{j=1}^{\infty} \rho(z, z_j) \geq \prod_{j=1}^{p_l} \rho(r_1, |z_j|) \times \prod_{j > p_l} \rho(r_1, |z_j|) := P_1 \times P_2,$$

and we estimate $P_1$ and $P_2$ separately.

We first observe that $\frac{\rho_l}{\varepsilon_{p_l}} \leq \frac{\alpha \sigma^{l-1}\varepsilon_1}{2\sigma^{l-1}\varepsilon_1} \leq \frac{\alpha}{2}$, and then:

$$\frac{\rho_l}{\varepsilon_j} = \frac{\rho_l}{\varepsilon_{p_l} \varepsilon_j} \leq \frac{\alpha}{2} \sigma^{p_l-j}.$$

The inequality $\rho(1 - u, 1 - v) \geq \frac{|u - v|}{(u + v)}$ for $0 < u, v \leq 1$ now gives us:

$$\rho(r_1, |z_j|) \geq \frac{\varepsilon_j - \rho_l}{\varepsilon_j + \rho_l} = \frac{1 - \rho_l/\varepsilon_j}{1 + \rho_l/\varepsilon_j} \geq \frac{1 - (\alpha/2) \sigma^{p_l-j}}{1 + (\alpha/2) \sigma^{p_l-j}}, \quad \text{for } j \leq p_l,$$

and:

$$P_1 \geq \prod_{k=0}^{\infty} \left(1 - \frac{\alpha}{2} \sigma^k\right).$$
Similarly:
\[ \frac{\varepsilon_{p_l + 1}}{\rho_l} \leq \frac{\sigma^{l-1} \varepsilon_1}{\alpha} \leq \frac{1}{\alpha} \]
and:
\[ \frac{\varepsilon_j}{\rho_l} \leq \frac{1}{\alpha} \sigma^{j-p_l-1} \quad \text{for } j > p_l ; \]
so that:
\[ \rho(r_l, |z_j|) \geq \frac{\rho_l - \varepsilon_j}{\rho_l + \varepsilon_j} = \frac{1 - \varepsilon_j / \rho_l}{1 + \varepsilon_j / \rho_l} \geq \frac{1 - \alpha^{-1} \sigma^{j-p_l-1}}{1 + \alpha^{-1} \sigma^{j-p_l-1}}, \quad \text{for } j > p_l , \]
and
\[ P_2 \geq \prod_{k=0}^{\infty} \left( \frac{1 - \alpha^{-1} \sigma^k}{1 + \alpha^{-1} \sigma^k} \right). \]

Finally, the condition of lower and upper bound for \( \rho_l \) is fulfilled by construction.

**Case 2:** \( \varepsilon_{p_l} \leq 2 \sigma^{l-1} \varepsilon_1 \).

Then, we choose \( \rho_l = a \varepsilon_{p_l} \) with \( \sigma < a < 1 \) fixed. Computations exactly similar to those of Case 1 give us:

\[ |B(z)| \geq \prod_{k=0}^{\infty} \left( \frac{1 - a \sigma^k}{1 + a \sigma^k} \right) \times \prod_{k=0}^{\infty} \left( \frac{1 - a^{-1} \sigma^k}{1 + a^{-1} \sigma^k} \right) =: \delta > 0 , \quad \text{for } |z| = r_l . \]

Moreover, in this case:
\[ a \sigma^{l-1} \varepsilon_1 \leq \rho_l \leq 2 a \sigma^{l-1} \varepsilon_1 , \]
and the proof is ended. \( \square \)

Now, we have the following estimation.

**Theorem 5.9.** Let \( \phi, \psi : \mathbb{D} \rightarrow \mathbb{D} \) be two non-constant analytic self-maps and \( \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)) \), where \( h \) is inner.

Let \( (r_l)_{l \geq 1} \) be an increasing sequence of positive numbers with limit 1 such that:
\[ \inf_{|z| = r_l} |\psi(z)| \geq \delta_l > 0 , \]
with \( \delta_l \leq e^{-1/\text{Cap}(A_l)} \), where \( A_l = \phi(r_l \mathbb{D}) \).

Then the approximation numbers \( a_N(C_\Phi) \), \( N \geq 1 \), of the composition operator \( C_\Phi : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2) \) satisfy:
\[ a_N(C_\Phi) \geq \sup_{l \geq 1} \left[ \sqrt{1 - r_l} \exp \left( -8 \sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)} \right) \right] , \]
where:
\[ \Gamma_l = e^{-1/\text{Cap}(A_l)} . \]
Proof. Since $h$ is inner, the sequence $(h^k)_{k \geq 0}$ is orthonormal in $H^2$ and hence $a_n(C_\Phi) = a_n(T)$ for all $n \geq 1$, where $T = \bigoplus_{k=0}^{\infty} T_k$ and $T_k = M_{\psi^k} C_\phi$. Then Lemma 5.6 gives:

\begin{equation}
(5.20) \quad a_n(T_k) \gtrsim \sqrt{1 - r_l \delta_k^l \Gamma^n_l}
\end{equation}

for all $n \geq 1$ and all $k \geq 0$.

Let now:

\begin{equation}
(5.21) \quad p_l = \left[ \log(1/\delta_l) / \log(1/\Gamma_l) \right],
\end{equation}

where $[.]$ stands for the integer part, and:

\begin{equation}
(5.22) \quad n_k = p_l k, \quad \text{for} \quad k = 1, \ldots, K.
\end{equation}

By Lemma 5.2, applied with $m_k = k$ (i.e. to $H_1, \ldots, H_K$), we have, if $N = n_1 + \cdots + n_K$:

\[
a_N(T) \gtrsim \inf_{1 \leq k \leq K} \alpha \sqrt{1 - r_l \delta_k^l \Gamma^n_l} = \alpha \sqrt{1 - r_l \delta^K_l \Gamma^n_K}.
\]

But, since $p_l \leq \log(1/\delta_l)/\log(1/\Gamma_l)$:

\[
\delta^K_l \Gamma^n_K = \exp \left[ - \left( K \log(1/\delta_l) + p_l K \log(1/\Gamma_l) \right) \right] \geq \exp[-2K \log(1/\delta_l)].
\]

Since:

\[
N = p_l K \frac{K + 1}{2} \geq p_l \frac{K^2}{4} \geq \frac{K^2 \log(1/\delta_l)}{16 \log(1/\Gamma_l)};
\]

we get:

\[
\delta^K_l \Gamma^n_K \geq \exp \left[ - 8 \sqrt{N} \sqrt{\log(1/\delta_l) \log(1/\Gamma_l)} \right],
\]

and the result ensues. \hfill \qed

Example 1. We take $\phi = \lambda_\theta$, a lens map, and $\psi = B$, a Blaschke product associated to a strongly regular sequence, as defined in (5.10); then we get:

Theorem 5.10. Let $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ be defined by:

\[
\Phi(z_1, z_2) = (\lambda_\theta(z_1), c B(z_1) h(z_2)),
\]

where $B$ is a Blaschke product as in (5.10), $0 < c < 1$, and $h$ is an arbitrary inner function, we have, for some positive constant $b$, for all $N \geq 1$:

\begin{equation}
(5.23) \quad a_N(C_\Phi) \gtrsim \exp(-b N^{1/3}) = \exp(-b \sqrt{N}/N^{1/6}).
\end{equation}

In particular $\beta_2(C_\Phi) = \beta_2^+ (C_\Phi) = 1$.

Remark. We saw in Theorem 5.4 that this is the exact size, since we have:

\[a_N(C_\Phi) \lesssim e^{-b N^{1/3}}.\]
Proof. By Lemma 5.8, there is a sequence of numbers \( r_l \approx \sigma \) such that \(|B(z)| \geq \delta\) for \(|z| = r_l\), where \( \delta \) is a positive constant (depending on \( \sigma \)). Since \( \lambda_{b}(0) = 0 \), we have:

\[
\text{diam}_{\mu}(A_l) \geq \lambda_{b}(r_l) \gtrsim 1 - (1 - r_l)^{\theta};
\]

hence, by [21], Theorem 3.13, we have:

\[
\text{Cap} (A_l) \gtrsim \log \frac{1}{1 - r_l} \gtrsim l,
\]

or, equivalently: \( \Gamma_l \gtrsim e^{-b/l} \), some some \( b > 0 \). Then (5.18) gives, for all \( l \geq 1 \) (with another \( b \)):

\[
a_{N}(C_{\Phi}) \gtrsim \exp \left[ -b \left( l + \frac{\sqrt{N}}{\sqrt{l}} \right) \right].
\]

Taking \( l = N^{1/3} \), we get the result. \( \square \)

Example 2. By taking the cusp instead of a lens map, we obtain a better result, close to the extremal one.

Theorem 5.11. Let \( \Phi(z_1, z_2) = (\chi(z_1), cB(z_1), h(z_2)) \), where \( \chi \) is the cusp map, \( B \) a Blaschke product as in (5.10), \( 0 < c < 1 \), and \( h \) an arbitrary inner function. Then, for all \( N \geq 1 \):

\[
a_{N}(C_{\Phi}) \gtrsim e^{-b \sqrt{N}/\log N}.
\]

In particular \( \beta_2(C_{\Phi}) = 1 \).

Remark. We saw in Theorem 5.5 that this is the exact size, since we have:

\[
a_{N}(C_{\Phi}) \lesssim e^{-\beta \sqrt{N}/\log N}.
\]

Proof. The proof is the same as that of Proposition 5.10, except that, for the cusp map, we have (note that \( \chi(0) = 0 \)):

\[
\text{diam}_{\mu}(A_l) \geq \chi(r_l).
\]

But when \( r \) goes to 1:

\[
1 - \chi(r) \sim \frac{\pi (\sqrt{2} - 1)}{2} \frac{1}{\log \left( 1/(1 - r) \right)}
\]

(see [20], Lemma 4.2). Hence, by [21], Theorem 3.13, again, we have:

\[
\text{Cap} (A_l) \gtrsim \log \left( \log \left( 1/(1 - r_l) \right) \right),
\]

so \( \Gamma_l \gtrsim e^{-b/\log l} \). Then, (5.18) gives (with another \( b \)):

\[
a_{N}(C_{\Phi}) \gtrsim \exp \left[ -b \left( l + \frac{\sqrt{N}}{\log l} \right) \right].
\]

In taking \( l = \sqrt{N}/\log N \), we get the announced result. \( \square \)
5.3 Upper bounds

All previous results point in the direction that, if $\|\Phi\|_\infty = 1$, then however small $a_n(C_\Phi)$ is, it will always be larger than $\alpha e^{-\beta \varepsilon_n \sqrt{n}}$ with $\varepsilon_n \to 0^+$, as this is the case in dimension one (with $n$ instead of $\sqrt{n}$). But Theorem 5.12 to follow shows that we cannot hope, in full generality, to get the same result in dimension $d \geq 2$, and that other phenomena await to be understood. Here is our main result. It shows that, even for a truly 2-dimensional symbol $\Phi$, we can have $\|\Phi\|_\infty = 1$ and nevertheless $\beta^2_2(C_\Phi) < 1$, in contrast to the 1-dimensional case where (1.1) holds.

**Theorem 5.12.** There exist a map $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ such that:

1) the composition operator $C_\Phi : H^2(D^2) \to H^2(D^2)$ is bounded and compact;

2) we have $\|\Phi\|_\infty = 1$ and $\Phi$ is truly 2-dimensional, so that $\beta^2_2(C_\Phi) > 0$;

3) the singular numbers satisfy $a_n(C_\Phi) \leq \alpha e^{-\beta \sqrt{n}}$ for some positive constants $\alpha, \beta$, in other terms $\beta^2_2(C_\Phi) < 1$.

**Proof.** Let $\theta < 1$ be fixed, and $\lambda_\theta$ be the corresponding lens map. We set:

$$
\begin{align*}
\phi &= \frac{1 + \lambda_\theta}{2} \\
S(z) &= \left(\frac{1 + z}{1 - z}\right)^\theta \\
w &= e^{-S} \\
\psi &= \frac{1}{2} (w \circ \phi).
\end{align*}
$$

Note that $\|\phi\|_\infty = 1$.

Setting $\delta = \cos \theta \pi / 2 > 0$, we have for $z \in \mathbb{D}$:

$$
|1 - \phi(z)| = \frac{1}{2} |1 - \lambda_\theta(z)| = \left| \frac{(1 - z)^\theta}{(1 - z)^\theta + (1 + z)^\theta} \right| \leq \frac{|1 - z|^\theta}{\delta}.
$$

Indeed, the argument $\alpha$ of $(1 \pm z)^\theta$ satisfies $|\alpha| \leq \theta \pi / 2$ for $z \in \mathbb{D}$, and we get:

$$
|(1 - z)^\theta + (1 + z)^\theta| \geq \text{Re}[(1 - z)^\theta + (1 + z)^\theta] \geq \delta |(1 + z)^\theta + |1 - z|^\theta| \geq \delta.
$$

We also see that $\phi(\mathbb{D})$ touches the boundary $\partial \mathbb{D}$ only at 1 in a non-tangential way, meaning that for some constant $C > 1$:

$$
1 - |\phi(z)| \geq \frac{1}{C} |1 - \phi(z)|, \quad \forall z \in \mathbb{D}.
$$

Now, we have the following two inequalities:

$$
\begin{align*}
\text{Re} z \geq 0 &\implies |w(z)| \leq \exp \left( - \frac{\delta}{|1 - z|^\theta} \right) \\
z \in \mathbb{D} &\implies |\psi(z)| \leq \frac{1}{2} \exp \left( - \frac{\delta^2}{|1 - z|^\theta} \right).
\end{align*}
$$
Indeed, \( \Re S(z) \geq \delta |S(z)| \geq \delta |1-z|^{-\theta} \) when \( \Re z \geq 0 \), giving (5.25), and (5.24) and (5.25) imply, since \( \Re \phi(z) \geq 0 \):

\[
|\psi(z)| = \frac{1}{2} |w(\phi(z))| \leq \frac{1}{2} \exp \left( -\frac{\delta}{1 - \phi(z)|^\theta} \right) \leq \frac{1}{2} \exp \left( -\frac{\delta^2}{|1 - z|^\theta} \right).
\]

We now set:

\[
\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))\]

with \( h \) a Rudin function.

Observe that the components \( \phi \) and \( \psi \otimes h \) of \( \Phi \) can be in the bidisk algebra \( A(D^2) \), because we can take for \( h \) a finite Blaschke product, and moreover \( \phi \in A(D) \) while \( \psi = (1/2) w \circ \phi \in A(D) \) as well, since \( w \in A(D) \) with \( w(1) = 0 \); this is due to the presence of the parameter \( \theta < 1 \).

Now, 1) follows from the orthogonal model presented in Section 5.1, because \( \|\psi\|_\infty < 1 \).

The assertion 2) follows from [2], Theorem 3.1, since \( \|\phi\|_\infty = 1 \).

We now prove 3).

As observed, \( C_\phi \) can be viewed as a direct sum \( T = \bigoplus_{k=0}^{\infty} T_k \) acting on a Hilbertian sum \( H = \bigoplus_{k=0}^{\infty} H_k \), where \( T_k \) acts on a copy \( H_k \) of \( H^2(D) \) with:

\[
T_k = M_{\psi^k} C_\phi.
\]

We fix the positive integer \( n \). The rest of the proof will consist of three lemmas.

**Lemma 5.13.** We have \( \|T_k\| \leq 2^{-n} \) for \( k > n \).

**Proof.** Indeed, since \( \phi(0) = 1/2 \), we know that \( \|C_\phi\| \leq \sqrt{\frac{1+\phi(0)}{1-\phi(0)}} = \sqrt{3} \leq 2 \), so that \( \|T_k\| \leq \|\psi^k\|_\infty \|C_\phi\| \leq 2^{-k} \times 2 \).

**Lemma 5.14.** Set \( b = a/\delta^2 \) where \( a > 0 \) is given by \( e^{-a} = 4C/\sqrt{16C^2 + 1} \) and \( C \) is as in (2.1). Let \( m_k \) be the smallest integer such that \( k \delta^2 2^{m_k} \theta^{a'} \geq an \); namely:

\[
m_k = \left\lfloor \frac{\log(bn/k)}{\theta^2 \log 2} \right\rfloor + 1,
\]

where \( \lfloor . \rfloor \) stands for the integer part. Then, with \( a' = \min(\log 2, a) \):

\[
a_{nm_k+1}(T_k) \lesssim e^{-a'n}.
\]

**Proof.** This follows from Theorem 2.3 applied with \( w = \psi^k \), \( R = k \delta^2 \) and \( \theta \) changed into \( \theta^2 \). This is possible thanks to (5.26) and to Lemma 5.13. Moreover we have adjusted \( m_k \) so as to make the two terms in Theorem 2.3 of the same order.

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Lemma 5.15. The dimension $d := \sum_{k=0}^{n} n m_k$ satisfies, for some positive constant $\alpha$:
\[ d \leq \alpha n^2. \]

Proof. Indeed, it is well-known that:
\[ \sum_{k=1}^{n} \log k = n \log n - n + O(\log n), \]
and, in view of (5.28), we have $m_k \leq \alpha'_\theta \log(b n/k) \leq \alpha''_\theta (\log n)$; hence:
\[ \sum_{k=1}^{n} m_k \leq \alpha''_\theta [n \log n - (n \log n - n + O(\log n))] = \alpha''_\theta n + O(\log n), \]
and we get $d \leq \alpha''_\theta n^2 + O(n \log n) \leq \alpha n^2$.

Alternatively, we could have used a Riemann sum for the function $\log(1/x)$ on $(0, 1]$.

Finally, putting things together and using as well Proposition 5.3 with $K = n$ and $n_k = nm_k + 1$ so that $(\sum_{k=0}^{n} n_k) - n = (\sum_{k=0}^{n} n m_k) + 1 = d + 1$, we get:
\[ a_n^*(T) \lesssim a_d(T) \leq \alpha e^{-\beta n} \]
with positive constants $\alpha$, $\beta$. This ends the proof of Theorem 5.12.

6 Monge-Ampère capacity and applications

6.1 Definition

Let $K$ be a compact subset of $\mathbb{D}^m$ (in this section, for notational reasons, we denote the dimension by $m$ instead of $d$). The Monge-Ampère capacity of $K$ has been defined by Bedford and Taylor ([3]; see also [13], § 5 or [11], Chapter 1) as:
\[ \text{Cap}_m(K) = \sup \left\{ \int_K (dd^c u)^m ; u \in \text{PSH} \text{ and } 0 \leq u \leq 1 \right\}, \]
where $\text{PSH}$ is the set of plurisubharmonic functions on $\mathbb{D}^m$, $dd^c = 2i\partial\bar{\partial}$, and $(dd^c)^m = dd^c \wedge \cdots \wedge dd^c$ ($m$ times). When $u \in \text{PSH} \cap C^2(\mathbb{D}^m)$, we have:
\[ (dd^c u)^m = 4^m m! \det \left( \frac{\partial^2 u}{\partial z_i \partial \bar{z}_k} \right) dV(z), \]
where $dV(z) = (i/2)^m dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m$ is the usual volume in $\mathbb{C}^m$. A more convenient formula (because $\mathbb{D}^m$ is bounded and hyperconvex: see [11], p. 80, for the definition) is:
\[ \text{Cap}_m(K) = \int_K (dd^c u_K)^m, \]

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where $u_K^*$ is called the extremal function of $K$ and is the upper semi-continuous regularization of:

$$u_K = \sup\{ v \in PSH ; v \leq 0 \text{ and } v \leq -1 \text{ on } K \},$$

but we will not need that.

As in [27], we set:

$$\tau_m(K) = \frac{1}{(2\pi)^m} \text{Cap}_m(K).$$

For $m = 1$, $\tau(K) := \tau_1(K)$ is equal to the Green capacity $\text{Cap}(K)$ of $K$ with respect to $\mathbb{D}$, with the definition used in [21] (see [13], Theorem 8.1, where a factor $2\pi$ is introduced).

We further set:

$$\Gamma_m(K) = \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right].$$

We proved in [21] that, for $m = 1$, and $\varphi : \mathbb{D} \to r\mathbb{D}$, with $0 < r < 1$, we have:

$$\beta_1(C_\varphi) = \Gamma_1(\varphi(\mathbb{D})).$$

The goal of this section is to see that Theorem 5.12 shows that this no longer holds for $m = 2$.

### 6.2 A seminal example

In one variable, our initial motivation had been the simple-minded example $\varphi(z) = rz$, $0 < r < 1$, for which $C_\varphi(z^n) = r^n z^n$, implying $a_n(C_\varphi) = r^{n-1}$ and $\beta_1(C_\varphi) = r$. If $K = \varphi(\mathbb{D}) = \mathbb{D}(0, r)$, we have $\text{Cap}(K) = \frac{1}{\log(1/r)}$ and $\Gamma_1(K) = r$, so that $\beta_1(C_\varphi) = \Gamma_1(K)$. Let us examine the multivariate example (where $0 < r_j < 1$):

$$\Phi(z_1, z_2, ..., z_m) = (r_1 z_1, r_2 z_2, ..., r_m z_m).$$

If $K = \Phi(\mathbb{D}^m)$, we have $K = \prod_{k=1}^m \mathbb{D}(0, r_k)$, and hence ([5], Theorem 3):

$$\tau_m(K) = \prod_{k=1}^m \frac{1}{\log(1/r_k)}.$$

On the other hand, $C_\Phi(z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}) = r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}$ so that the sequence $(a_n)$ of approximation numbers of $C_\Phi$ is the non-increasing rearrangement of the numbers $r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m}$. It is convenient to state the following simple lemma.

**Lemma 6.1.** Let $\lambda_1, \ldots, \lambda_m$ be positive numbers. Let $N_A$ be the number of $m$-tuples of non-negative integers $(n_1, \ldots, n_m)$ such that $\sum_{k=1}^m \lambda_k n_k \leq A$. Then, as $A \to \infty$:

$$N_A \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}.$$
Indeed, just apply Karamata’s tauberian theorem (see [12] p. 30) to the
generalized Dirichlet series:

\[ S(\varepsilon) := \prod_{k=1}^{m} \frac{1}{1 - e^{-\lambda_k \varepsilon}} = \sum_{n_1, \ldots, n_m \geq 0} e^{-(\sum_{k=1}^{m} \lambda_k n_k) \varepsilon}; \]

we have \( S(\varepsilon) \sim \frac{\varepsilon^{-m}}{(\lambda_1 \cdots \lambda_m) m!} \) as \( \varepsilon \to 0^+ \).

Let now \( N \) be a positive integer and \( \varepsilon = a_N \). Setting \( \lambda_k = \log(1/r_k) \) and \( A = \log(1/\varepsilon) \), we see that \( N \) is the number of \( m \)-tuples \( (n_1, \ldots, n_m) \) of non-negative integers such that \( r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \geq \varepsilon \), i.e. such that \( \sum_{k=1}^{m} \lambda_k n_k \leq A \). This number \( N \) is hence nothing but the number \( N_A \) of the previous lemma, so that:

\[ N \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}. \]

Inverting this formula, we get:

\[ a_N(C_{\Phi}) = \exp \left[ - (1+o(1)) (m!/(\lambda_1 \lambda_2 \cdots \lambda_m) N)^{1/m} \right] \]

and:

\[ \beta_m(C_{\Phi}) = \exp \left[ - (m!/(\lambda_1 \lambda_2 \cdots \lambda_m)^{1/m}) \right] = \Gamma_m(K), \]

in view of (6.2) and (6.4).

On the view of the simple-minded previous example, the extension of the spectral radius formula (6.3) to the multivariate case holds, and it is tempting to conjecture that this is a general phenomenon as in dimension one, all the more as the following extension of Widom’s theorem was proved by Zakharyuta, based on the solution by S. Nivoche of Zakharyuta’s conjecture ([22]); see also [27], Theorem 5.4. A compact subset \( K \) of \( \mathbb{D}^m \) is said to be regular if its extremal function \( u^*_K \) is continuous on \( \mathbb{D}^m \).

**Theorem 6.2** ([27], Theorem 5.6). Let \( K \) be a regular compact subset of \( \mathbb{D}^m \) and \( J : H^\infty(\mathbb{D}^m) \to C(K) \) the canonical injection; then the Kolmogorov numbers \( d_n(J) \) satisfy:

\[ \lim_{n \to \infty} \left[ d_n(J) \right]^{1/n} = \exp \left[ - \left( \frac{m!}{r_m(K)} \right)^{1/m} \right]. \]

Note that the right side is nothing but \( \Gamma_m(K) \).

We shall see in section 6.4 that, for the symbol \( \Phi \) of Theorem 5.12, the compact subsets \( \Phi(r \mathbb{D}^m), 0 < r < 1 \), are not regular.

### 6.3 Upper bound

For the upper bound, the situation behaves better, as stated in the following theorem.
\begin{theorem}[\cite{27}, Proposition 6.1] \label{thm:6.3}
Let $K$ be a compact subset of $\mathbb{D}^m$ with non-void interior. Then:
\begin{equation}
\limsup_{n \to \infty} \left[ d_n(J) \right]^{1/n^{1/m}} \leq \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right].
\end{equation}

Note that $(K, \mathbb{D}^m)$ is a condenser since $K$ has non-void interior. We deduce the following upper bound.
\begin{theorem}
Let $\Phi$ be an analytic self-map of $\mathbb{D}^m$ with $\|\Phi\|_\infty = \rho < 1$, thus inducing a compact composition operator on $H^2(\mathbb{D}^m)$. Then we have:
\begin{equation}
\beta_m^+ (C_\Phi) \leq \Gamma_m (\overline{\Phi(\mathbb{D}^m)}).
\end{equation}
\end{theorem}
\begin{proof}
This proof provides in particular a simplification of that given in [21] in dimension $m = 1$.
\end{proof}

Changing $n$ into $n^m$, Theorem 6.3 means that for every $\varepsilon > 0$, there exists an $(n^m - 1)$-dimensional subspace $V$ of $\mathcal{C}(K)$ such that, for any $g \in H^\infty(\mathbb{D}^m)$, there exists $h \in V$ such that:

\begin{equation}
\|g - h\|_{\mathcal{C}(K)} \leq C_\varepsilon (1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n \|g\|_\infty.
\end{equation}

Let $l$ be an integer to be adjusted later, and $f(z) = \sum_\alpha b_\alpha z^\alpha \in B_{HZ}$, as well as $g(z) = \sum_{|\alpha| \leq l} b_\alpha z^\alpha$. We first note that (with $M_m$ depending only on $m$ and $\rho$, and since the number of $\alpha$'s such that $|\alpha| \leq p$ is $O(p^m)$):
\begin{equation}
\sum_{|\alpha| > l} \rho^{2|\alpha|} \leq M_m \sum_{p > l} \rho^{mp} \rho^{2p} \leq M_m l^m \frac{\rho^{2l}}{(1 - \rho^2)^{m+1}}.
\end{equation}

We next observe that, by the Cauchy-Schwarz and Parseval inequalities:
\begin{equation}
\|g\|_\infty \leq M_m l^{m/2},
\end{equation}
and
\begin{equation}
|f(z) - g(z)| \leq M_m l^{m/2} \frac{|z|^l}{(1 - |z|^2)^{(m+1)/2}}, \quad \forall z \in \mathbb{D}^m.
\end{equation}

where $|z| := \max_{1 \leq j \leq m} |z_j|$ if $z = (z_1, \ldots, z_m)$.

The subspace $F$ formed by functions $v \circ \Phi$, for $v \in V$, can be viewed as a subspace of $L^\infty(\mathbb{T}^m) \subseteq L^2(\mathbb{T}^m)$ with respect to the Haar measure of $\mathbb{T}^m$, the distinguished boundary of $\mathbb{D}^m$ (indeed, we can write $(v \circ \Phi)^* = v \circ \Phi^*$, where $\Phi^*$ denotes the almost everywhere existing radial limits of $\Phi(rz)$, which belong to $K$). Let finally $E = P(F) \subseteq H^2(\mathbb{D}^m)$ where $P : L^2(\mathbb{T}^m) \to H^2(\mathbb{T}^m) = H^2(\mathbb{D}^m)$ is the orthogonal projection. This is a subspace of $H^2$ with dimension $< n^m$.

Set temporarily $\eta = C_\varepsilon (1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n$. It follows from (6.7) and (6.8) that, for some $h \in V$:
\begin{equation}
\|g - h\|_{\mathcal{C}(K)} \leq \eta \|g\|_\infty \leq \eta M_m l^{m/2}
\end{equation}
and hence:
\[ \|g \circ \Phi - h \circ \Phi\|_2 \leq \|g \circ \Phi - h \circ \Phi\|_\infty \leq \eta M_m l^{m/2}, \]

implying by orthogonal projection:
\[ \text{dist} (C_\Phi g, E) \leq \|g \circ \Phi - P(h \circ \Phi)\|_2 \leq \eta M_m l^{m/2}. \]

Now, since \( C_\Phi f(z) - C_\Phi g(z) = f(\Phi(z)) - g(\Phi(z)) \), (6.9) gives:
\[ \|C_\Phi f - C_\Phi g\|_2 \leq \|C_\Phi f - C_\Phi g\|_\infty \leq M_m l^{m/2} \]
and hence:
\[ \text{dist} (C_\Phi f, E) \leq M_m l^{m/2} \left( \frac{\rho^j}{(1 - \rho^2)(m+1)/2} + C_\varepsilon (1 + \varepsilon)^n \left[ \Gamma_m(K) \right]^n \right). \]

It ensues, since \( a_N(C_\Phi) = d_N(C_\Phi) \), that:
\[ \left[ a_n(C_\Phi) \right]^{1/n} \leq (M_m l^{m/2})^{1/n} \left( \frac{\rho^j/n}{(1 - \rho^2)(m+1)/2} + C_\varepsilon^{1/n} (1 + \varepsilon) \Gamma_m(K) \right). \]

Taking now for \( l \) the integer part of \( n \log n \), and passing to the upper limit as \( n \to \infty \), we obtain (since \( l/n \to \infty \) and \( (\log l)/n \to 0 \)):
\[ \beta_m^+(C_\Phi) \leq (1 + \varepsilon) \Gamma_m(K), \]
and Theorem 6.4 follows. \( \square \)

### 6.4 Exact asymptotic

From Theorem 6.2 we deduce the following result.

**Theorem 6.5.** Let \( \Phi: D^m \to D^m \) be an analytic self-map such that \( \|\Phi\|_\infty = 1 \). Assume moreover that for \( 0 < r < 1 \), the compact subsets \( K_r = \Phi(rD^m) \) are regular. Then:
\[ \beta_m^-(C_\Phi) = \beta_m(C_\Phi) = 1. \]

**Proof.** Without loss of generality, we may assume that \( \Phi(0) = 0 \).

Let \( r_j < 1 \) with \( r_j \to 1^- \), and \( K_j = \Phi(r_jD^m) \). We can apply Theorem 6.2 to \( K_j \). Let \( E \) be a subspace of \( H^2 \) with dimension \( < n^m \). This space can be considered as a subspace of \( C(K_j) \). By Theorem 6.2, and repeating word for word the proof of [21], we can find \( g \in B_{H^2} \) with:
\[ \|g - h\|_{C(K_j)} \geq c_\varepsilon (1 - \varepsilon)^n \left[ \Gamma_m(K_j) \right]^n \]
for all \( h \in E \). Equivalently:
\[ \|C_\Phi(g) - C_\Phi(h)\|_{C(r_jD^m)} \geq c_\varepsilon (1 - \varepsilon)^n \left[ \Gamma_m(K_j) \right]^n. \]
This implies that:
\[ \| C_\Phi(g) - C_\Phi(h) \|_2 \geq \eta_j c_\varepsilon (1 - \varepsilon)^n \left[ \Gamma_m(K_j) \right]^{n}, \]
where \( \eta_j > 0 \) only depends on \( j \).

Since \( a_n(C_\Phi) = d_n(C_\Phi) \), we thus get:
\[ a_n(C_\Phi) \geq \eta_j c_\varepsilon (1 - \varepsilon)^n \left[ \Gamma_m(K_j) \right]^{n}. \]

Taking \( n \)-th roots and passing to the lower limit, we obtain that \( \beta_m^{-}(C_\Phi) \geq (1 - \varepsilon) \Gamma_m(K_j) \). Finally, letting \( \varepsilon \) going to 0, we get:
\[ (6.10) \quad \beta_m^{-}(C_\Phi) \geq \Gamma_m(K_j). \]

Now, since \( \| \Phi \|_\infty = 1 \), there exists a sequence of points \( a_j \in K_j \) with \( \| a_j \|_\infty \to 1 \) and we can assume that \( |\pi_1(a_j)| \to 1 \), where \( \pi_1 \) is the projection on the first coordinate. By [13], Theorem 13.1, we have \( \text{Cap}_m(K_j) \geq \text{Cap}_1[\pi_1(K_j)] \). Since \( \text{Cap}_1[\pi_1(K_j)] = 2\pi \text{ Cap}[\pi_1(K_j)] \to +\infty \) (see [21], Theorem 3.13), we get that \( \Gamma_m(K_j) \to 1 \), and that ends the proof.

**Corollary 6.6.** For the symbol \( \Phi \) constructed in Theorem 5.12, the compact subsets \( K = \Phi(rD^2) \) are not all regular.

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**References**


Daniel Li
Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec
Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59 655 VILLENEUVE D’ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Apartado de Correos 1160 41 080 SEVILLA, SPAIN
piazza@us.es