

Some examples of composition operators and their approximation numbers on the Hardy space of the bi-disk

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Abstract. We give examples of composition operators C_Φ on $H^2(\mathbb{D}^2)$ showing that the condition $\|\Phi\|_\infty = 1$ is not sufficient for their approximation numbers $a_n(C_\Phi)$ to satisfy $\lim_{n \rightarrow \infty} [a_n(C_\Phi)]^{1/\sqrt{n}} = 1$, contrary to the 1-dimensional case. We also give a situation where this implication holds. We make a link with the Monge-Ampère capacity of the image of Φ .

Key-words: approximation numbers; Bergman space; bidisk; composition operator; Green capacity; Hardy space; Monge-Ampère capacity; weighted composition operator.

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1 Introduction and notation

1.1 Introduction

The purpose of this paper is to continue the study of composition operators on the polydisk initiated in [2], and in particular to examine to what extent one of the main results of [21] still holds.

Let H be a Hilbert space and $T: H \rightarrow H$ a bounded operator. Recall that the *approximation numbers* of T are defined as:

$$a_n(T) = \inf_{\text{rank } R < n} \|T - R\|, \quad n \geq 1,$$

and we have:

$$\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq \cdots$$

The operator T is compact if and only if $a_n(T) \xrightarrow{n \rightarrow \infty} 0$.

For $d \geq 1$, we define:

$$\begin{cases} \beta_d^-(T) &= \liminf_{n \rightarrow \infty} [a_{n^d}(T)]^{1/n} \\ \beta_d^+(T) &= \limsup_{n \rightarrow \infty} [a_{n^d}(T)]^{1/n} \end{cases}$$

We have:

$$0 \leq \beta_d^-(T) \leq \beta_d^+(T) \leq 1,$$

and we simply write $\beta_d(T)$ in case of equality.

It may well happen in general (consider diagonal operators) that $\beta_d^-(T) = 0$ and $\beta_d^+(T) = 1$.

When $H = H^2(\mathbb{D})$ is the Hardy space on the open unit disk \mathbb{D} of \mathbb{C} , and $T = C_\Phi$ is a composition operator, with $\Phi: \mathbb{D} \rightarrow \mathbb{D}$ a non-constant analytic function, we always have ([19]):

$$\beta_1^-(C_\Phi) > 0,$$

and one of the main results of [19] is the equivalence:

$$(1.1) \quad \beta_1^+(C_\Phi) < 1 \iff \|\Phi\|_\infty < 1.$$

An alternative proof was given in [21], as a consequence of a so-called ‘‘spectral radius formula’’, which moreover shows that:

$$\beta_1^-(C_\Phi) = \beta_1^+(C_\Phi).$$

In [2], for $d \geq 2$, it is proved that, for a bounded symmetric domain $\Omega \subseteq \mathbb{C}^d$, if $\Phi: \Omega \rightarrow \Omega$ is analytic, such that $\Phi(\Omega)$ has a non-void interior, and the composition operator $C_\Phi: H^2(\Omega) \rightarrow H^2(\Omega)$ is compact, then:

$$\beta_d^-(C_\Phi) > 0.$$

On the other hand, if Ω is a product of balls, then:

$$\|\Phi\|_\infty < 1 \implies \beta_d^+(C_\Phi) < 1.$$

We do not know whether the converse holds and the purpose of this paper is to study some examples towards an answer.

The paper is organized as follows. Section 1 is this short introduction, as well as some notations and definitions on singular numbers of operators and Hardy spaces of the polydisk to follow. Section 2 contains preliminary results on weighted composition operators in one variable, which surprisingly play an important role in the study of non-weighted composition operators in two variables. Section 3 studies the case of symbols with ‘‘separated’’ variables. Our main one variable result extends in this case. Section 4 studies the ‘‘glued case’’ $\Phi(z_1, z_2) = (\phi(z_1), \phi(z_1))$ for which even boundedness is an issue. Here, the

Bergman space $B^2(\mathbb{D})$ enters the picture. Section 5 studies the case of “triangularly separated” variables. This section lets direct Hilbertian sums of weighted composition operators in one variable appear, and it contains our main result: an example of a symbol Φ satisfying $\|\Phi\|_\infty = 1$ and yet $\beta_2^+(C_\Phi) < 1$. The final Section 6 discusses the role of the Monge-Ampère pluricapacity, which is a multivariate extension of the Green capacity in the disk. Even though, as evidenced by our counterexample of Section 5, this capacity will not capture all the behavior of the parameter $\beta_m(C_\Phi)$, some partial results are obtained, relying on theorems of S. Nivoche and V. Zakharyuta.

1.2 Notation

We say that an analytic map $\Phi: \mathbb{D}^d \rightarrow \mathbb{D}^d$ is a *symbol* if its associated composition operator $C_\Phi: H^2(\mathbb{D}^d) \rightarrow H^2(\mathbb{D}^d)$, defined by $C_\Phi(f) = f \circ \Phi$, is bounded.

We say that Φ is *truly d -dimensional* if $\Phi(\mathbb{D}^d)$ has a non-void interior.

We will make use of two kinds of symbols defined on \mathbb{D} .

The *lens map* $\lambda_\theta: \mathbb{D} \rightarrow \mathbb{D}$ is defined, for $\theta \in (0, 1)$, by:

$$(1.2) \quad \lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}$$

(see [25], p. 27, or [16], for more information), and corresponds to $u \mapsto u^\theta$ in the right half-plane.

The *cuspidal map* $\chi: \mathbb{D} \rightarrow \mathbb{D}$ was first defined in [15] and in a slightly different form in [20]; we actually use here the modified form introduced in [17], and then used in [18]. We first define:

$$\chi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1};$$

we note that $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, $\chi_0(-i) = i$, and $\chi_0(0) = \sqrt{2}-1$. Then we set:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where:

$$(1.3) \quad a = 1 - \frac{2}{\pi} \log(\sqrt{2}-1) \in (1, 2)$$

is chosen in order that $\chi(0) = 0$. The image Ω of the (univalent) cuspidal map is formed by the intersection of the inside of the disk $D(1 - \frac{a}{2}, \frac{a}{2})$ and the outside of the two disks $D(1 + \frac{ia}{2}, \frac{a}{2})$ and $D(1 - \frac{ia}{2}, \frac{a}{2})$.

Besides the approximation numbers, we need other singular numbers for an operator $S: X \rightarrow Y$ between Banach spaces X and Y .

The *Bernstein numbers* $b_n(S)$, $n \geq 1$, which are defined by:

$$(1.4) \quad b_n(S) = \sup_E \min_{x \in S_E} \|Sx\|,$$

where the supremum is taken over all n -dimensional subspaces of X and S_E is the unit sphere of E .

The *Gelfand numbers* $c_n(S)$, $n \geq 1$, which are defined by:

$$(1.5) \quad c_n(S) = \inf\{\|S|_M\|; \text{codim } M < n\}.$$

The *Kolmogorov numbers* $d_n(S)$, $n \geq 1$, which are defined by:

$$(1.6) \quad d_n(S) = \inf_{\dim E < n} \left[\sup_{x \in B_X} \text{dist}(Sx, E) \right].$$

Pietsch showed that all s -numbers on Hilbert spaces are equal (see [23], § 2, Corollary, or [24], Theorem 11.3.4); hence:

$$(1.7) \quad a_n(S) = b_n(S) = c_n(S) = d_n(S).$$

We denote m the normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$. If $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, m_φ is the pull-back measure on $\overline{\mathbb{D}}$ defined by $m_\varphi(E) = m[\varphi^{*-1}(E)]$, where φ^* stands for the non-tangential boundary values of φ .

The notation $A \lesssim B$ means that $A \leq CB$ for some positive constant C and we write $A \approx B$ if we have both $A \lesssim B$ and $B \lesssim A$.

2 Preliminary results on weighted composition operators on $H^2(\mathbb{D})$

We see in this section that the presence of a ‘‘rapidly decaying’’ weight allows simpler estimates for the approximation numbers of a corresponding weighted composition operator. Such a study, but a bit different, is made in [14].

Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ a non-constant analytic self-map in the disk algebra $A(\mathbb{D})$ such that, for some constant $C > 1$ and for all $z \in \mathbb{D}$:

$$(2.1) \quad \varphi(1) = 1, \quad |1 - \varphi(z)| \leq 1, \quad |1 - \varphi(z)| \leq C(1 - |\varphi(z)|)$$

as well as $\varphi(z) \neq 1$ for $z \neq 1$. We can take for example $\varphi = \frac{1+\lambda_\theta}{2}$ where λ_θ is the lens map with parameter θ .

Let $w \in H^\infty$ and let T be the weighted composition operator

$$T = M_{w \circ \varphi} C_\varphi: H^2 \rightarrow H^2.$$

Note that $M_{w \circ \varphi} C_\varphi = C_\varphi M_w$. We first show that:

Theorem 2.1. *Let $T = M_{w \circ \varphi} C_\varphi: H^2 \rightarrow H^2$ be as above and let B be a Blaschke product with length $< N$. Then, with the implied constant depending only on the number C in (2.1) (and of φ):*

$$a_N(T) \lesssim \sup_{|z-1| \leq 1, z \in \varphi(\mathbb{D})} |B(z)| |w(z)|.$$

Proof. The following preliminary observation (see also [16], p. 809), in which we denote by $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}$ the Carleson window with center $\xi \in \mathbb{T}$ and size h , and by K_φ the support of the pull-back measure m_φ , will be useful.

$$(2.2) \quad u \in S(\xi, h) \cap K_\varphi \implies u \in S(1, Ch) \cap K_\varphi.$$

Indeed, if $|u - \xi| \leq h$ and $u \in K_\varphi$, (2.1) implies:

$$1 - |u| \leq |u - \xi| \leq h \quad \text{and} \quad |u - 1| \leq C(1 - |u|) \leq Ch.$$

Set $E = BH^2$. This is a subspace of codimension $< N$. If $f = Bg \in E$, with $\|g\| = \|f\|$ (isometric division by B in BH^2), we have $Tf = (wBg) \circ \varphi$, whence:

$$\|T(f)\|^2 = \int_{\mathbb{D}} |B|^2 |w|^2 |g|^2 dm_\varphi,$$

implying $\|T(f)\|^2 \leq \|f\|^2 \|J\|^2$ where $J: H^2 \rightarrow L^2(\sigma)$ is the natural embedding and where

$$\sigma = |B|^2 |w|^2 dm_\varphi.$$

Now, Carleson's embedding theorem for the measure σ and (2.2) show that (the implied constants being absolute):

$$\begin{aligned} \|J\|^2 &\lesssim \sup_{\xi \in \mathbb{T}, 0 < h < 1} \frac{1}{h} \int_{S(\xi, h) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi \\ &\lesssim \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi \\ &\lesssim \left(\sup_{|z-1| \leq 1, z \in \varphi(\mathbb{D})} |B(z)|^2 |w(z)|^2 \right) \left(\sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} dm_\varphi \right) \\ &\lesssim \sup_{|z-1| \leq 1, z \in \varphi(\mathbb{D})} |B(z)|^2 |w(z)|^2, \end{aligned}$$

since m_φ is a Carleson measure for H^2 and where we used that, according to (2.1):

$$K_\varphi \subseteq \overline{\varphi(\mathbb{D})} \subseteq \{z \in \mathbb{D}; |z - 1| \leq 1\}.$$

This ends the proof of Theorem 2.1 with help of the equality of $a_N(T)$ with the Gelfand number $c_N(T)$ recalled in (1.7). \square

In order to specialize efficiently the general Theorem 2.1, we recall the following simple Lemma 2.3 of [16], where:

$$(2.3) \quad \rho(a, b) = \left| \frac{a - b}{1 - \bar{a}b} \right|, \quad a, b \in \mathbb{D},$$

is the *pseudo-hyperbolic distance*:

Lemma 2.2 ([16]). *Let $a, b \in \mathbb{D}$ such that $|a - b| \leq L \min(1 - |a|, 1 - |b|)$. Then:*

$$\rho(a, b) \leq \frac{L}{\sqrt{L^2 + 1}} =: \kappa < 1.$$

We can now state:

Theorem 2.3. *Assume that φ is as in (2.1) and that the weight w satisfies, for some parameters $0 < \theta \leq 1$ and $R > 0$:*

$$|w(z)| \leq \exp\left(-\frac{R}{|1 - z|^\theta}\right), \quad \forall z \in \mathbb{D}.$$

Then, the approximation numbers of $T = M_{w \circ \varphi} C_\varphi$ satisfy:

$$a_{nm+1}(T) \lesssim \max[\exp(-an), \exp(-R2^{m\theta})],$$

for all integers $n, m \geq 1$, where $a = \log[\sqrt{16C^2 + 1}/(4C)] > 0$ and C is as in (2.1).

Proof. Let $p_l = 1 - 2^{-l}$, $0 \leq l < m$ and let B be the Blaschke product:

$$B(z) = \prod_{0 \leq l < m} \left(\frac{z - p_l}{1 - \bar{p}_l z} \right)^n.$$

Let $z \in K_\varphi \cap \mathbb{D}$ so that $0 < |z - 1| \leq 1$. Let l be the non-negative integer such that $2^{-l-1} < |z - 1| \leq 2^{-l}$. We separate two cases:

Case 1: $l \geq m$. Then, *the weight does the job.* Indeed, majorizing $|B(z)|$ by 1 and using the assumption on w , we get:

$$\begin{aligned} |B(z)|^2 |w(z)|^2 &\leq |w(z)|^2 \leq \exp\left(-\frac{2R}{|1 - z|^\theta}\right) \\ &\leq \exp(-2R2^{l\theta}) \leq \exp(-2R2^{m\theta}). \end{aligned}$$

Case 2: $l < m$. Then, *the Blaschke product does the job.* Indeed, majorize $|w(z)|$ by 1 and estimate $|B(z)|$ more accurately with help of Lemma 2.2; we observe that

$$|z - p_l| \leq |z - 1| + 1 - p_l \leq 2 \times 2^{-l} = 2(1 - p_l) \leq 4C(1 - p_l)$$

and then, since $z \in K_\varphi$, we can write with $C \geq 1$ as in (2.1):

$$1 - |z| \geq \frac{1}{C} |1 - z| \geq \frac{1}{2C} 2^{-l} \geq \frac{1}{4C} |z - p_l|,$$

so that the assumptions of Lemma 2.2 are verified with $L = 4C$, giving:

$$\rho(z, p_l) \leq \frac{4C}{\sqrt{16C^2 + 1}} = \exp(-a) < 1.$$

Hence, by definition, since $l < m$:

$$|B(z)| \leq [\rho(z, p_l)]^n \leq \exp(-an).$$

Putting both cases together, and observing that our Blaschke product has length $nm < nm + 1$, we get the result by applying Theorem 2.1 with $N = nm + 1$. \square

2.1 Some remarks

1. Twisting a composition operator by a weight may improve the compactness of this composition operator, or even may make this weighted composition operator compact though the non-weighted was not (see [8] or [14]). However, this is not possible for all symbols, as seen in the following proposition.

Proposition 2.4. *Let $w \in H^\infty$. If φ is inner, or more generally if $|\varphi| = 1$ on a subset of \mathbb{T} of positive measure, then $M_w C_\varphi$ is never compact (unless $w \equiv 0$).*

Proof. Indeed, suppose $T = M_w C_\varphi$ compact. Since $(z^n)_n$ converges weakly to 0 in H^2 and since $T(z^n) = w \varphi^n$, we should have, since $|\varphi| = 1$ on E , with $m(E) > 0$:

$$\int_E |w|^2 dm = \int_E |w|^2 |\varphi|^{2n} dm \leq \int_{\mathbb{T}} |w|^2 |\varphi|^{2n} dm = \|T(z^n)\|^2 \xrightarrow{n \rightarrow \infty} 0,$$

but this would imply that w is null a.e. on E and hence $w \equiv 0$ (see [7], Theorem 2.2), which was excluded. \square

Note that É. Amar and A. Lederer proved in [1] that $|\varphi| = 1$ on a set of positive measure if and only if φ is an exposed point of the unit ball of H^∞ ; hence the following proposition can be viewed as the (almost) opposite case.

Proposition 2.5. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ such that $\|\varphi\|_\infty = 1$. Assume that:*

$$\int_{\mathbb{T}} \log(1 - |\varphi|) dm > -\infty$$

(meaning that φ is not an extreme point of the unit ball of H^∞ : see [7], Theorem 7.9). Then, if w is an outer function such that $|w| = 1 - |\varphi|$, the weighted composition operator $T = M_w C_\varphi$ is Hilbert-Schmidt.

Proof. We have:

$$\sum_{n=0}^{\infty} \|T(z^n)\|^2 = \sum_{n=0}^{\infty} \int_{\mathbb{T}} (1 - |\varphi|)^2 |\varphi|^{2n} dm = \int_{\mathbb{T}} \frac{1 - |\varphi|}{1 + |\varphi|} dm < +\infty,$$

and T is Hilbert-Schmidt, as claimed. \square

2. In [14], Theorem 2.5, it is proved that we always have, for some constants $\delta, \rho > 0$:

$$(2.4) \quad a_n(M_w C_\varphi) \geq \delta \rho^n, \quad n = 1, 2, \dots$$

(if $w \not\equiv 0$). We give here an alternative proof, based on a result of Gunatillake ([9]), this result holding in a wider context.

Theorem 2.6 (Gunatillake). *Let $T = M_w C_\varphi$ be a compact weighted composition operator on H^2 and assume that φ has a fixed point $a \in \mathbb{D}$. Then the spectrum of T is the set:*

$$\sigma(T) = \{0, w(a), w(a) \varphi'(a), w(a) [\varphi'(a)]^2, \dots, w(a) [\varphi'(a)]^n, \dots\}$$

Proof of (2.4). First observe that, in view of Proposition 2.4, φ cannot be an automorphism of \mathbb{D} so that the point a is the Denjoy-Wolff point of φ and is attractive. Theorem 2.6 is interesting only when $w(a) \varphi'(a) \neq 0$.

Now, we can give a new proof Theorem 2.5 of [14] as follows. Let $a \in \mathbb{D}$ be such that $w(a) \varphi'(a) \neq 0$ ($H(\mathbb{D})$ is a division ring and $\varphi' \not\equiv 0, w \not\equiv 0$). Let $b = \varphi(a)$ and $\tau \in \text{Aut } \mathbb{D}$ with $\tau(b) = a$. We set:

$$\psi = \tau \circ \varphi \quad \text{and} \quad S = M_w C_\psi = T C_\tau.$$

This operator S is compact because T is.

Since $\psi(a) = a$ and $\psi'(a) = \tau'(b) \varphi'(a) \neq 0$, Theorem 2.6 says that the non-zero eigenvalues of S , arranged in non-increasing order, are the numbers $\lambda_n = w(a) [\psi'(a)]^{n-1}$, $n \geq 1$. As a consequence of Weyl's inequalities, we know that:

$$a_1(S) a_n(S) \geq |\lambda_{2n}|^2 \geq \delta \rho^n,$$

with:

$$\delta = |w(a)|^2 > 0 \quad \text{and} \quad \rho = |\psi'(a)|^4 > 0.$$

To finish, it is enough to observe that $a_n(S) \leq a_n(T) \|C_\tau\|$ by the ideal property of approximation numbers. \square

3 The splitted case

Theorem 3.1. *Let $\Phi = (\phi, \psi): \mathbb{D}^d \rightarrow \mathbb{D}^d$ be a truly d -dimensional symbol with $\phi: \mathbb{D} \rightarrow \mathbb{D}$ depending only on z_1 and $\psi: \mathbb{D}^{d-1} \rightarrow \mathbb{D}^{d-1}$ only on z_2, \dots, z_d , i.e. $\Phi(z_1, z_2, \dots, z_d) = (\phi(z_1), \psi(z_2, \dots, z_d))$. Then, whatever ψ behaves:*

$$\|\phi\|_\infty = 1 \quad \implies \quad \beta_d(C_\Phi) = 1.$$

Proof. The proof is based on the following simple lemma, certainly well-known.

Lemma 3.2. *Let $S: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$ be two compact linear operators, where H_1 and H_2 are Hilbert spaces. Let $S \otimes T$ be their tensor product, acting on the tensor product $H_1 \otimes H_2$. Then:*

$$a_{mn}(S \otimes T) \geq a_m(S) a_n(T)$$

for all positive integers m, n .

We postpone the proof of the lemma and show how to conclude.

We can assume C_Φ to be compact, so that C_ϕ is compact as well. Since $\|\phi\|_\infty = 1$, we have, thanks to (1.1):

$$a_m(C_\phi) \geq e^{-m\varepsilon_m} \quad \text{with} \quad \varepsilon_m \xrightarrow{m \rightarrow \infty} 0.$$

Replacing ε_m by $\delta_m := \sup_{p \geq m} \varepsilon_p$, we can assume that $(\varepsilon_m)_m$ is non-increasing. Moreover,

$$m \varepsilon_m \rightarrow \infty$$

since C_ϕ is compact and hence $a_m(C_\phi) \xrightarrow{m \rightarrow \infty} 0$. We next observe that, due to the separation of variables in the definition of ϕ and ψ , we can write:

$$(3.1) \quad C_\Phi = C_\phi \otimes C_\psi.$$

Indeed, write $z = (z_1, w)$ with $z_1 \in \mathbb{D}$ and $w \in \mathbb{D}^{d-1}$. If $f \in H^2(\mathbb{D})$ and $g \in H^2(\mathbb{D}^{d-1})$, we see that:

$$\begin{aligned} C_\Phi(f \otimes g)(z) &= (f \otimes g)(\phi(z_1), \psi(w)) = f(\phi(z_1)) g(\psi(w)) \\ &= [C_\phi f(z_1)] [C_\psi g(w)] = (C_\phi f \otimes C_\psi g)(z). \end{aligned}$$

Since tensor products $f \otimes g$ generate $H^2(\mathbb{D}^d) = H^2(\mathbb{D}) \otimes H^2(\mathbb{D}^{d-1})$, this proves (3.1).

Let now m be a large positive integer. Set ($[\cdot]$ stands for the integer part):

$$(3.2) \quad n_m = [m\varepsilon_m]^{d-1} \quad \text{and} \quad N_m = m n_m.$$

From what we know in dimension $d-1$ (see [2], Theorem 3.1) and from the preceding, we can write (observe that ψ has to be truly $(d-1)$ -dimensional since Φ is truly d -dimensional):

$$a_m(C_\phi) \geq \exp(-m\varepsilon_m) \quad \text{and} \quad a_n(C_\psi) \geq a \exp(-C n^{1/(d-1)}),$$

for some positive constant C , which will be allowed to vary from one formula to another. Lemma 3.2 implies:

$$a_{N_m}(C_\Phi) \geq a \exp[-C(m\varepsilon_m + n_m^{1/(d-1)})].$$

Since $n_m \lesssim (m\varepsilon_m)^{d-1}$, we get:

$$a_{N_m}(C_\Phi) \geq a \exp(-C m \varepsilon_m).$$

Observe that $N_m = m n_m \sim m^d \varepsilon_m^{d-1}$ and so $N_m^{1/d} \sim m \varepsilon_m^{1-1/d}$. As a consequence:

$$\begin{aligned} a_{N_m}(C_\Phi) &\geq a \exp(-C m \varepsilon_m) = a \exp[-(C \varepsilon_m^{1/d})(m \varepsilon_m^{1-1/d})] \\ &\geq a \exp(-\eta_m N_m^{1/d}) \end{aligned}$$

with $\eta_m := C \varepsilon_m^{1/d}$.

Now, for $N > N_1$, let m be the smallest integer satisfying $N_m \geq N$ (so that $N_{m-1} < N \leq N_m$), and set $\delta_N = \eta_m$. We have $\lim_{N \rightarrow \infty} \delta_N = 0$. Next, we note that $\lim_{m \rightarrow \infty} N_m/N_{m-1} = 1$, because $N_m \geq N_{m-1}$ and:

$$\frac{N_m}{N_{m-1}} \leq \frac{m}{m-1} \left(\frac{m \varepsilon_m + 1}{(m-1) \varepsilon_{m-1}} \right)^{d-1} \sim \left(\frac{\varepsilon_m}{\varepsilon_{m-1}} \right)^{d-1} \leq 1.$$

Finally, if N is an arbitrary integer and $N_{m-1} < N \leq N_m$, we obtain:

$$a_N(C_\Phi) \geq a_{N_m}(C_\Phi) \geq a \exp(-\eta_m N_m^{1/d}) \geq a \exp(-C \delta_N N^{1/d}),$$

since we observed that $\lim_{m \rightarrow \infty} N_m/N_{m-1} = 1$.

This amounts to say that $\beta_d(C_\Phi) = 1$. \square

Proof of Lemma 3.2. It is rather formal. Start from the Schmidt decompositions of S and T respectively (recall that Hilbert spaces, the approximation numbers are equal to the singular ones):

$$S = \sum_{m=1}^{\infty} a_m(S) u_m \odot v_m, \quad T = \sum_{n=1}^{\infty} a_n(T) u'_n \odot v'_n,$$

where $(u_m), (v_m)$ are two orthonormal sequences of H_1 , $(u'_n), (v'_n)$ two orthonormal sequences of H_2 , and $u_m \odot v_m$ and $u'_n \odot v'_n$ denote the rank one operators defined by $(u_m \odot v_m)(x) = \langle x, v_m \rangle u_m$, $x \in H_1$, and $(u'_n \odot v'_n)(x) = \langle x, v'_n \rangle u'_n$, $x \in H_2$.

We clearly have:

$$(u_m \odot v_m) \otimes (u'_n \odot v'_n) = (u_m \otimes u'_n) \odot (v_m \otimes v'_n),$$

so that the Schmidt decomposition of $S \otimes T$ is (with SOT-convergence):

$$S \otimes T = \sum_{m,n \geq 1} a_m(S) a_n(T) (u_m \otimes u'_n) \odot (v_m \otimes v'_n),$$

since the two sequences $(u_m \otimes u'_n)_{m,n}$ and $(v_m \otimes v'_n)_{m,n}$ are orthonormal: for instance, we have by definition:

$$\langle u_{m_1} \otimes u'_{n_1}, u_{m_2} \otimes u'_{n_2} \rangle = \langle u_{m_1}, u_{m_2} \rangle \langle u'_{n_1}, u'_{n_2} \rangle.$$

This shows that the singular values of $S \otimes T$ are the non-increasing rearrangement of the positive numbers $a_m(S) a_n(T)$ and ends the proof of the lemma: the mn numbers $a_k(S) a_l(T)$, for $1 \leq k \leq m$, $1 \leq l \leq n$ all satisfy $a_k(S) a_l(T) \geq a_m(S) a_n(T)$, so that $a_{mn}(S \otimes T) \geq a_m(S) a_n(T)$. \square

4 The glued case

Here we consider symbols of the form:

$$(4.1) \quad \Phi(z_1, z_2) = (\phi(z_1), \phi(z_1)),$$

where $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is a non-constant analytic map.

Note that such maps Φ are not truly 2-dimensional.

4.1 Preliminary

We begin by remarking the following fact.

Let $B^2(\mathbb{D})$ be the Bergman space of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that:

$$\|f\|_{B^2}^2 := \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where dA is the normalized area measure on \mathbb{D} .

Proposition 4.1. *Assume that the composition operator C_ϕ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$. Then $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$, defined by (4.1), is bounded.*

Proof. If we write $f \in H^2(\mathbb{D}^2)$ as:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k, \quad \text{with} \quad \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2}^2,$$

we formally (or assuming that f is a polynomial) have:

$$[C_\Phi f](z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\phi(z_1)]^k = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} c_{j,k} \right) [\phi(z_1)]^n.$$

Hence, if we set $g(z) = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} c_{j,k} \right) z^n$, we get:

$$[C_\Phi(f)](z_1, z_2) = [C_\phi(g)](z_1),$$

so that, by integrating:

$$\|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} = \|C_\phi(g)\|_{H^2(\mathbb{D})}.$$

By hypothesis, there is a positive constant M such that:

$$\|C_\phi(g)\|_{H^2(\mathbb{D})} \leq M \|g\|_{B^2(\mathbb{D})}.$$

But, by the Cauchy-Schwarz inequality:

$$\begin{aligned} \|g\|_{B^2(\mathbb{D})}^2 &= \sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{j+k=n} c_{j,k} \right|^2 \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{j+k=n} |c_{j,k}|^2 \right) = \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2(\mathbb{D}^2)}^2, \end{aligned}$$

and we obtain $\|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} \leq M \|f\|_{H^2(\mathbb{D}^2)}$. □

4.2 Lens maps

Let λ_θ be a lens map of parameter θ , $0 < \theta < 1$. We consider $\Phi_\theta: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ defined by:

$$(4.2) \quad \Phi_\theta(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_2)).$$

We have the following result.

Theorem 4.2. *The composition operator $C_{\Phi_\theta}: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ is:*

- 1) *not bounded for $\theta > 1/2$;*
- 2) *bounded, but not compact for $\theta = 1/2$;*
- 3) *compact, and even Hilbert-Schmidt, for $0 < \theta < 1/2$.*

Proof. The reproducing kernel of $H^2(\mathbb{D}^2)$ is, for $(a, b) \in \mathbb{D}^2$:

$$(4.3) \quad K_{a,b}(z_1, z_2) = \frac{1}{1 - \bar{a}z_1} \frac{1}{1 - \bar{b}z_2}, \quad (z_1, z_2) \in \mathbb{D}^2,$$

and:

$$\|K_{a,b}\|^2 = \frac{1}{(1 - |a|^2)(1 - |b|^2)}.$$

- 1) If C_{Φ_θ} were bounded, we should have, for some $M < \infty$:

$$\|C_{\Phi_\theta}^*(K_{a,b})\|_{H^2} \leq M \|K_{a,b}\|_{H^2}, \quad \text{for all } a, b \in \mathbb{D}.$$

Since $C_{\Phi_\theta}^*(K_{a,b}) = K_{\Phi_\theta(a,b)} = K_{\lambda_\theta(a), \lambda_\theta(b)}$, we would get, with $b = 0$:

$$\left(\frac{1}{1 - |\lambda_\theta(a)|^2} \right)^2 \leq M^2 \frac{1}{1 - |a|^2};$$

but this is not possible for $\theta > 1/2$, since $1 - |\lambda_\theta(a)|^2 \approx 1 - |\lambda_\theta(a)| \sim (1 - a)^\theta$ when a goes to 1, with $0 < a < 1$.

For 2) and 3), let us consider the pull-back measure m_θ of the normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$ by λ_θ . It is easy to see that:

$$(4.4) \quad \sup_{\xi \in \mathbb{T}} m_\theta[D(\xi, h) \cap \mathbb{D}] = m_\theta[D(1, h) \cap \mathbb{D}] \approx h^{1/\theta}.$$

In particular, for $\theta \leq 1/2$, m_θ is a 2-Carleson measure, and hence (see [15], Theorem 2.1, for example) the canonical injection $j: B^2(\mathbb{D}) \rightarrow L^2(m_\theta)$ is bounded, meaning that, for some positive constant $M < \infty$:

$$\int_{\mathbb{D}} |f(z)|^2 dm_\theta(z) \leq M^2 \|f\|_{B^2}^2.$$

Since

$$\int_{\mathbb{D}} |f(z)|^2 dm_\theta(z) = \int_{\mathbb{T}} |f[\lambda_\theta(u)]|^2 dm(u) = \|C_{\lambda_\theta}(f)\|_{H^2}^2,$$

we get that C_{λ_θ} maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$.

It follows from Proposition 4.1 that $C_{\Phi_\theta} : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ is bounded.

However, $C_{\Phi_{1/2}}$ is not compact since $C_{\Phi_{1/2}}^*(K_{a,b})/\|K_{a,b}\|$ does not converge to 0 as $a, b \rightarrow 1$, by the calculations made in 1).

For 3), let $e_{j,k}(z_1, z_2) = z_1^j z_2^k$, $j, k \geq 0$, be the canonical orthonormal basis of $H^2(\mathbb{D}^2)$; we have $[C_{\Phi_\theta}(e_{j,k})](z_1, z_2) = [\lambda_\theta(z_1)]^{j+k}$. Hence:

$$\sum_{j,k \geq 0} \|C_{\Phi_\theta}(e_{j,k})\|_{H^2(\mathbb{D}^2)}^2 \leq \sum_{n=0}^{\infty} (2n+1) \int_{\mathbb{T}} |\lambda_\theta|^{2n} dm \leq \int_{\mathbb{T}} \frac{2}{(1-|\lambda_\theta|^2)^2} dm.$$

Since, by Lemma 4.3 below, $1 - |\lambda_\theta(e^{it})|^2 \gtrsim |1 - e^{it}|^\theta \geq t^\theta$ for $|t| \leq \pi/2$, we get:

$$\sum_{j,k \geq 0} \|C_{\Phi_\theta}(e_{j,k})\|_{H^2(\mathbb{D}^2)}^2 \lesssim \int_0^{\pi/2} \frac{dt}{t^{2\theta}} < \infty,$$

since $\theta < 1/2$. Therefore C_{Φ_θ} is Hilbert-Schmidt for $\theta < 1/2$. \square

For sake of completeness, we recall the following elementary fact (see [25], p. 28, or also [16], Lemma 2.5)).

Lemma 4.3. *With $\delta = \cos(\theta\pi/2)$, we have, for $|z| \leq 1$ and $\Re z \geq 0$:*

$$1 - |\lambda_\theta(z)|^2 \geq \frac{\delta}{2} |1 - z|^\theta.$$

Proof. We can write:

$$\lambda_\theta(z) = \frac{1-w}{1+w} \quad \text{with} \quad w = \left(\frac{1-z}{1+z}\right)^\theta \quad \text{and} \quad |w| \leq 1.$$

Then:

$$\Re w \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta.$$

Hence:

$$1 - |\lambda_\theta(z)|^2 = \frac{4 \Re w}{|1+w|^2} \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta,$$

as announced \square

We now improve the result 3) of Theorem 4.2 by estimating the approximation numbers of C_{Φ_θ} and get that C_{Φ_θ} is in all Schatten classes of $H^2(\mathbb{D}^2)$ when $\theta < 1/2$.

Theorem 4.4. *For $0 < \theta < 1/2$, there exists $b = b_\theta > 0$ such that:*

$$(4.5) \quad a_n(C_{\Phi_\theta}) \lesssim e^{-b\sqrt{n}}.$$

In particular $\beta_2^+(C_{\Phi_\theta}) \leq e^{-b} < 1$, though $\|\Phi_\theta\|_\infty = 1$, and even $\Phi_\theta(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$.

Proof. Proposition 4.1 (and its proof) can be rephrased in the following way: if C_ϕ maps boundedly $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$, then, we have the following factorization:

$$(4.6) \quad C_{\Phi}: H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_\phi} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2),$$

where $I: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^2)$ is the canonical injection given by $(If)(z_1, z_2) = f(z_1)$ for $f \in H^2(\mathbb{D})$, and $J: H^2(\mathbb{D}^2) \rightarrow B^2(\mathbb{D})$ is the contractive map defined by:

$$(Jf)(z) = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} c_{j,k} \right) z^n,$$

for $f \in H^2(\mathbb{D}^2)$ with $f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k$.

In the proof of Theorem 4.2, we have seen that, for $0 < \theta \leq 1/2$, the composition operator C_{λ_θ} is bounded from $B^2(\mathbb{D})$ into $H^2(\mathbb{D})$; we get hence the factorization:

$$C_{\Phi_\theta}: H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_{\lambda_\theta}} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2),$$

Now, the lens maps have a semi-group property:

$$(4.7) \quad \lambda_{\theta_1 \theta_2} = \lambda_{\theta_1} \lambda_{\theta_2},$$

giving $C_{\lambda_{\theta_1 \theta_2}} = C_{\lambda_{\theta_1}} \circ C_{\lambda_{\theta_2}}$.

For $0 < \theta < 1/2$, we therefore can write $C_{\lambda_\theta} = C_{\lambda_{2\theta}} \circ C_{\lambda_{1/2}}$ (note that $2\theta < 1$, so $C_{\lambda_{2\theta}}: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ is bounded), and we get:

$$C_{\Phi_\theta} = I C_{\lambda_{2\theta}} C_{\lambda_{1/2}} J.$$

Consequently:

$$a_n(C_{\Phi_\theta}) \leq \|I\| \|J\| \|C_{\lambda_{1/2}}\|_{B^2 \rightarrow H^2} a_n(C_{\lambda_{2\theta}}).$$

Now, we know ([16], Theorem 2.1) that $a_n(C_{\lambda_{2\theta}}) \lesssim e^{-b\sqrt{n}}$, so we get that $a_n(C_{\Phi_\theta}) \lesssim e^{-b\sqrt{n}}$. \square

5 Triangularly separated variables

In this section, we consider symbols of the form:

$$(5.1) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2),$$

where $\phi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ are non-constant analytic maps.

Such maps Φ are truly 2-dimensional.

More generally, if $h \in H^\infty$, with $h(0) = 0$ and $\|h\|_\infty \leq 1$, has its powers h^k , $k \geq 0$, orthogonal in H^2 (for convenience, we shall say that h is a *Rudin function*), we can consider:

$$(5.2) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$$

For such h we can take for example an inner function vanishing at the origin, but there are other such functions, as shown by C. Bishop:

Theorem (Bishop [4]). *The function h is a Rudin function if and only if the pull-back measure $\mu = \mu_h$ is radial and Jensen, i.e for every Borel set E :*

$$\mu(e^{i\theta} E) = \mu(E) \quad \text{and} \quad \int_{\mathbb{D}} \log(1/|z|) d\mu(z) < \infty.$$

Conversely, for every probability measure μ supported by $\overline{\mathbb{D}}$, which is radial and Jensen, there exists h in the unit ball of H^∞ , with $h(0) = 0$, such that $\mu = \mu_h$.

If we take for μ the Lebesgue measure of \mathbb{T} , we get an inner function. But, as remarked in [4], we can take for μ the Lebesgue measure on the union $\mathbb{T} \cup (1/2)\mathbb{T}$, normalized in order that $\mu(T) = \mu((1/2)\mathbb{T}) = 1/2$. Then the corresponding h is not inner since $|h| = 1/2$ on a subset of \mathbb{T} of positive measure. He also showed that $h(z)/z$ may be a non-constant outer function. Also, P. Bourdon ([6]) showed that the powers of h are orthogonal if and only if its Nevanlinna counting function is almost everywhere constant on each circle centered on the origin.

5.1 General facts

We first observe that if $f \in H^2(\mathbb{D}^2)$ and:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k,$$

then we can write:

$$f(z_1, z_2) = \left(\sum_{k \geq 0} f_k(z_1) \right) z_2^k$$

with:

$$f_k(z_1) = \sum_{j \geq 0} c_{j,k} z_1^j,$$

and:

$$\|f\|_{H^2(\mathbb{D}^2)}^2 = \sum_{j,k \geq 0} |c_{j,k}|^2 = \sum_{k \geq 0} \|f_k\|_{H^2(\mathbb{D})}^2.$$

That means that we have an isometric isomorphism:

$$J: H^2(\mathbb{D}^2) \longrightarrow \bigoplus_{k=0}^{\infty} H^2(\mathbb{D}),$$

defined by $Jf = (f_k)_{k \geq 0}$.

Now, for symbols Φ as in (5.1), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j, k \geq 0} c_{j, k} [\phi(z_1)]^j [\psi(z_1)]^k z_2^k,$$

so that $J C_\Phi J^{-1}$ appears as the operator $\bigoplus_k M_{\psi^k} C_\phi$ on $\bigoplus_k H^2(\mathbb{D})$, where M_{ψ^k} is the multiplication operator by ψ^k :

$$[(M_{\psi^k} C_\phi) f_k](z_1) = [\psi(z_1)]^k [(f_k \circ \phi)(z_1)].$$

When Φ is as in (5.2), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j, k \geq 0} c_{j, k} [\phi(z_1)]^j [\psi(z_1)]^k [h(z_2)]^k,$$

with:

$$\|C_\Phi f\|^2 \leq \sum_{k=0}^{\infty} \|T_k f_k\|^2$$

and:

$$T_k = M_{\psi^k} C_\phi;$$

hence $J C_\Phi J^{-1}$ appears as pointwise dominated by the operator $T = \bigoplus_k T_k$ on $\bigoplus_k H^2(\mathbb{D})$. This implies a factorization $C_\Phi = AT$ with $\|A\| \leq 1$, so that $a_n(C_\Phi) \leq a_n(T)$ for all $n \geq 1$.

We recall the following elementary fact.

Lemma 5.1. *Let $(H_k)_{k \geq 0}$ be a sequence of Hilbert spaces and $T_k: H_k \rightarrow H_k$ be bounded operators. Let $H = \bigoplus_{k=0}^{\infty} H_k$ and $T: H \rightarrow H$ defined by $Tx = (T_k x_k)_k$. Then:*

- 1) *T is bounded on H if and only if $\sup_k \|T_k\| < \infty$;*
- 2) *T is compact on H if and only if each T_k is compact and $\|T_k\| \xrightarrow[k \rightarrow \infty]{} 0$.*

Going back to the symbols of the form (5.1), we have $\|M_{\psi^k}\| \leq \|\psi^k\|_\infty \leq 1$, since $\|\psi\|_\infty \leq 1$; hence $\|M_{\psi^k} C_\phi\| \leq \|C_\phi\|$ and the operator $(M_{\psi^k} C_\phi)_k$ is bounded on $\bigoplus_k H^2(\mathbb{D})$. Therefore C_Φ is bounded on $H^2(\mathbb{D}^2)$.

For approximation numbers, we have the following two facts.

Lemma 5.2. *Let $T_k: H_k \rightarrow H_k$ be bounded linear operators between Hilbert spaces H_k , $k \geq 0$. Let $H = \bigoplus_k H_k$ and $T = (T_k)_k: H \rightarrow H$, assumed to be compact. Then, for every $n_1, \dots, n_K \geq 1$, and $0 \leq m_1 < \dots < m_K$, $K \geq 1$, we have:*

$$(5.3) \quad a_N(T) \geq \inf_{1 \leq k \leq K} a_{n_k}(T_{m_k}),$$

where $N = n_1 + \dots + n_K$.

Proof. We use the Bernstein numbers b_n (see (1.4)), which are equal to the approximation numbers (see (1.7)).

For $k = 1, \dots, K$, there is an n_k -dimensional subspace E_k of H_{m_k} such that:

$$b_{n_k}(T_{m_k}) \leq \|T_{m_k}x\|, \quad \text{for all } x \in S_{E_k}.$$

Then $E = \bigoplus_{k=1}^K E_k$ is an N -dimensional subspace of H and for every $x = (x_1, x_2, \dots) \in E$, we have:

$$\begin{aligned} \|Tx\|^2 &= \sum_{k \leq K} \|T_{m_k}x_{m_k}\|^2 \geq \sum_{k \leq K} [b_{n_k}(T_{m_k})]^2 \|x_{m_k}\|^2 \\ &\geq \inf_{k \leq K} [b_{n_k}(T_{m_k})]^2 \sum_{k \leq K} \|x_{m_k}\|^2 = \inf_{k \leq K} [b_{n_k}(T_{m_k})]^2 \|x\|^2; \end{aligned}$$

hence $b_N(T) \geq \inf_{k \leq K} b_{n_k}(T_{m_k})$, and we get the announced result. \square

Lemma 5.3. *Let $T = \bigoplus_{k=0}^{\infty} T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^{\infty} H_k$. Let n_0, \dots, n_K be positive integers and $N = n_0 + \dots + n_K - K$. Then, the approximation numbers of T satisfy:*

$$(5.4) \quad a_N(T) \leq \max \left(\max_{0 \leq k \leq K} a_{n_k}(T_k), \sup_{k > K} \|T_k\| \right).$$

Proof. Denote by S the right-hand side of (5.4). Let R_k , $0 \leq k \leq K$ be operators on H_k of respective rank $< n_k$ such that $\|T_k - R_k\| = a_{n_k}(T_k)$ and let $R = \bigoplus_{k=0}^K R_k$ be an operator of rank $\leq n_0 + \dots + n_K - K - 1 < N$. If $f = \sum_{k=0}^{\infty} f_k \in H$, we see that:

$$\begin{aligned} \|Tf - Rf\|^2 &= \sum_{k=0}^K \|T_k f_k - R_k f_k\|^2 + \sum_{k > K} \|T_k f_k\|^2 \\ &\leq \sum_{k=0}^K a_{n_k}(T_k)^2 \|f_k\|^2 + \sum_{k > K} \|T_k f_k\|^2 \leq S^2 \sum_{k=0}^{\infty} \|f_k\|^2 = S^2 \|f\|^2, \end{aligned}$$

hence the result. \square

We give now two corollaries of Lemma 5.3.

Example 1. We first use lens maps. We get:

Theorem 5.4. *Let λ_θ the lens map of parameter θ and let $\psi: \mathbb{D} \rightarrow \mathbb{D}$ such that $\|\psi\|_\infty := c < 1$ and h a Rudin function. We consider:*

$$\Phi(z_1, z_2) = (\lambda_\theta(z_1), \psi(z_1)h(z_2)).$$

Then, for some positive constant β , we have, for all $N \geq 1$:

$$(5.5) \quad a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}.$$

Proof. Let $T_k = M_{\psi^k} C_{\lambda_\theta}$. We have $\|T_k\| \leq c^k$, so $\sup_{k > K} \|T_k\| \leq c^K$. On the other hand, we have $a_n(T_k) \leq c^k a_n(C_{\lambda_\theta}) \leq a_n(C_{\lambda_\theta}) \lesssim e^{-\beta_\theta \sqrt{n}}$ ([16], Theorem 2.1). Taking $n_0 = n_1 = \dots = n_K = K^2$ in Lemma 5.3, we get:

$$\max_{0 \leq k \leq K} a_{n_k}(T_k) \lesssim e^{-\beta_\theta K}.$$

Since $n_0 + \dots + n_K - K \approx K^3$, we obtain $a_{K^3} \lesssim e^{-\beta K}$, which gives the claimed result, by taking $\beta = \max(\beta_\theta, \log(1/c))$. \square

Example 2. We consider the cusp map χ . We have:

Theorem 5.5. *Let χ be the cusp map, h a Rudin function, and ψ in the unit ball of H^∞ , with $\|\psi\|_\infty := c < 1$. Let:*

$$\Phi(z_1, z_2) = (\chi(z_1), \psi(z_1) h(z_2)).$$

Then, for positive constant β , we have, for all $N \geq 1$:

$$a_N(C_\Phi) \lesssim e^{-\beta \sqrt{N}/\sqrt{\log N}}.$$

Proof. Let $T_k = M_{\psi^k} C_\chi$. As above, we have $\sup_{k > K} \|T_k\| \leq c^K$. For the cusp map, we have $a_n(C_\chi) \lesssim e^{-\alpha n/\log n}$ ([20], Theorem 4.3); hence $a_n(T_k) \lesssim e^{-\alpha n/\log n}$. We take $n_0 = n_1 = \dots = n_K = K \lfloor \log K \rfloor$ (where $\lfloor \log K \rfloor$ is the integer part of $\log K$). Since $n_0 + \dots + n_K \approx K^2 \lfloor \log K \rfloor$, we get, for another $\alpha > 0$:

$$a_{K^2 \lfloor \log K \rfloor}(C_\Phi) \lesssim e^{-\alpha K},$$

which reads: $a_N(C_\Phi) \lesssim e^{-\beta \sqrt{N}/\sqrt{\log N}}$, as claimed. \square

5.2 Lower bounds

In this subsection, we give lower bounds for approximation numbers of composition operators on H^2 of the bidisk, attached to a symbol Φ of the previous form $\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$ where h is a Rudin function. The sharpness of those estimates will be discussed in the next subsection. We first need some lemmas in dimension one.

Lemma 5.6. *Let $u, v: \mathbb{D} \rightarrow \mathbb{D}$ be two non-constant analytic self-maps and $T = M_v C_u: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ be the associated weighted composition operator. For $0 < r < 1$, we set $A = u(r\overline{\mathbb{D}})$ and $\Gamma = \exp(-1/\text{Cap}(A))$. Then, for $0 < \delta \leq \inf_{|z|=r} |v(z)|$, we have:*

$$(5.6) \quad a_n(T) \gtrsim \sqrt{1-r} \delta \Gamma^n.$$

In this lemma, $\text{Cap}(A)$ denotes the Green capacity of the compact subset $A \subseteq \mathbb{D}$ (see [21], § 2.3 for the definition).

For the proof, we need the following result ([26], Theorem 7, p. 353).

Theorem 5.7 (Widom). *Let A be a compact subset of \mathbb{D} and $\mathcal{C}(A)$ be the space of continuous functions on A with its natural norm. Set:*

$$\tilde{d}_n(A) = \inf_E \left[\sup_{f \in B_{H^\infty}} \text{dist}(f, E) \right],$$

where E runs over all $(n-1)$ -dimensional subspaces of $\mathcal{C}(A)$ and $\text{dist}(f, E) = \inf_{h \in E} \|f - h\|_{\mathcal{C}(A)}$. Then

$$(5.7) \quad \tilde{d}_n(A) \geq \alpha e^{-n/\text{Cap}(A)}$$

for some positive constant α .

Proof of Lemma 5.6. We apply Theorem 5.7 to the compact set $A = u(r\overline{\mathbb{D}})$.

Let E be an $(n-1)$ -dimensional subspace of $H^2 = H^2(\mathbb{D})$; it can be viewed as a subspace of $\mathcal{C}(A)$, so, by Theorem 5.7, there exists $f \in H^\infty \subseteq H^2$ with $\|f\|_2 \leq \|f\|_\infty \leq 1$ such that:

$$\|f - h\|_{\mathcal{C}(A)} \geq \alpha \Gamma^n, \quad \forall h \in E.$$

Then:

$$\|v(f \circ u - h \circ u)\|_{\mathcal{C}(r\mathbb{T})} \geq \delta \|(f - h) \circ u\|_{\mathcal{C}(r\mathbb{T})} = \delta \|f - h\|_{\mathcal{C}(A)} \geq \alpha \delta \Gamma^n.$$

But:

$$\|v(f \circ u - h \circ u)\|_{\mathcal{C}(r\mathbb{T})} \leq \frac{1}{\sqrt{1-r^2}} \|v(f \circ u - h \circ u)\|_{H^2};$$

Hence:

$$\|Tf - Th\|_{H^2} \geq \alpha \sqrt{1-r^2} \delta \Gamma^n \geq \alpha \sqrt{1-r} \delta \Gamma^n.$$

Since h is an arbitrary function of E , we get (B_{H^2} being the unit ball of H^2):

$$\inf_{\dim E < n} \left[\sup_{f \in B_{H^2}} \text{dist}(Tf, T(E)) \right] \geq \alpha \sqrt{1-r} \delta \Gamma^n.$$

But the left-hand side is equal to the Kolmogorov number $d_n(T)$ of T (see [21], Lemma 3.12), and, as recalled in (1.7), in Hilbert spaces, the Kolmogorov numbers are equal to the approximation numbers; hence we obtain:

$$(5.8) \quad a_n(T) \geq \alpha \sqrt{1-r} \delta \Gamma^n, \quad n = 1, 2, \dots,$$

as announced. □

The next lemma shows that some Blaschke products are far away from 0 on some circles centered at 0.

We consider a *strongly interpolating sequence* $(z_j)_{j \geq 1}$ of \mathbb{D} in the sense that, if $\varepsilon_j := 1 - |z_j|$, then:

$$(5.9) \quad \varepsilon_{j+1} \leq \sigma \varepsilon_j$$

and so $\varepsilon_j \leq \sigma^{j-1}\varepsilon_1$, where $0 < \sigma < 1$ is fixed. Equivalently, the sequence $(|z_j|)_{j \geq 1}$ is interpolating. We consider the corresponding interpolating Blaschke product:

$$(5.10) \quad B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - z_j z}.$$

The following lemma is probably well-known, but we could find no satisfactory reference (see yet [10] for related estimates) and provide a simple proof.

Lemma 5.8. *Let $(z_j)_{j \geq 1}$ be a strongly interpolating sequence as in (5.9) and B the associated Blaschke product (5.10).*

Then there exists a sequence $r_l := 1 - \rho_l$ such that:

$$(5.11) \quad C_1 \sigma^l \leq \rho_l \leq C_2 \sigma^l,$$

where C_1, C_2 are positive constants, and for which:

$$(5.12) \quad |z| = r_l \implies |B(z)| \geq \delta,$$

where $\delta > 0$ does not depend on l .

Proof. Let us denote by p_l , $1 \leq p_l \leq l$, the biggest integer such that $\varepsilon_{p_l} \geq \sigma^{l-1}\varepsilon_1$.

We separate two cases.

Case 1: $\varepsilon_{p_l} \geq 2\sigma^{l-1}\varepsilon_1$.

Then, we choose $\rho_l = \alpha \sigma^{l-1}\varepsilon_1$ with α fixed, $1 < \alpha < 2$. Since $\rho(\xi, \zeta) \geq \rho(|\xi|, |\zeta|)$ for all $\xi, \zeta \in \mathbb{D}$ (recall that ρ is the pseudo-hyperbolic distance on \mathbb{D}), we have the following lower bound for $|z| = r_l$:

$$|B(z)| = \prod_{j=1}^{\infty} \rho(z, z_j) \geq \prod_{j=1}^{\infty} \rho(r_l, |z_j|) = \prod_{j \leq p_l} \rho(r_l, |z_j|) \times \prod_{j > p_l} \rho(r_l, |z_j|) := P_1 \times P_2,$$

and we estimate P_1 and P_2 separately.

We first observe that $\frac{\rho_l}{\varepsilon_{p_l}} \leq \frac{\alpha \sigma^{l-1}\varepsilon_1}{2\sigma^{l-1}\varepsilon_1} \leq \frac{\alpha}{2}$, and then:

$$\frac{\rho_l}{\varepsilon_j} = \frac{\rho_l}{\varepsilon_{p_l}} \frac{\varepsilon_{p_l}}{\varepsilon_j} \leq \frac{\alpha}{2} \sigma^{p_l-j}.$$

The inequality $\rho(1-u, 1-v) \geq \frac{|u-v|}{u+v}$ for $0 < u, v \leq 1$ now gives us:

$$(5.13) \quad \rho(r_l, |z_j|) \geq \frac{\varepsilon_j - \rho_l}{\varepsilon_j + \rho_l} = \frac{1 - \rho_l/\varepsilon_j}{1 + \rho_l/\varepsilon_j} \geq \frac{1 - (\alpha/2) \sigma^{p_l-j}}{1 + (\alpha/2) \sigma^{p_l-j}}, \quad \text{for } j \leq p_l,$$

and:

$$(5.14) \quad P_1 \geq \prod_{k=0}^{\infty} \left(\frac{1 - (\alpha/2) \sigma^k}{1 + (\alpha/2) \sigma^k} \right).$$

Similarly:

$$\frac{\varepsilon_{p_l+1}}{\rho_l} \leq \frac{\sigma^{l-1}\varepsilon_1}{\alpha\sigma^{l-1}\varepsilon_1} \leq \frac{1}{\alpha}$$

and:

$$\frac{\varepsilon_j}{\rho_l} \leq \frac{1}{\alpha}\sigma^{j-p_l-1} \quad \text{for } j > p_l;$$

so that:

$$(5.15) \quad \rho(r_l, |z_j|) \geq \frac{\rho_l - \varepsilon_j}{\rho_l + \varepsilon_j} = \frac{1 - \varepsilon_j/\rho_l}{1 + \varepsilon_j/\rho_l} \geq \frac{1 - \alpha^{-1}\sigma^{j-p_l-1}}{1 + \alpha^{-1}\sigma^{j-p_l-1}}, \quad \text{for } j > p_l,$$

and

$$(5.16) \quad P_2 \geq \prod_{k=0}^{\infty} \left(\frac{1 - \alpha^{-1}\sigma^k}{1 + \alpha^{-1}\sigma^k} \right).$$

Finally, the condition of lower and upper bound for ρ_l is fulfilled by construction.

Case 2: $\varepsilon_{p_l} \leq 2\sigma^{l-1}\varepsilon_1$.

Then, we choose $\rho_l = a\varepsilon_{p_l}$ with $\sigma < a < 1$ fixed. Computations exactly similar to those of Case 1 give us:

$$(5.17) \quad |B(z)| \geq \prod_{k=0}^{\infty} \left(\frac{1 - a\sigma^k}{1 + a\sigma^k} \right) \times \prod_{k=0}^{\infty} \left(\frac{1 - a^{-1}\sigma^k}{1 + a^{-1}\sigma^k} \right) =: \delta > 0, \quad \text{for } |z| = r_l.$$

Moreover, in this case:

$$a\sigma^{l-1}\varepsilon_1 \leq \rho_l \leq 2a\sigma^{l-1}\varepsilon_1,$$

and the proof is ended. \square

Now, we have the following estimation.

Theorem 5.9. *Let $\phi, \psi: \mathbb{D} \rightarrow \mathbb{D}$ be two non-constant analytic self-maps and $\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1)h(z_2))$, where h is inner.*

Let $(r_l)_{l \geq 1}$ be an increasing sequence of positive numbers with limit 1 such that:

$$\inf_{|z|=r_l} |\psi(z)| \geq \delta_l > 0,$$

with $\delta_l \leq e^{-1/\text{Cap}(A_l)}$, where $A_l = \phi(r_l\mathbb{D})$.

Then the approximation numbers $a_N(C_\Phi)$, $N \geq 1$, of the composition operator $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ satisfy:

$$(5.18) \quad a_N(C_\Phi) \gtrsim \sup_{l \geq 1} \left[\sqrt{1 - r_l} \exp(-8\sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)}) \right],$$

where:

$$(5.19) \quad \Gamma_l = e^{-1/\text{Cap}(A_l)}.$$

Proof. Since h is inner, the sequence $(h^k)_{k \geq 0}$ is orthonormal in H^2 and hence $a_n(C_\Phi) = a_n(T)$ for all $n \geq 1$, where $T = \bigoplus_{k=0}^{\infty} T_k$ and $T_k = M_{\psi^k} C_\phi$. Then Lemma 5.6 gives:

$$(5.20) \quad a_n(T_k) \gtrsim \sqrt{1-r_l} \delta_l^k \Gamma_l^n$$

for all $n \geq 1$ and all $k \geq 0$.

Let now:

$$(5.21) \quad p_l = \left\lfloor \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ stands for the integer part, and:

$$(5.22) \quad n_k = p_l k, \quad \text{for } k = 1, \dots, K.$$

By Lemma 5.2, applied with $m_k = k$ (i.e. to H_1, \dots, H_K), we have, if $N = n_1 + \dots + n_K$:

$$a_N(T) \geq \inf_{1 \leq k \leq K} \alpha \sqrt{1-r_l} \delta_l^k \Gamma_l^n = \alpha \sqrt{1-r_l} \delta_l^K \Gamma_l^{n_K}.$$

But, since $p_l \leq \log(1/\delta_l)/\log(1/\Gamma_l)$:

$$\delta_l^K \Gamma_l^{n_K} = \exp \left[- \left(K \log(1/\delta_l) + p_l K \log(1/\Gamma_l) \right) \right] \geq \exp[-2K \log(1/\delta_l)].$$

Since:

$$N = p_l \frac{K(K+1)}{2} \geq p_l \frac{K^2}{4} \geq \frac{K^2}{16} \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)},$$

we get:

$$\delta_l^K \Gamma_l^{n_K} \geq \exp \left[- 8 \sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)} \right],$$

and the result ensues. \square

Example 1. We take $\phi = \lambda_\theta$, a lens map, and $\psi = B$, a Blaschke product associated to a strongly regular sequence, as defined in (5.10); then we get:

Theorem 5.10. *Let $\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ be defined by:*

$$\Phi(z_1, z_2) = (\lambda_\theta(z_1), c B(z_1) h(z_2)),$$

where B is a Blaschke product as in (5.10), $0 < c < 1$, and h is an arbitrary inner function, we have, for some positive constant b , for all $N \geq 1$:

$$(5.23) \quad a_N(C_\Phi) \gtrsim \exp(-b N^{1/3}) = \exp(-b \sqrt{N}/N^{1/6}).$$

In particular $\beta_2(C_\Phi) = \beta_2^\pm(C_\Phi) = 1$.

Remark. We saw in Theorem 5.4 that this is the exact size, since we have: $a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}$.

Proof. By Lemma 5.8, there is a sequence of numbers $r_l \approx \sigma^l$ such that $|B(z)| \geq \delta$ for $|z| = r_l$, where δ is a positive constant (depending on σ). Since $\lambda_\theta(0) = 0$, we have:

$$\text{diam}_\rho(A_l) \geq \lambda_\theta(r_l) \gtrsim 1 - (1 - r_l)^\theta;$$

hence, by [21], Theorem 3.13, we have:

$$\text{Cap}(A_l) \gtrsim \log \frac{1}{1 - r_l} \gtrsim l,$$

or, equivalently: $\Gamma_l \geq e^{-b/l}$, some $b > 0$. Then (5.18) gives, for all $l \geq 1$ (with another b):

$$a_N(C_\Phi) \gtrsim \exp \left[-b \left(l + \frac{\sqrt{N}}{\sqrt{l}} \right) \right].$$

Taking $l = N^{1/3}$, we get the result. \square

Example 2. By taking the cusp instead of a lens map, we obtain a better result, close to the extremal one.

Theorem 5.11. *Let $\Phi(z_1, z_2) = (\chi(z_1), cB(z_1), h(z_2))$, where χ is the cusp map, B a Blaschke product as in (5.10), $0 < c < 1$, and h an arbitrary inner function. Then, for all $N \geq 1$:*

$$a_N(C_\Phi) \gtrsim e^{-b\sqrt{N}/\sqrt{\log N}}.$$

In particular $\beta_2(C_\Phi) = 1$.

Remark. We saw in Theorem 5.5 that this is the exact size, since we have: $a_N(C_\Phi) \lesssim e^{-\beta\sqrt{N}/\log N}$.

Proof. The proof is the same as that of Proposition 5.10, except that, for the cusp map, we have (note that $\chi(0) = 0$):

$$\text{diam}_\rho(A_l) \geq \chi(r_l).$$

But when r goes to 1:

$$1 - \chi(r) \sim \frac{\pi(\sqrt{2} - 1)}{2} \frac{1}{\log(1/(1 - r))}$$

(see [20], Lemma 4.2). Hence, by [21], Theorem 3.13, again, we have:

$$\text{Cap}(A_l) \gtrsim \log(\log(1/(1 - r_l))),$$

so $\Gamma_l \geq e^{-b/\log l}$. Then, (5.18) gives (with another b):

$$a_N(C_\Phi) \gtrsim \exp \left[-b \left(l + \frac{\sqrt{N}}{\sqrt{\log l}} \right) \right].$$

In taking $l = \sqrt{N/\log N}$, we get the announced result. \square

5.3 Upper bounds

All previous results point in the direction that, if $\|\Phi\|_\infty = 1$, then however small $a_n(C_\Phi)$ is, it will always be larger than $\alpha e^{-\beta \varepsilon_n \sqrt{n}}$ with $\varepsilon_n \rightarrow 0^+$, as this is the case in dimension one (with n instead of \sqrt{n}). But Theorem 5.12 to follow shows that we cannot hope, in full generality, to get the same result in dimension $d \geq 2$, and that other phenomena await to be understood. Here is our main result. It shows that, even for a truly 2-dimensional symbol Φ , we can have $\|\Phi\|_\infty = 1$ and nevertheless $\beta_2^+(C_\Phi) < 1$, in contrast to the 1-dimensional case where (1.1) holds.

Theorem 5.12. *There exist a map $\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ such that:*

- 1) *the composition operator $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ is bounded and compact;*
- 2) *we have $\|\Phi\|_\infty = 1$ and Φ is truly 2-dimensional, so that $\beta_2^-(C_\Phi) > 0$;*
- 3) *the singular numbers satisfy $a_n(C_\Phi) \leq \alpha e^{-\beta \sqrt{n}}$ for some positive constants α, β , in other terms $\beta_2^+(C_\Phi) < 1$.*

Proof. Let $0 < \theta < 1$ be fixed, and λ_θ be the corresponding lens map. We set:

$$\left\{ \begin{array}{l} \phi = \frac{1 + \lambda_\theta}{2} \\ S(z) = \left(\frac{1+z}{1-z} \right)^\theta \\ w = e^{-S} \\ \psi = \frac{1}{2}(w \circ \phi). \end{array} \right.$$

Note that $\|\phi\|_\infty = 1$.

Setting $\delta = \cos \theta\pi/2 > 0$, we have for $z \in \mathbb{D}$:

$$(5.24) \quad |1 - \phi(z)| = \frac{1}{2} |1 - \lambda_\theta(z)| = \left| \frac{(1-z)^\theta}{(1-z)^\theta + (1+z)^\theta} \right| \leq \frac{|1-z|^\theta}{\delta}.$$

Indeed, the argument α of $(1 \pm z)^\theta$ satisfies $|\alpha| \leq \theta\pi/2$ for $z \in \mathbb{D}$, and we get:

$$|(1-z)^\theta + (1+z)^\theta| \geq \Re[(1-z)^\theta + (1+z)^\theta] \geq \delta(|1+z|^\theta + |1-z|^\theta) \geq \delta.$$

We also see that $\phi(\mathbb{D})$ touches the boundary $\partial\mathbb{D}$ only at 1 in a non-tangential way, meaning that for some constant $C > 1$:

$$1 - |\phi(z)| \geq \frac{1}{C} |1 - \phi(z)|, \quad \forall z \in \mathbb{D}.$$

Now, we have the following two inequalities:

$$(5.25) \quad \Re z \geq 0 \implies |w(z)| \leq \exp\left(-\frac{\delta}{|1-z|^\theta}\right)$$

$$(5.26) \quad z \in \mathbb{D} \implies |\psi(z)| \leq \frac{1}{2} \exp\left(-\frac{\delta^2}{|1-z|^{\theta^2}}\right).$$

Indeed, $\Re S(z) \geq \delta |S(z)| \geq \delta |1-z|^{-\theta}$ when $\Re z \geq 0$, giving (5.25), and (5.24) and (5.25) imply, since $\Re \phi(z) \geq 0$:

$$|\psi(z)| = \frac{1}{2} |w(\phi(z))| \leq \frac{1}{2} \exp\left(-\frac{\delta}{|1-\phi(z)|^\theta}\right) \leq \frac{1}{2} \exp\left(-\frac{\delta^2}{|1-z|^{\theta^2}}\right).$$

We now set:

$$(5.27) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)),$$

with h a Rudin function.

Observe that the components ϕ and $\psi \otimes h$ of Φ can be in the bidisk algebra $A(\mathbb{D}^2)$, because we can take for h a finite Blaschke product, and moreover $\phi \in A(\mathbb{D})$ while $\psi = (1/2) w \circ \phi \in A(\mathbb{D})$ as well, since $w \in A(\mathbb{D})$ with $w(1) = 0$; this is due to the presence of the parameter $\theta < 1$.

Now, 1) follows from the orthogonal model presented in Section 5.1, because $\|\psi\|_\infty < 1$.

The assertion 2) follows from [2], Theorem 3.1, since $\|\phi\|_\infty = 1$.

We now prove 3).

As observed, C_Φ can be viewed as a direct sum $T = \bigoplus_{k=0}^\infty T_k$ acting on a Hilbertian sum $H = \bigoplus_{k=0}^\infty H_k$, where T_k acts on a copy H_k of $H^2(\mathbb{D})$ with:

$$T_k = M_{\psi^k} C_\phi.$$

We fix the positive integer n . The rest of the proof will consist of three lemmas.

Lemma 5.13. *We have $\|T_k\| \leq 2^{-n}$ for $k > n$.*

Proof. Indeed, since $\phi(0) = 1/2$, we know that $\|C_\phi\| \leq \sqrt{\frac{1+\phi(0)}{1-\phi(0)}} = \sqrt{3} \leq 2$, so that $\|T_k\| \leq \|\psi^k\|_\infty \|C_\phi\| \leq 2^{-k} \times 2$. \square

Lemma 5.14. *Set $b = a/\delta^2$ where $a > 0$ is given by $e^{-a} = 4C/\sqrt{16C^2 + 1}$ and C is as in (2.1). Let m_k be the smallest integer such that $k \delta^2 2^{m_k \theta^2} \geq an$; namely:*

$$(5.28) \quad m_k = \left\lceil \frac{\log(bn/k)}{\theta^2 \log 2} \right\rceil + 1,$$

where $\lceil \cdot \rceil$ stands for the integer part. Then, with $a' = \min(\log 2, a)$:

$$a_{nm_k+1}(T_k) \lesssim e^{-a'n}.$$

Proof. This follows from Theorem 2.3 applied with $w = \psi^k$, $R = k \delta^2$ and θ changed into θ^2 . This is possible thanks to (5.26) and to Lemma 5.13. Moreover we have adjusted m_k so as to make the two terms in Theorem 2.3 of the same order. \square

Lemma 5.15. *The dimension $d := \sum_{k=0}^n n m_k$ satisfies, for some positive constant α :*

$$d \leq \alpha n^2.$$

Proof. Indeed, it is well-known that:

$$\sum_{k=1}^n \log k = n \log n - n + O(\log n),$$

and, in view of (5.28), we have $m_k \leq \alpha'_\theta \log(bn/k) \leq \alpha''_\theta(\log n - \log k)$; hence:

$$\sum_{k=1}^n m_k \leq \alpha''_\theta [n \log n - (n \log n - n + O(\log n))] = \alpha''_\theta n + O(\log n),$$

and we get $d \leq \alpha''_\theta n^2 + O(n \log n) \leq \alpha_\theta n^2$.

Alternatively, we could have used a Riemann sum for the function $\log(1/x)$ on $(0, 1]$. \square

Finally, putting things together and using as well Proposition 5.3 with $K = n$ and $n_k = nm_k + 1$ so that $(\sum_{k=0}^n n_k) - n = (\sum_{k=0}^n n m_k) + 1 = d + 1$, we get ignoring once more multiplicative constants:

$$a_{n^2}(T) \lesssim a_d(T) \leq \alpha e^{-\beta n}$$

with positive constants α, β . This ends the proof of Theorem 5.12. \square

6 Monge-Ampère capacity and applications

6.1 Definition

Let K be a compact subset of \mathbb{D}^m (in this section, for notational reasons, we denote the dimension by m instead of d). The Monge-Ampère capacity of K has been defined by Bedford and Taylor ([3]; see also [13], § 5 or [11], Chapter 1) as:

$$\text{Cap}_m(K) = \sup \left\{ \int_K (dd^c u)^m; u \in PSH \text{ and } 0 \leq u \leq 1 \right\},$$

where PSH is the set of plurisubharmonic functions on \mathbb{D}^m , $dd^c = 2i\partial\bar{\partial}$, and $(dd^c)^m = dd^c \wedge \dots \wedge dd^c$ (m times). When $u \in PSH \cap \mathcal{C}^2(\mathbb{D}^m)$, we have:

$$(dd^c u)^m = 4^m m! \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV(z),$$

where $dV(z) = (i/2)^m dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_m \wedge d\bar{z}_m$ is the usual volume in \mathbb{C}^m . A more convenient formula (because \mathbb{D}^m is bounded and hyperconvex: see [11], p. 80, for the definition) is:

$$\text{Cap}_m(K) = \int_K (dd^c u_K^*)^m,$$

where u_K^* is called the *extremal function of K* and is the upper semi-continuous regularization of:

$$u_K = \sup\{v \in PSH; v \leq 0 \text{ and } v \leq -1 \text{ on } K\},$$

but we will not need that.

As in [27], we set:

$$(6.1) \quad \tau_m(K) = \frac{1}{(2\pi)^m} \text{Cap}_m(K).$$

For $m = 1$, $\tau(K) := \tau_1(K)$ is equal to the Green capacity $\text{Cap}(K)$ of K with respect to \mathbb{D} , with the definition used in [21] (see [13], Theorem 8.1, where a factor 2π is introduced).

We further set:

$$(6.2) \quad \Gamma_m(K) = \exp \left[- \left(\frac{m!}{\tau_m(K)} \right)^{1/m} \right].$$

We proved in [21] that, for $m = 1$, and $\varphi: \mathbb{D} \rightarrow r\mathbb{D}$, with $0 < r < 1$, we have:

$$(6.3) \quad \beta_1(C_\varphi) = \Gamma_1(\overline{\varphi(\mathbb{D})}).$$

The goal of this section is to see that Theorem 5.12 shows that this no longer holds for $m = 2$.

6.2 A seminal example

In one variable, our initial motivation had been the simple-minded example $\varphi(z) = rz$, $0 < r < 1$, for which $C_\varphi(z^n) = r^n z^n$, implying $a_n(C_\varphi) = r^{n-1}$ and $\beta_1(C_\varphi) = r$. If $K = \overline{\varphi(\mathbb{D})} = \overline{D}(0, r)$, we have $\text{Cap}(K) = \frac{1}{\log 1/r}$ and $\Gamma_1(K) = r$, so that $\beta_1(C_\varphi) = \Gamma_1(K)$. Let us examine the multivariate example (where $0 < r_j < 1$):

$$\Phi(z_1, z_2, \dots, z_m) = (r_1 z_1, r_2 z_2, \dots, r_m z_m).$$

If $K = \overline{\Phi(\mathbb{D}^m)}$, we have $K = \prod_{k=1}^m \overline{D}(0, r_k)$, and hence ([5], Theorem 3):

$$(6.4) \quad \tau_m(K) = \prod_{k=1}^m \frac{1}{\log(1/r_k)}.$$

On the other hand, $C_\Phi(z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}) = r_1^{n_1} r_2^{n_2} \dots r_m^{n_m} z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}$ so that the sequence $(a_n)_n$ of approximation numbers of C_Φ is the non-increasing rearrangement of the numbers $r_1^{n_1} r_2^{n_2} \dots r_m^{n_m}$. It is convenient to state the following simple lemma.

Lemma 6.1. *Let $\lambda_1, \dots, \lambda_m$ be positive numbers. Let N_A be the number of m -tuples of non-negative integers (n_1, \dots, n_m) such that $\sum_{k=1}^m \lambda_k n_k \leq A$. Then, as $A \rightarrow \infty$:*

$$N_A \sim \frac{A^m}{(\lambda_1 \dots \lambda_m) m!}.$$

Indeed, just apply Karamata's tauberian theorem (see [12] p. 30) to the generalized Dirichlet series:

$$S(\varepsilon) := \prod_{k=1}^m \frac{1}{1 - e^{-\lambda_k \varepsilon}} = \sum_{n_1, \dots, n_m \geq 0} e^{-(\sum_{k=1}^m \lambda_k n_k) \varepsilon};$$

we have $S(\varepsilon) \sim \frac{\varepsilon^{-m}}{(\lambda_1 \cdots \lambda_m)}$ as $\varepsilon \rightarrow 0^+$.

Let now N be a positive integer and $\varepsilon = a_N$. Setting $\lambda_k = \log(1/r_k)$ and $A = \log(1/\varepsilon)$, we see that N is the number of m -tuples (n_1, \dots, n_m) of non-negative integers such that $r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \geq \varepsilon$, i.e. such that $\sum_{k=1}^m \lambda_k n_k \leq A$. This number N is hence nothing but the number N_A of the previous lemma, so that:

$$N \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}.$$

Inverting this formula, we get:

$$a_N(C_\Phi) = \exp \left[- (1 + o(1)) (m! (\lambda_1 \lambda_2 \cdots \lambda_m) N)^{1/m} \right]$$

and:

$$\beta_m(C_\Phi) = \exp \left[- (m! \lambda_1 \lambda_2 \cdots \lambda_m)^{1/m} \right] = \Gamma_m(K),$$

in view of (6.2) and (6.4).

On the view of the simple-minded previous example, the extension of the spectral radius formula (6.3) to the multivariate case holds, and it is tempting to conjecture that this is a general phenomenon as in dimension one, all the more as the following extension of Widom's theorem was proved by Zakharyuta, based on the solution by S. Nivoche of Zakharyuta's conjecture ([22]); see also [27], Theorem 5.4. A compact subset K of \mathbb{D}^m is said to be *regular* if its extremal function u_K^* is continuous on \mathbb{D}^m .

Theorem 6.2 ([27], Theorem 5.6). *Let K be a regular compact subset of \mathbb{D}^m and $J: H^\infty(\mathbb{D}^m) \rightarrow \mathcal{C}(K)$ the canonical injection; then the Kolmogorov numbers $d_n(J)$ satisfy:*

$$(6.5) \quad \lim_{n \rightarrow \infty} [d_n(J)]^{1/n^{1/m}} = \exp \left[- \left(\frac{m!}{\tau_m(K)} \right)^{1/m} \right].$$

Note that the right side is nothing but $\Gamma_m(K)$.

We shall see in section 6.4 that, for the symbol Φ of Theorem 5.12, the compact subsets $\overline{\Phi(r\mathbb{D}^m)}$, $0 < r < 1$, are not regular.

6.3 Upper bound

For the upper bound, the situation behaves better, as stated in the following theorem.

Theorem 6.3 ([27], Proposition 6.1). *Let K be a compact subset of \mathbb{D}^m with non-void interior. Then:*

$$(6.6) \quad \limsup_{n \rightarrow \infty} [d_n(J)]^{1/n^{1/m}} \leq \exp \left[- \left(\frac{m!}{\tau_m(K)} \right)^{1/m} \right].$$

Note that (K, \mathbb{D}^m) is a condenser since K has non-void interior. We deduce the following upper bound.

Theorem 6.4. *Let Φ be an analytic self-map of \mathbb{D}^m with $\|\Phi\|_\infty = \rho < 1$, thus inducing a compact composition operator on $H^2(\mathbb{D}^m)$. Then we have:*

$$\beta_m^+(C_\Phi) \leq \Gamma_m(\overline{\Phi(\mathbb{D}^m)}).$$

Proof. This proof provides in particular a simplification of that given in [21] in dimension $m = 1$.

Changing n into n^m , Theorem 6.3 means that for every $\varepsilon > 0$, there exists an $(n^m - 1)$ -dimensional subspace V of $\mathcal{C}(K)$ such that, for any $g \in H^\infty(\mathbb{D}^m)$, there exists $h \in V$ such that:

$$(6.7) \quad \|g - h\|_{\mathcal{C}(K)} \leq C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n \|g\|_\infty.$$

Let l be an integer to be adjusted later, and $f(z) = \sum_\alpha b_\alpha z^\alpha \in B_{H^2}$, as well as $g(z) = \sum_{|\alpha| \leq l} b_\alpha z^\alpha$. We first note that (with M_m depending only on m and ρ , and since the number of α 's such that $|\alpha| \leq p$ is $O(p^m)$):

$$\sum_{|\alpha| > l} \rho^{2|\alpha|} \leq M_m \sum_{p > l} p^m \rho^{2p} \leq M_m l^m \frac{\rho^{2l}}{(1 - \rho^2)^{m+1}}.$$

We next observe that, by the Cauchy-Schwarz and Parseval inequalities:

$$(6.8) \quad \|g\|_\infty \leq M_m l^{m/2},$$

and

$$(6.9) \quad |f(z) - g(z)| \leq M_m l^{m/2} \frac{|z|_\infty^l}{(1 - |z|_\infty^2)^{(m+1)/2}}, \quad \forall z \in \mathbb{D}^m.$$

where $|z|_\infty := \max_{j \leq m} |z_j|$ if $z = (z_1, \dots, z_m)$.

The subspace F formed by functions $v \circ \Phi$, for $v \in V$, can be viewed as a subspace of $L^\infty(\mathbb{T}^m) \subseteq L^2(\mathbb{T}^m)$ with respect to the Haar measure of \mathbb{T}^m , the distinguished boundary of \mathbb{D}^m (indeed, we can write $(v \circ \Phi)^* = v \circ \Phi^*$, where Φ^* denotes the almost everywhere existing radial limits of $\Phi(rz)$, which belong to K). Let finally $E = P(F) \subseteq H^2(\mathbb{D}^m)$ where $P: L^2(\mathbb{T}^m) \rightarrow H^2(\mathbb{T}^m) = H^2(\mathbb{D}^m)$ is the orthogonal projection. This is a subspace of H^2 with dimension $< n^m$. Set temporarily $\eta = C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n$. It follows from (6.7) and (6.8) that, for some $h \in V$:

$$\|g - h\|_{\mathcal{C}(K)} \leq \eta \|g\|_\infty \leq \eta M_m l^{m/2}$$

and hence:

$$\|g \circ \Phi - h \circ \Phi\|_2 \leq \|g \circ \Phi - h \circ \Phi\|_\infty \leq \eta M_m l^{m/2},$$

implying by orthogonal projection:

$$\text{dist}(C_\Phi g, E) \leq \|g \circ \Phi - P(h \circ \Phi)\|_2 \leq \eta M_m l^{m/2}.$$

Now, since $C_\Phi f(z) - C_\Phi g(z) = f(\Phi(z)) - g(\Phi(z))$, (6.9) gives:

$$\|C_\Phi f - C_\Phi g\|_2 \leq \|C_\Phi f - C_\Phi g\|_\infty \leq M_m l^{m/2} \frac{\rho^l}{(1 - \rho^2)^{(m+1)/2}}$$

and hence:

$$\text{dist}(C_\Phi f, E) \leq M_m l^{m/2} \left(\frac{\rho^l}{(1 - \rho^2)^{(m+1)/2}} + C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n \right).$$

It ensues, since $a_N(C_\Phi) = d_N(C_\Phi)$, that:

$$[a_{n^m}(C_\Phi)]^{1/n} \leq (M_m l^{m/2})^{1/n} \left[\frac{\rho^{l/n}}{(1 - \rho^2)^{(m+1)/2n}} + C_\varepsilon^{1/n} (1 + \varepsilon) \Gamma_m(K) \right].$$

Taking now for l the integer part of $n \log n$, and passing to the upper limit as $n \rightarrow \infty$, we obtain (since $l/n \rightarrow \infty$ and $(\log l)/n \rightarrow 0$):

$$\beta_m^+(C_\Phi) \leq (1 + \varepsilon) \Gamma_m(K),$$

and Theorem 6.4 follows. \square

6.4 Exact asymptotic

From Theorem 6.2 we deduce the following result.

Theorem 6.5. *Let $\Phi: \mathbb{D}^m \rightarrow \mathbb{D}^m$ be an analytic self-map such that $\|\Phi\|_\infty = 1$. Assume moreover that for $0 < r < 1$, the compact subsets $K_r = \overline{\Phi(r\mathbb{D}^m)}$ are regular. Then:*

$$\beta_m^-(C_\Phi) = \beta_m(C_\Phi) = 1.$$

Proof. Without loss of generality, we may assume that $\Phi(0) = 0$.

Let $r_j < 1$ with $r_j \rightarrow 1^-$, and $K_j = \overline{\Phi(r_j\mathbb{D}^m)}$. We can apply Theorem 6.2 to K_j . Let E be a subspace of H^2 with dimension $< n^m$. This space can be considered as a subspace of $\mathcal{C}(K_j)$. By Theorem 6.2, and repeating word for word the proof of [21], we can find $g \in B_{H^2}$ with:

$$\|g - h\|_{\mathcal{C}(K_j)} \geq c_\varepsilon (1 - \varepsilon)^n [\Gamma_m(K_j)]^n$$

for all $h \in E$. Equivalently:

$$\|C_\Phi(g) - C_\Phi(h)\|_{\mathcal{C}(\overline{r_j\mathbb{D}^m})} \geq c_\varepsilon (1 - \varepsilon)^n [\Gamma_m(K_j)]^n.$$

This implies that:

$$\|C_\Phi(g) - C_\Phi(h)\|_2 \geq \eta_j c_\varepsilon (1 - \varepsilon)^n [\Gamma_m(K_j)]^n,$$

where $\eta_j > 0$ only depends on j .

Since $a_{n^m}(C_\Phi) = d_{n^m}(C_\Phi)$, we thus get:

$$a_{n^m}(C_\Phi) \geq \eta_j c_\varepsilon (1 - \varepsilon)^n [\Gamma_m(K_j)]^n.$$

Taking n -th roots and passing to the lower limit, we obtain that $\beta_m^-(C_\Phi) \geq (1 - \varepsilon) \Gamma_m(K_j)$. Finally, letting ε going to 0, we get:

$$(6.10) \quad \beta_m^-(C_\Phi) \geq \Gamma_m(K_j).$$

Now, since $\|\Phi\|_\infty = 1$, there exists a sequence of points $a_j \in K_j$ with $\|a_j\|_\infty \xrightarrow{j \rightarrow \infty} 1$ and we can assume that $|\pi_1(a_j)| \xrightarrow{j \rightarrow \infty} 1$, where π_1 is the projection on the first coordinate. By [13], Theorem 13.1, we have $\text{Cap}_m(K_j) \geq \text{Cap}_1[\pi_1(K_j)]$. Since $\text{Cap}_1[\pi_1(K_j)] = 2\pi \text{Cap}[\pi_1(K_j)] \xrightarrow{j \rightarrow \infty} +\infty$ (see [21], Theorem 3.13), we get that $\Gamma_m(K_j) \xrightarrow{j \rightarrow \infty} 1$, and that ends the proof. \square

Corollary 6.6. *For the symbol Φ constructed in Theorem 5.12, the compact subsets $K = \overline{\Phi(r\mathbb{D}^2)}$ are not all regular.*

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