Approximation numbers of composition operators on the Hardy space of the infinite polydisk

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Abstract. We study the composition operators of the Hardy space on $\mathbb{D}^\infty \cap \ell_1$, the ℓ_1 part of the infinite polydisk, and the behavior of their approximation numbers.

1 Introduction

Recently, in [2], we investigated approximation numbers $a_n(C_\varphi), n \geq 1$, of composition operators C_φ, $C_\varphi(f) = f \circ \varphi$, on the Hardy or Bergman spaces $H^2(\Omega)$, $B^2(\Omega)$ over a bounded symmetric domain $\Omega \subseteq \mathbb{C}^d$. Assuming that $\varphi(\Omega)$ has non-empty interior, one of the main results of this study was the following theorem.

Theorem 1.1 ([2]). *Let $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$ be compact. Then:*

1) *we always have $a_n(C_\varphi) \geq c\,\mathrm{e}^{-C\,n^{1/d}}$ where c, C are positive constants;*

2) *if Ω is a product of balls and if $\varphi(\Omega) \subseteq r\,\Omega$ for some $r < 1$, then:*

$$a_n(C_\varphi) \leq C\,\mathrm{e}^{-c\,n^{1/d}}\,.$$

As a result, the minimal decay of approximation numbers is slower and slower as the dimension d increases, which might lead one to think that, in infinite-dimension, no compact composition operators can exist, since their approximation numbers will not tend to 0. After all, this is the case for the Hardy space of a half-plane, which supports no compact composition operator ([12], Theorem 3.1; in [9], it is moreover proved that $\|C_\varphi\|_e = \|C_\varphi\|$ as soon as C_φ is bounded; see also [15] for a necessary and sufficient condition for $H^2(\Omega)$ has compact composition operators, where Ω is a domain of $\mathbb{C}$). We will see that this is not quite the case here, even though the decay will be severely limited. In particular, we will never have a decay of the form $C\,\mathrm{e}^{-c\,n^{\delta}}$ for some $c, C, \delta > 0$.

2 Framework and reminders

2.1 Hardy spaces on $\mathbb{D}^\infty$

Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle of the set of complex numbers. We consider $\mathbb{T}^\infty$ and equip it with its Haar measure m. This is a compact Abelian group with dual $\mathbb{Z}^{(\infty)}$, the set of eventually zero sequences $\alpha = (\alpha_j)_{j\geq 1}$ of integers. We denote $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty)$ the Hilbert subspace of $L^2(\mathbb{T}^\infty)$ formed by the functions f whose Fourier spectrum is contained in $\mathbb{N}^{(\infty)}$:

$$\widehat{f}(\alpha) := \int_{\mathbb{T}^\infty} f(z)\,\overline{z}^\alpha\,dm(z) = 0 \qquad \text{if } \alpha \notin \mathbb{N}^{(\infty)}\,.$$

The set $E := \mathbb{N}^{(\infty)}$ is called the *narrow cone of Helson*, and we also denote $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty) = L^2_E(\mathbb{T}^\infty)$. Any element of that subspace can be formally written as:

$$f = \sum_{\alpha\geq 0} c_\alpha\, e_\alpha \qquad \text{with } c_\alpha = \widehat{f}(\alpha) \quad \text{and} \quad \sum_{\alpha\geq 0} |c_\alpha|^2 < \infty\,.$$

Here, $(e_\alpha)_{\alpha\in\mathbb{Z}^{(\infty)}}$ is the canonical basis of $L^2(\mathbb{T}^\infty)$ formed by characters, and accordingly $(e_\alpha)_{\alpha\in\mathbb{N}^{(\infty)}}$ is the canonical basis of $L^2_E(\mathbb{T}^\infty)$.

Now we consider $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$.

Any $f \sim \sum_{\alpha\geq 0} c_\alpha\, e_\alpha \in L^2_E(\mathbb{T}^\infty)$ defines an analytic function on the infinite-dimensional Reinhardt domain Ω_2 by the formula:

$$f(z) = \sum_{\alpha\geq 0} c_\alpha\, z^\alpha \tag{2.1}$$

where the series is absolutely convergent for each $z = (z_j)_{j\geq 1} \in \Omega_2$, as the pointwise product of two square-summable sequences. Indeed, using an Euler type formula, we get for $z \in \Omega_2$:

$$\sum_{\alpha\geq 0} |z^\alpha|^2 = \prod_{j=1}^{\infty} (1 - |z_j|^2)^{-1} < \infty\,,$$

and hence, by the Cauchy-Schwarz inequality:

$$\sum_{\alpha\geq 0} |c_\alpha\, z^\alpha| \leq \Big(\sum_{\alpha\geq 0} |c_\alpha|^2\Big)^{1/2} \Big(\sum_{\alpha\geq 0} |z^\alpha|^2\Big)^{1/2} < \infty\,.$$

If $\alpha \in E$ and $z \in \Omega_2$, we have set, as usual, $z^\alpha = \prod_{j\geq 1} z_j^{\alpha_j}$.

This shows that $L^2_E(\mathbb{T}^\infty)$ can be identified with $H^2(\Omega_2)$, the Hardy-Hilbert space of analytic functions $f(z) = \sum_{\alpha\geq 0} c_\alpha\, z^\alpha$ on Ω_2 with

$$\|f\|^2 := \sum_{\alpha\geq 0} |c_\alpha|^2 < \infty\,.$$

This setting is customary in connection with Dirichlet series (see [7]).

In this paper, for a technical reason appearing below in the proof of Proposition 2.5, we will consider, instead of $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$, the sub-domain:

$$\Omega = \mathbb{D}^\infty \cap \ell_1 \,,$$

i.e. the *open* subset of ℓ^1 formed by the sequences:

$$z = (z_n)_{n\geq 1} \quad \text{such that} \quad |z_n| < 1 \,, \forall\, n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} |z_n| < \infty \,,$$

and the restrictions to Ω of the functions $f \in H^2(\Omega_2)$. We denote $H^2(\Omega)$ the space of such restrictions.

Hence $f \in H^2(\Omega)$ if and only if:

$$f(z) = \sum_{\alpha \geq 0} c_\alpha \, z^\alpha \quad \text{with } z \in \Omega \,,$$

and $\|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty$.

We now identify the space $L^2_E(\mathbb{T}^\infty)$ with the space $H^2(\Omega)$.

We more generally define Hardy spaces $H^p(\Omega)$, for $1 \leq p < \infty$, in the usual way:

$$H^p = H^p(\Omega) = \{ f \colon \Omega \to \mathbb{C} \,;\ \|f\|_p < \infty \} \,,$$

where f is analytic in Ω and $\|f\|_p = \sup_{0<r<1} M_p(r, f) = \lim_{r \to 1^-} M_p(r, f)$ with:

$$M_p(r, f) = \left(\int_{\mathbb{T}^\infty} |f(rz)|^p \, dm(z) \right)^{1/p} , \quad 0 < r < 1 \,.$$

We have $\|f\| = \|f\|_2$. Moreover, H^q contractively embeds into H^p for $p < q$.

2.2 Singular numbers

We begin with a reminder of operator-theoretic facts. We recall that the approximation numbers $a_n(T) = a_n$ of an operator $T \colon H \to H$ (with H a Hilbert space) are defined by:

$$a_n = \inf_{\operatorname{rank} R < n} \|T - R\| \,.$$

According to a 1957's result of Allahverdiev (see [3], page 155), we have $a_n = s_n$, the n-th singular number of T. We also recall a basic result due to H. Weyl and one obvious consequence:

Theorem 2.1. *Let $T \colon H \to H$ be a compact operator with eigenvalues (λ_n) rearranged in decreasing order and singular numbers (a_n). Then:*

$$\prod_{j=1}^{n} |\lambda_j| \leq \prod_{j=1}^{n} a_j \quad \textit{for all } n \geq 1 \,.$$

As a consequence:

$$|\lambda_{2n}|^2 \leq a_1 a_n.$$

2.3 Spectra of projective tensor products

The following operator-theoretic result will play a basic role in the sequel. Let $E_1, \ldots, E_n$ be Banach spaces and let $E = \otimes_{i=1}^n E_i$ their *projective* tensor product (the only tensor product we shall use). If $T_i \in \mathcal{L}(E_i)$, we define as usual their projective tensor product $T = \otimes_{i=1}^n T_i \in \mathcal{L}(E)$ by its action on the atoms of E, namely:

$$T(\otimes_{i=1}^n x_i) = \otimes_{i=1}^n T_i(x_i)\,.$$

Denote in general $\sigma(x)$ the spectrum of $x \in \mathcal{A}$ where $\mathcal{A}$ is a unital Banach algebra. We recall ([13], chap.11, Theorem 11.23) the following result.

Lemma 2.2. *Let $\mathcal{A}$ be a unital Banach algebra, and $x_1, \ldots, x_n$ be* pairwise commuting *elements of $\mathcal{A}$. Then:*

$$\sigma(x_1 \cdots x_n) \subseteq \prod_{i=1}^n \sigma(x_i)\,.$$

Here, $\prod_{i=1}^n \sigma(x_i)$ is the product in the Minkowski sense, namely:

$$\prod_{i=1}^n \sigma(x_i) = \left\{ \prod_{i=1}^n \lambda_i : \lambda_i \in \sigma(x_i) \right\}.$$

As a consequence, we then have the following lemma due to Schechter, which we prove under a weakened form, sufficient here, and which is indeed already in [1] (we just add a few details because this is a central point in our estimates).

Lemma 2.3. *Let F be a Banach space, $T_1, \ldots, T_n \in \mathcal{L}(F)$ and $T = \otimes_{i=1}^n T_i$. Then $\sigma(T) \subset \prod_{i=1}^n \sigma(T_i)$.*

Proof. To save notation, we assume $n = 2$. Let $x_1 = T_1 \otimes I_2$ and $x_2 = I_1 \otimes T_2$ where I_i is the identity of E_i. Clearly,

$$x_1 x_2 = x_2 x_1 = T_1 \otimes T_2 = T \quad \text{and} \quad \sigma(x_i) = \sigma(T_i)$$

where the spectrum of x_i is in the Banach algebra $\mathcal{L}(E)$ and that of T_i in $\mathcal{L}(E_i)$. Lemma 2.2 now gives:

$$\sigma(T) = \sigma(x_1 x_2) \subseteq \sigma(x_1)\,\sigma(x_2) = \sigma(T_1)\,\sigma(T_2)\,,$$

hence the result. □

2.4 Schur maps and composition operators

We now pass to some general facts on composition operators C_φ, defined by $C_\varphi(f) = f \circ \varphi$, associated with a Schur map, namely a *non-constant* analytic self-map φ of Ω. We say that φ is a *symbol* for $H^2(\Omega)$ if C_φ is a bounded linear operator from $H^2(\Omega)$ into itself.

The differential $\varphi'(a)$ of φ at some point $a \in \Omega$ is a bounded linear map $\varphi'(a) \colon \ell^1 \to \ell^1$.

Definition 2.4. *The symbol φ is said to be* truly infinite-dimensional *if the differential $\varphi'(a)$ is an* injective *linear map from ℓ^1 into itself for at least one point $a \in \Omega$.*

In finite dimension, this amounts to saying that $\varphi(\Omega)$ has non-void interior.

We have the following general result.

Proposition 2.5. *Let $(\varphi_j)_{j\geq 1}$ be a sequence of analytic self-maps of $\mathbb{D}$ such that $\sum_{j\geq 1} |\varphi_j(0)| < \infty$. Then, the mapping $\varphi\colon \Omega \to \mathbb{C}^\infty$ defined by the formula $\varphi(z) = (\varphi_j(z_j))_{j\geq 1}$ maps Ω to itself and is a symbol for $H^2(\Omega)$.*

Proof. First, the Schwarz inequality:

$$|\varphi_j(z_j) - \varphi_j(0)| \leq 2\,|z_j|$$

shows that $\varphi(z) \in \Omega$ when $z \in \Omega$. To see that φ is moreover a symbol for $H^2(\Omega)$, we use the fact ([8]) that:

$$\|C_{\varphi_j}\| \leq \sqrt{\frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|}}. \tag{2.2}$$

Now, by the separation of variables and Fubini's theorem, we easily get:

$$\|C_\varphi\| \leq \prod_{j=1}^{\infty} \|C_{\varphi_j}\| < \infty\,. \tag{2.3}$$

As $\sum_{j\geq 1} |\varphi_j(0)| < \infty$, by hypothesis, the infinite product

$$\prod_{j\geq 1} \sqrt{\frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|}}$$

converges and, in view of (2.2) and (2.3), C_φ is bounded. $\square$

We also have the following useful fact.

Lemma 2.6. *The automorphisms of Ω act transitively on Ω and define bounded composition operators on $H^2(\Omega)$.*

Proof. Let $a = (a_j)_j \in \Omega$ and let $\Psi_a\colon \Omega \to \mathbb{C}^\infty$ be defined by:

$$\Psi_a(z) = \big(\Phi_{a_j}(z_j)\big)_{j\geq 1}$$

where in general $\Phi_u\colon \mathbb{D} \to \mathbb{D}$ is defined by $\Phi_u(z) = (z - u)/(1 - \overline{u}z)$. The Schwarz lemma gives $|\Phi_{a_j}(z_j) + a_j| \leq 2|z_j|$, and shows that Ψ_a maps Ω to itself. Clearly, Ψ_a is an automorphism of Ω with inverse Ψ_{-a} and $\Psi_a(a) = 0$. The fact that the composition operator C_{Ψ_a} is bounded on $H^2(\Omega)$ is a consequence of Proposition 2.5. $\square$

3 Spectrum of compact composition operators

We begin with the following definition, following [10].

Definition 3.1. *Let $\varphi\colon \Omega \to \Omega$ be a truly infinite-dimensional symbol. We say that φ is* compact *if $\overline{\varphi(\Omega)}$ is a compact subset of Ω.*

We then have the following result.

Lemma 3.2. *If $\varphi\colon \Omega \to \Omega$ is a* compact mapping*, then:*

1) *$C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ is bounded and moreover compact.*

2) *If $a \in \Omega$ a fixed point of φ, $\varphi'(a) \in \mathcal{L}(\ell^1)$ is a compact operator.*

Proof. 1) follows from a H. Schwarz type criterion via an Ascoli-Montel type theorem: every sequence (f_n) of $H^2(\Omega)$ bounded in norm contains a subsequence which converges uniformly on compact subsets of Ω. Indeed, we have the following ([4], chap. 17, p. 274): if A is a locally bounded set of holomorphic functions on Ω, then A is locally equi-Lipschitz, namely every point $a \in \Omega$ has a neighourhood $U \subset \Omega$ such that:

$$z, w \in U \quad \text{and} \quad f \in A \quad \Longrightarrow \quad |f(z) - f(w)| \leq C_{A,U}\, \|z - w\|\,.$$

The Ascoli-Montel theorem easily follows from this. Then, if $f_n \in H^2(\Omega)$ converges weakly to 0, it converges uniformly to 0 on compact subsets of Ω; in particular on $\overline{\varphi(\Omega)}$. This means that $\|C_\varphi(f_n)\|_\infty = \|f_n \circ \varphi\|_\infty \to 0$, implying $\|f_n \circ \varphi\|_2 \to 0$ and the compactness of C_φ.

Actually, C_φ is compact on every Hardy space $H^p(\Omega)$, $1 \leq p \leq \infty$. This observation will be useful later on.

For 2), we may indeed dispense ourselves with the invariance of a, and force $a = 0$ to be a fixed point of φ. Indeed, we can replace φ by $\psi = \Psi_b \circ \varphi \circ \Psi_a$ where $b = \varphi(a)$ is arbitrary, and use Lemma 2.6 as well as the ideal property of compact linear operators. We set $A = \varphi'(0)$. Expanding each coordinate φ_j of φ in a series of homogeneous polynomials, we may write (since $\varphi(0) = 0$):

$$\varphi(z) = \sum_{|\alpha|=1} c_\alpha z^\alpha + \sum_{s=2}^{\infty} \Big(\sum_{|\alpha|=s} c_\alpha z^\alpha \Big) = A(z) + \sum_{s=2}^{\infty} \Big(\sum_{|\alpha|=s} c_\alpha z^\alpha \Big)\,,$$

where $c_\alpha = (c_{\alpha,j})_{j\geq 1} \in \mathbb{C}^\infty$. We clearly have (looking at the Fourier series of $\varphi(z\,\mathrm{e}^{i\theta})$):

$$(3.1) \qquad \|z\|_1 < 1 \quad \Longrightarrow \quad z \in \Omega \quad \Longrightarrow \quad A(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z\,\mathrm{e}^{i\theta})\,\mathrm{e}^{-i\theta}\, d\theta\,.$$

Since φ is compact, this clearly implies, with B the open unit ball of ℓ^1, that $A(B)$ is totally bounded, proving the compactness of A. □

The following extension of results of [11], then [1] and [6], which themselves extend a theorem of G. Königs ([14], p. 93) will play an essential role for lower bounds of approximation numbers.

Theorem 3.3. *Let $\varphi\colon \Omega \to \Omega$ be a compact symbol. Assume there is $a \in \Omega$ such that $\varphi(a) = a$ and that $\varphi'(a) \in \mathcal{L}(\ell^1)$ is* injective. *Then, the spectrum of $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ is exactly formed by the numbers λ^α, $\alpha \in \mathbb{N}^{(\infty)}$, and $0, 1$, where $(\lambda_j)_{j\geq 1}$ denote the eigenvalues of $A := \varphi'(a)$ and:*

$$\lambda^\alpha = \prod_{j\geq 1} \lambda_j^{\alpha_j} \qquad \textit{if} \quad \alpha = (\alpha_j)_{j\geq 1} \in \mathbb{N}^{(\infty)}\,.$$

Proof. This is proved in [1] for the unit ball B_E of an arbitrary Banach space E and for the space $H^\infty(B_E)$, in four steps which are the following:

1. If $\varphi(B_E)$ lies strictly inside B_E (namely if $\varphi(B_E) \subseteq rB_E$ for some $r < 1$), in particular when φ is compact, φ has a unique fixed point $a \in B_E$, according to a theorem of Earle and Hamilton.

2. The spectrum of C_φ contains the numbers λ where λ is an eigenvalue of $\varphi'(a)$ or $\lambda = 0, 1$.

3. It is then proved that the spectrum of C_φ contains the numbers λ^α and $0, 1$.

4. It is finally proved that spectrum of C_φ is contained in the numbers λ^α and $0, 1$.

Here, handling with the domain Ω, we see that:

1. True or not for Ω, the Earle-Hamilton theorem is not needed since we will force, by a change of the compact symbol φ in another compact symbol $\psi = \Psi_b \circ \varphi \circ \Psi_a$, the point 0 to be a fixed point. Moreover $A = \psi'(0)$ is injective if $\varphi'(a)$ is, since Ψ_a' and Ψ_b' are invertible.

2. The second step (non-surjectivity) is valid for any domain and for $H^2(\Omega)$, or $H^p(\Omega)$, in exactly the same way.

3. The third step consists of proving $\{\lambda^\alpha\} \subseteq \sigma(C_\varphi)$.

For that purpose, assume that $\lambda^\alpha = \prod_{l=1}^m \lambda_l \neq 0$ with λ_l an eigenvalue of $\varphi'(0)$ and with repetitions allowed. As we already mentioned, under the assumption of compactness of φ, C_φ is compact on $H^p(\Omega)$ as well, for any $p \geq 1$. We take here $p = 2m$. Step 2 provides us with non-zero functions $f_i \in H^p(\Omega)$ such that $f_i \circ \varphi = \lambda_i f_i$, $1 \leq i \leq m$, since for the compact operator $C_\varphi\colon H^p \to H^p$, non-surjectivity implies non-injectivity. Let $f = \prod_{1\leq i\leq m} f_i$. Then, using the integral representation of the norm and the Hölder inequality, we see that $f \in H^2(\Omega)$, $f \neq 0$ and $f \circ \varphi = \lambda^\alpha f$, proving our claim.

4. The fourth step is valid as well, with a slight simplification: we have to show that, if $\mu \neq 1$ is not of the form λ^α, then $C_\varphi - \mu I$ is injective. Let $f \in H^2(\Omega)$ satisfying $f \circ \varphi = \mu f$ and let:

$$f(z) = \sum_{m=0}^\infty \frac{d^m f(0)}{m!}(z^m)$$

be the Taylor expansion of f about $z = 0$ (observe that Ω is a Reinhardt domain). As usual, $d^m f(0) =: L_m$ is an m-linear symmetric form on $F = \ell^1$ and the notation $L_m(z^m)$ means $L_m(z, z, \ldots, z)$.

Observe that L_m can be isometrically identified with an element (denoted $\overline{L_m}$) of $\mathcal{L}(F^{\otimes n})$ defined by the formula:

$$\overline{L_m}(x_1 \otimes \cdots \otimes x_n) = L_m(x_1, \ldots, x_m)\,.$$

We will prove by induction that $L_n = 0$ for each n. For this, we can avoid the appeal to transposes of [1] as follows: if the result holds for L_m with $m < n$, one gets (comparing the terms in z^n in both members of $f \circ \varphi = \mu f$):

$$\mu A = A \circ B \quad \text{where} \quad A = \overline{L_n} \quad \text{and} \quad B = \varphi'(0)^{\otimes n}\,. \tag{3.2}$$

That is $A(B - \mu I) = 0$ where I is the identity map of $F^{\otimes n}$. Now, $B - \mu I$ in invertible in $\mathcal{L}(F)$ by Lemma 3.3, so that $A = A(B - \mu I)(B - \mu I)^{-1} = 0$.

The proof is complete. □

The following theorem summarizes and exploits the preceding theorem. Possibly, some restrictions can be removed, and we could only assume the compactness of C_φ, not of φ itself. After all, in dimension one, there are symbols φ with $\|\varphi\|_\infty = 1$ for which $C_\varphi \colon H^2 \to H^2$ is compact.

Theorem 3.4. *Let $\varphi\colon \Omega \to \Omega$ be a truly infinite-dimensional* compact *mapping of Ω. Then:*

1) *$C_\varphi : H^2(\Omega) \to H^2(\Omega)$ is bounded and even compact.*

2) *$A = \varphi'(0)$ is compact.*

3) *No $\delta > 0$ can exist such that $a_n(C_\varphi) \leq C\,\mathrm{e}^{-c\,n^\delta}$ for all $n \geq 1$. More precisely, the numbers a_n satisfy:*

$$\sum_{n\geq 1} \frac{1}{\log^p(1/a_n)} = \infty \quad \textit{for all} \quad p < \infty\,. \tag{3.3}$$

Proof. The proof is based on the previous Theorem 3.3. Without loss of generality, we can assume that $\varphi(0) = 0$ and $\varphi'(0)$ is injective, by using a point a at which $\varphi'(a)$ is injective, and then the fact that automorphisms of Ω act transitively on Ω, act boundedly on $H^2(\Omega)$, and the ideal property of approximation numbers. More precisely, we pass to $\Psi = \Psi_b \circ \varphi \circ \Psi_a$ with $b = \varphi(a)$ and get:

$$\Psi(0) = 0 \quad \text{and} \quad \Psi'(b) = \Psi_b'(b)\,\varphi'(a)\,\Psi_a'(0)$$

injective, since $\Psi_b'(b)$ and $\Psi_a'(0)$ are, and Ψ_a and Ψ_b are automorphisms of Ω.

We now have the following simple but crucial lemma.

Lemma 3.5. *Whatever the choice of the numbers λ_j with $0 < |\lambda_j| < 1$, denoting by $(\delta_n)_{n\geq 1}$ the non-increasing rearrangement of the numbers λ^α, one has:*

$$\sum_{n\geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty \quad \textit{for all} \quad p < \infty\,.$$

Proof of the Lemma. For any positive integer p, we set:

$$q = 2p\,, \quad \log 1/|\lambda_j| = A_j\,,$$

and we use that:

$$\sum_{1\leq j\leq q} \alpha_j\, A_j \leq \Big(\sum_{1\leq j\leq q} \alpha_j^2\Big)\Big(\sum_{1\leq j\leq q} A_j^2\Big) =: C_q \Big(\sum_{1\leq j\leq q} \alpha_j^2\Big) = C_q \|\alpha\|^2\,,$$

where $\|\,.\,\|$ stands for the euclidean norm in $\mathbb{R}^q$. We then get:

$$\begin{aligned}
\sum_{n\geq 1} \frac{1}{\log^p(1/\delta_n)} &= \sum_{\alpha>0} \frac{1}{\log^p(1/|\lambda^\alpha|)} \\
&\geq \sum_{\alpha_j\geq 1,\, 1\leq j\leq q} \frac{1}{\log^p(1/|\lambda_1^{\alpha_1}|\cdots 1/|\lambda_q^{\alpha_q}|)} \\
&= \sum_{\alpha_j\geq 1,\, 1\leq j\leq q} \frac{1}{(\alpha_1 A_1 + \cdots + \alpha_q A_q)^p} \\
&\geq C_q^{-p} \sum_{\alpha_j\geq 1,\, 1\leq j\leq q} \frac{1}{(\alpha_1^2 + \cdots + \alpha_q^2)^p} \\
&= C_q^{-p} \sum_{\alpha_j\geq 1,\, 1\leq j\leq q} \frac{1}{\|\alpha\|^q} = \infty\,,
\end{aligned}$$

because:

$$\int_{x\in\mathbb{R}^q,\, \|x\|\geq 1} \frac{1}{\|x\|^q}\, dx = c_q \int_1^\infty \frac{r^{q-1}}{r^q}\, dr = \infty\,.$$

This proves the lemma. □

This can be transferred to the approximation numbers $a_n = a_n(C_\varphi)$ to end the proof of Theorem 3.4. Indeed, we know from Lemma 3.5 that the non-increasing rearrangement (δ_n) of the eigenvalues λ^α of C_φ satisfies

$$\sum_{n\geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty\,.$$

Since a divergent series of non-negative and non-increasing numbers u_n satisfies $\sum u_{2n} = \infty$, we further see that:

$$\sum_{n\geq 1} \frac{1}{\log^p(1/\delta_{2n})} = \infty \quad \text{for all} \quad p < \infty\,.$$

Moreover, by Theorem 2.1 we have:

$$\Big(\frac{1}{2\log 1/\delta_{2n}}\Big)^p \leq \Big(\frac{1}{\log 1/(a_1 a_n)}\Big)^p. \tag{3.4}$$

Since $1/(\log 1/a_1 a_n) \sim 1/(\log 1/a_n)$, Lemma 3.5 then gives the result. This clearly prevents an inequality of the form $a_n \leq C\, \mathrm{e}^{-c\, n^{\delta}}$ for some positive numbers c, C, δ and all $n \geq 1$. Indeed, this would imply:

$$\sum_{n\geq 1} \frac{1}{\log^p (1/a_n)} < \infty \quad \text{for} \quad p > 1/\delta \,,$$

contradicting (3.3). $\square$

Remarks. Let us briefly comment on the assumptions in Theorem 3.4.

1) We do not need the Earle-Hamilton theorem under our assumptions. The Schauder-Tychonoff theorem gives the existence (if not the uniqueness) of a fixed point for φ in Ω (bounded and convex).

2) The Earle-Hamilton theorem is in some sense more general (for analytic maps) since it remains valid when $\overline{\varphi(\Omega)}$ is only assumed to lie strictly inside Ω, i.e. when $\varphi(\Omega) \subseteq r\Omega$ for some $r < 1$. But this assumption does not ensure the compactness of C_φ as indicated by the simple example $\varphi(z) = rz$, $0 < r < 1$. The coordinate functions $z \mapsto z_n$ converge weakly to 0, while $\|C_\varphi(z_n)\|_{H^2(\Omega)} = r$.

3) The mere assumption that $\overline{\varphi(\Omega)}$ is compact is not sufficient either. Juste take:

$$\varphi(z) = \left(\frac{1 + z_1}{2}, 0, \ldots, 0, \ldots \right).$$

Since the composition operator C_{φ_1} associated with $\varphi_1(z) = \frac{1+z}{2}$ is notoriously non-compact on $H^2(\mathbb{D})$, neither is C_φ on $H^2(\Omega)$. Yet, $\overline{\varphi(\Omega)}$ is obviously compact in ℓ^1.

4 Possible upper bounds

Recall that $\Omega = \mathbb{D}^\infty \cap \ell^1$.

4.1 A general example

Theorem 4.1. *Let $\varphi((z_j)_j) = (\lambda_j z_j)_j$ with $|\lambda_j| < 1$ for all j, so that $\varphi(\Omega) \subseteq \Omega$ and $\varphi'(0)$ is the diagonal operator with eigenvalues λ_j, $j \geq 1$, on the canonical basis of ℓ^1. Let $p > 0$. Then:*

$$(\lambda_j)_j \in \ell^p \quad \Longrightarrow \quad C_\varphi \in S_p \,.$$

In particular, there exist truly infinite-dimensional symbols on Ω such that the composition operator $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$ is in all Schatten classes S_p, $p > 0$.

Proof. Since C_φ is diagonal on the orthonormal basis $(z^\alpha)_\alpha$ of the Hilbert space $H^2(\Omega)$, with $C_\varphi(z^\alpha) = \varphi^\alpha$, its approximation numbers are the non-increasing rearrangement of the moduli of eigenvalues λ^α, so that an Euler product-type computation gives:

$$\sum_{n=1}^{\infty} a_n^p = \sum_{\alpha \in E} |\lambda^\alpha|^p = \sum_{\alpha_j \in \mathbb{N}} \prod_{j \geq 1} |\lambda_j|^{p\alpha_j} = \prod_{j=1}^{\infty} (1 - |\lambda_j|^p)^{-1} < \infty \, .$$

To obtain $C_\varphi \in \bigcap_{p>0} S_p$, just take $\lambda_n = \mathrm{e}^{-n}$. This ends the proof. $\square$

4.2 A sharper upper bound

By making a more quantitative study, we can prove the following result.

Theorem 4.2. *For any $0 < \delta < 1$, there exists a compact composition operator on $H^2(\Omega)$, with a truly infinite-dimensional symbol, such that, for some positive constants c, C, b, we have:*

$$a_n(C_\varphi) \leq C \, \exp\big(- c \, \mathrm{e}^{b \, (\log n)^\delta} \big) \, .$$

Proof. Take the same operator C_φ as in Theorem 4.1, with $\lambda_n = \mathrm{e}^{-A_n}$ where the positive numbers A_n have to be adjusted. Its approximation numbers a_N are then the non-increasing rearrangement of the sequence of numbers $(\varepsilon_n)_n := (\lambda^\alpha)_\alpha$. This suggests using a generating function argument, namely considering $\sum \varepsilon_n x^n$, but the rearrangement perturbs the picture. Accordingly, we follow a sligthly different route. Fix an integer $N \geq 1$ and a real number $r > 0$. Observe that, following the proof of Theorem 4.1:

$$N \, a_N^r \leq \sum_{n=1}^{N} a_n^r \leq \sum_{n=1}^{\infty} a_n^r = \prod_{n=1}^{\infty} (1 - e^{-rA_n})^{-1}.$$

First, consider the simple example $A_n = n$. We get:

$$N \, a_N^r \leq \eta \, (\mathrm{e}^{-r})$$

where η is the Dedekind eta function (see [5]) given by:

$$\eta(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n) \, x^n \, , \qquad |x| < 1 \, ,$$

where $p(n)$ is the number of partitions of the integer n. It is well-known ([5], Ch. 7, p. 169) that $\eta \, (\mathrm{e}^{-r}) \leq \mathrm{e}^{D/r}$ with $D = \pi^2/6$, so that:

$$a_N \leq \exp\left(\frac{D}{r^2} - \frac{\log N}{r} \right) .$$

Optimizing with $r = 2D/\log N$, we get:

$$a_N \leq \exp(-c \log^2 N) \, ,$$

with $c = 1/4D$. This is more precise than Theorem 4.1.

We now show that if A_n increases faster, we can achieve the decay of Theorem 4.2. As before, we get in general:

$$a_N \leq \inf_{x>1} \big(\exp\left[x(\log F(x^{-1}) - \log N)\right]\big)\,, \tag{4.1}$$

where

$$F(r) = \prod_{n=1}^{\infty} (1 - \mathrm{e}^{-rA_n})^{-1}\,.$$

We have:

$$\log F(r) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{\mathrm{e}^{-rmA_n}}{m} \right) = \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{n=1}^{\infty} \mathrm{e}^{-rmA_n} \right).$$

Now, take $A_n = \mathrm{e}^{n^\alpha}$ where $\alpha > 0$ is to be chosen. We have:

$$\sum_{n=1}^{\infty} \mathrm{e}^{-rm\,\mathrm{e}^{n^\alpha}} \leq \int_0^{\infty} \mathrm{e}^{-rm\,\mathrm{e}^{t^\alpha}}\, dt =: I_m(r)\,.$$

Standard estimates now give, for $r < 1$:

$$\begin{aligned} I_m(r) &= \int_1^{\infty} \mathrm{e}^{-rmx} \frac{1}{\alpha} (\log x)^{\frac{1}{\alpha}-1} \frac{dx}{x} = \int_{rm}^{\infty} \mathrm{e}^{-y} \frac{1}{\alpha} \left(\log \frac{y}{rm} \right)^{\frac{1}{\alpha}-1} \frac{dy}{y} \\ &\lesssim \left(\log \frac{1}{r} \right)^{\frac{1}{\alpha}-1} \int_{rm}^{\infty} \mathrm{e}^{-y} \frac{dy}{y} \lesssim \mathrm{e}^{-rm} \left(\log \frac{1}{r} \right)^{\frac{1}{\alpha}}, \end{aligned}$$

so that:

$$\log F(r) \lesssim (\log 1/r)^{\frac{1}{\alpha}} \sum_{m=1}^{\infty} m^{-1} \mathrm{e}^{-rm} \lesssim (\log 1/r)^{\frac{1}{\alpha}+1}\,.$$

Going back to (4.1), we get, for some constant $C > 0$, and for $x = 1/r > 1$:

$$a_N \leq C \exp\left[C\, x\big((\log x)^{\frac{1}{\alpha}+1} - \log N\big)\right].$$

Adjusting $x = x_N > 1$ so as to have $(\log x)^{\frac{1}{\alpha}+1} = \log N - 1$, that is:

$$x_N = \exp\left[(\log(N/\mathrm{e}))^{\frac{\alpha}{\alpha+1}}\right],$$

we get $a_N \leq C\,\mathrm{e}^{-c\,x_N}$, which is the claimed result with $\delta = \alpha/(\alpha+1)$.

This δ can be taken arbitrarily in $(0, 1)$ by choosing α suitable, and we are done. $\square$

Remark. Of course, $\delta = 1$ is forbidden, because this would give $a_n \leq C\,\mathrm{e}^{-c\,n^b}$, implying:

$$\sum_{n=1}^{\infty} \frac{1}{(\log 1/a_n)^p} \lesssim \sum_{n=1}^{\infty} n^{-b\,p} < \infty\,,$$

for large p, and contradicting Theorem 3.4.

References

[1] R. Aron, P. Galindo and M. Lindström, *Compact homomorphisms between algebras of analytic functions*, Studia Math. **123**(3) (1997), 235–247.

[2] F. Bayart, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Approximation numbers of composition operators on the Hardy and Bergman spaces of the ball and of the polydisk*, Math. Proc. of the Cambridge Philos. Soc., *to appear.*

[3] B. Carl and I. Stephani. *Entropy, compactness and the approximation of operators*, Cambridge Tracts in Mathematics 98, Cambridge University Press, Cambridge (1990).

[4] S. Chae, *Holomorphy and Calculus in Normed Spaces*, Monographs and Textbooks in Pure and Applied Mathematics 92, Marcel Dekker (1985).

[5] K. Chandrasekharan, *Arithmetical Functions*, Grundlehren Math. Wiss., Band 167, Berlin, Springer (1970).

[6] D. Clahane, *Spectra of compact composition operators over bounded symmetric domains*, Integral Equations Operator Theory **51** (2005), 41–56.

[7] B. Cole and T. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. Lond. Math. Society **53** (1986), 112–142.

[8] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, C.R.C. Press (1994).

[9] S. E. Elliott and M. T. Jury, *Composition operators on Hardy spaces of a half plane*, Bull. London Math. Soc. **44**, no. 3 (2012), 489–495.

[10] P. Lefèvre, *Generalized Essential Norm of Weighted Composition Operators on some Uniform Algebras of Analytic Functions*, Integral Equations Operator Theory **63** (2009), 557–569.

[11] B. MacCluer, *Spectra of compact composition operators on* $H^p(B_N)$, Analysis **4** (1984), 87–103.

[12] V. Matache, *Composition operators on Hardy spaces on a half-plane*, Proc. Amer. Math. Soc. **127**(5) (1999), 1483–1491.

[13] W. Rudin, *Functional Analysis*, Second edition, Internat. Ser. Pure Appl. Math., McGraw-Hill (1991).

[14] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext, Tracts in Mathematics, Springer-Verlag, New York (1993).

[15] J. H. Shapiro and W. Smith, *Hardy spaces that support no compact composition operators*, J. Funct. Anal. **205** no. 1 (2003), 62–89.

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