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DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÉ

Abstract. In the case of a compact orientable pseudomanifold, a well-known theorem of M. Goresky and R. MacPherson says that the cap-product with a fundamental class factorizes through the intersection homology groups. In this work, we show that this classical cap-product is compatible with a cap-product in intersection (co)-homology, that we have previously introduced. As a corollary, for any commutative ring of coefficients, the existence of a classical Poincaré duality isomorphism is equivalent to the existence of an isomorphism between the intersection homology groups corresponding to the zero and the top perversities. Our results answer a question asked by G. Friedman.

Let X be a compact oriented pseudomanifold and $[X] \in H_n(X;\mathbb{Z})$ be its fundamental class. In [9], M. Goresky and R. MacPherson prove that the *Poincaré duality map* defined by the cap-product $-\cap [X]\colon H^k(X;\mathbb{Z}) \to H_{n-k}(X;\mathbb{Z})$ can be factorized as

$$H^k(X;\mathbb{Z}) \xrightarrow{\alpha^{\overline{p}}} H^{\overline{p}}_{n-k}(X;\mathbb{Z}) \xrightarrow{\beta^{\overline{p}}} H_{n-k}(X;\mathbb{Z}), \tag{1}$$

where the groups $H^{\overline{p}}_i(X;\mathbb{Z})$ are the intersection homology groups for the perversity $\overline{p}$. The study of the Poincaré duality map via a filtration on homology classes is also considered in the thesis of C. McCrory [12], [13], using a Zeeman's spectral sequence.

In [7, Section 8.2.6], G. Friedman asks for a factorization of the Poincaré duality map through a cap-product defined in intersection (co)-homology. In this work we answer positively by using a cohomology $H^*_{\mathrm{TW},\overline{p}}(-)$ obtained via a simplicial blow-up and an intersection cap-product, $-\cap[X]\colon H^k_{\mathrm{TW},\overline{p}}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-k}(X;R)$, defined in [4, Section 11] for any commutative ring R and recalled in Section 2. Roughly, our main result consists in the fact that this "intersection cap-product" corresponds to the "classical cap-product". This property can be expressed as the commutativity of the next diagram, where $\overline{t}$ is the top perversity defined by $\overline{t}(i) = i-2$.

Theorem A. *Let X be a compact oriented n-dimensional pseudomanifold. For any perversity $\overline{p}$, there exists a commutative diagram,*

$$\begin{tikzcd}
H^k(X;R) \arrow[r, "-\cap[X]"] \arrow[d, "\mathcal{M}^*_{\overline{p}}"'] \arrow[dr, "\alpha^{\overline{p}}"] & H_{n-k}(X;R) \\
H^k_{\mathrm{TW},\overline{p}}(X;R) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{p}}_{n-k}(X;R). \arrow[u, "\beta^{\overline{p}}"']
\end{tikzcd} \tag{2}$$

In [9], the spaces and maps of (1) appear in the piecewise linear setting. In the previous statement we are working with singular homology and cohomology however, we keep the same letter $\alpha^{\overline{p}}$. The morphism $\beta^{\overline{p}}$ is generated by the inclusion of the corresponding complexes and the morphism $\mathcal{M}^*_{\overline{p}}$ is defined in Section 3.

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We specify now the previous statement to the constant perversity on 0. Then, for a normal compact oriented n-dimensional pseudomanifold, we get a commutative diagram,

$$\begin{tikzcd} H^k(X;R) \arrow[r, "-\cap[X]"] \arrow[dd, "\mathcal{M}^*_{\overline{0}}"', "\cong"] & H_{n-k}(X;R) \\ & H^{\overline{t}}_{n-k}(X;R) \arrow[u, "\beta^{\overline{t}}"', "\cong"'] \\ H^k_{\mathrm{TW},\overline{0}}(X;R) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{0}}_{n-k}(X;R). \arrow[u, "\beta^{\overline{0},\overline{t}}"'] \end{tikzcd} \tag{3}$$

This decomposition implies the next characterization.

Theorem B. *Let X be a normal compact oriented n-dimensional pseudomanifold. Then the following conditions are equivalent.*

(i) The Poincaré duality map $-\cap[X]\colon H^k(X;R)\to H_{n-k}(X;R)$ is an isomorphism.
(ii) The natural map $\beta^{\overline{0},\overline{t}}\colon H^{\overline{0}}_k(X;R)\to H^{\overline{t}}_k(X;R)$, induced by the canonical inclusion of the corresponding complexes, is an isomorphism.

We have chosen the setting of the original perversities of M. Goresky and R. MacPherson [9]. However, the previous results remain true in more general situations.

In the last section, we quote the existence of a cup-product structure on $H^*_{\mathrm{TW},\overline{\bullet}}(-)$ and detail how it combines with this factorization. We are also looking for a factorization involving an intersection cohomology defined from the dual of intersection chains (see [8]) and denoted $H^*_{\mathrm{GM},\overline{p}}(-)$. In the case of a locally free pseudomanifold (see [10]), in particular if R is a field, a factorization as in (3) has been established by G. Friedmann in [7, Section 8.2.6] for that cohomology. In this book, G. Friedmann asks also for such factorization in the general case. In the last section, we give an example showing that such factorization may not exist. Therefore the answer to the question asked by G. Friedman is double: yes a factorization of $\alpha^{\overline{p}}$ through an intersection cap-product exists but not through $H^*_{\mathrm{GM},\overline{p}}(-)$.

Contents

The coefficients for homology and cohomology are taken in a commutative ring R (with unity) and we do not mention it explicitly in the rest of this work. In all the text, X is a compact oriented n-dimensional pseudomanifold. The degree of an element x of a graded module is denoted $|x|$. All the maps β, with subscript and superscript, are induced by canonical inclusions of complexes.

1. Background on intersection homology and cohomology

We recall the basic definitions and properties we need, sending the reader to [9], [2] or [4] for more details.

Definition 1.1. An n-dimensional *pseudomanifold* is a topological space, X, filtered by closed subsets,

$$X_{-1} = \varnothing \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} = X_{n-1} \subsetneq X_n = X,$$

such that, for any $i \in \{0,\dots,n\}$, $X_i \backslash X_{i-1}$ is an i-dimensional topological manifold or the empty set. Moreover, for each point $x \in X_i \backslash X_{i-1}$, $i \neq n$, there exist

(i) an open neighborhood V of x in X, endowed with the induced filtration,
(ii) an open neighborhood U of x in $X_i \backslash X_{i-1}$,
(iii) a compact pseudomanifold L of dimension $n-i-1$, whose cone $\mathring{\mathtt{c}}L$ is endowed with the filtration $(\mathring{\mathtt{c}}L)_i = \mathring{\mathtt{c}}L_{i-1}$,
(iv) a homeomorphism, $\varphi\colon U \times \mathring{\mathtt{c}}L \to V$, such that
  (a) $\varphi(u,\mathtt{v}) = u$, for any $u \in U$, where $\mathtt{v}$ is the apex of the cone $\mathring{\mathtt{c}}L$,
  (b) $\varphi(U \times \mathring{\mathtt{c}}L_j) = V \cap X_{i+j+1}$, for all $j \in \{0,\dots,n-i-1\}$.

The pseudomanifold L is called a *link* of x. The pseudomanifold X is called *normal* if its links are connected.

As in [9], a *perversity* is a map $\overline{p}\colon \mathbb{N} \to \mathbb{Z}$ such that $\overline{p}(0) = \overline{p}(1) = \overline{p}(2) = 0$ and $\overline{p}(i) \leqslant \overline{p}(i+1) \leqslant \overline{p}(i)+1$ for all $i \geqslant 1$. Among them, we quote the null perversity $\overline{0}$ constant on 0 and the *top perversity* defined by $\overline{t}(i) = i-2$. For any perversity $\overline{p}$, the perversity $D\overline{p} := \overline{t} - \overline{p}$ is called the *complementary perversity* of $\overline{p}$.

In this work, we compute the intersection homology of a pseudomanifold X via *filtered simplices.* They are singular simplices $\sigma\colon \Delta \to X$ such that Δ admits a decomposition in join products, $\Delta = \Delta_0 * \cdots * \Delta_n$ with $\sigma^{-1}X_i = \Delta_0 * \cdots * \Delta_i$. The *perverse degree* of σ is defined by $\|\sigma\| = (\|\sigma\|_0,\dots,\|\sigma\|_n)$ with $\|\sigma_i\| = \dim(\Delta_0 * \cdots * \Delta_{n-i})$. A filtered simplex is called $\overline{p}$-*allowable* if

$$\|\sigma\|_i \leqslant \dim\Delta - i + \overline{p}(i), \tag{4}$$

for any $i \in \{0,\dots,n\}$. A singular chain ξ is $\overline{p}$-allowable if it can be written as a linear combination of $\overline{p}$-allowable filtered simplices, and of $\overline{p}$-*intersection* if ξ and $\partial\xi$ are $\overline{p}$-allowable. We denote by $C^{\overline{p}}_*(X)$ the complex of chains of $\overline{p}$-intersection. In [5, Théorème A], we have proved that $C^{\overline{p}}_*(X)$ is quasi-isomorphic to the original complex of [9].

Given an euclidean simplex Δ, we denote by $N_*(\Delta)$ and $N^*(\Delta)$ the associated simplicial chain and cochain complexes. For each face F of Δ, we write $\mathbf{1}_F$ the cochain of $N^*(\Delta)$ taking the value 1 on F and 0 otherwise. If F is a face of Δ, we denote by $(F,0)$ the same face viewed as face of the cone $\mathtt{c}\Delta = \Delta * [\mathtt{v}]$ and by $(F,1)$ the face $\mathtt{c}F$ of $\mathtt{c}\Delta$. By extension, we use also the notation $(\varnothing,1) = \mathtt{c}\varnothing = [\mathtt{v}]$ for the apex. The corresponding cochains are denoted $\mathbf{1}_{(F,\varepsilon)}$ for $\varepsilon = 0$ or 1.

A filtered simplex $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ is called *regular* if $\Delta_n \neq \varnothing$. The cochain complex we use for cohomology is built on the blow-up's of regular filtered simplices. More precisely, we set first

$$\widetilde{N}^*_\sigma = \widetilde{N}^*(\Delta) = N^*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N^*(\mathtt{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

With the previous convention, a basis of $\widetilde{N}^*(\Delta)$ is composed of elements of the form $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n} \in \widetilde{N}^*(\Delta)$, where F_i is a face of Δ_i for $i \in \{1,\dots,n\}$ or the empty set with $\varepsilon_i = 1$ if $i < n$. We set $|\mathbf{1}_{(F,\varepsilon)}|_{>s} = \sum_{i>s}(\dim F_i + \varepsilon_i)$.

Definition 1.2. Let ℓ be an element of $\{1,\dots,n\}$ and $\mathbf{1}_{(F,\varepsilon)} \in \widetilde{N}^*(\Delta)$. The ℓ-*perverse degree* of $\mathbf{1}_{(F,\varepsilon)} \in N^*(\Delta)$ is

$$\|\mathbf{1}_{(F,\varepsilon)}\|_\ell = \begin{cases} -\infty & \text{if } \varepsilon_{n-\ell} = 1, \\ |\mathbf{1}_{(F,\varepsilon)}|_{>n-\ell} & \text{if } \varepsilon_{n-\ell} = 0. \end{cases}$$

In the general case of a cochain $\omega = \sum_b \lambda_b \, \mathbf{1}_{(F_b,\varepsilon_b)} \in \widetilde{N}^*(\Delta)$ with $\lambda_b \neq 0$ for all b, the *ℓ-perverse degree* is

$$\|\omega\|_\ell = \max_b \|\mathbf{1}_{(F_b,\varepsilon_b)}\|_\ell.$$

By convention, we set $\|0\|_\ell = -\infty$.

If $\delta_\ell \colon \Delta' \to \Delta$ is a face operator, we denote by $\partial_\ell \sigma$ the filtered simplex defined by $\partial_\ell\sigma = \sigma \circ \delta_\ell \colon \Delta' \to X$. The *Thom-Whitney complex* of X is the cochain complex $\widetilde{N}^*(X)$ composed of the elements, ω, associating to each regular filtered simplex $\sigma \colon \Delta_0 * \cdots * \Delta_n \to X$, an element $\omega_\sigma \in \widetilde{N}^*_\sigma$, such that $\delta_\ell^*(\omega_\sigma) = \omega_{\partial_\ell \sigma}$, for any face operator $\delta_\ell \colon \Delta' \to \Delta$ with $\Delta'_n \neq \emptyset$. (Here $\Delta' = \Delta'_0 * \cdots * \Delta'_n$ is the induced filtration.) The differential $d\omega$ is defined by $(d\omega)_\sigma = d(\omega_\sigma)$. The *$\ell$-perverse degree* of $\omega \in \widetilde{N}^*(X)$ is the supremum of all the $\|\omega_\sigma\|_\ell$ for all regular filtered simplices $\sigma \colon \Delta \to X$.

A *cochain $\omega \in \widetilde{N}^*(X)$ is $\overline{p}$-allowable* if $\|\omega\|_\ell \leqslant \overline{p}(\ell)$ for any $\ell \in \{1, \ldots, n\}$, and of $\overline{p}$-intersection if ω and $d\omega$ are $\overline{p}$-allowable. We denote $\widetilde{N}^*_{\overline{p}}(X)$ the complex of $\overline{p}$-intersection cochains and by $H^*_{\mathrm{TW},\overline{p}}(X)$ its homology called *Thom-Whitney cohomology* (henceforth *TW-cohomology)* of X for the perversity $\overline{p}$.

2. Cap product and intersection homology

We first recall the definition and some basic properties of a cap-product in intersection (co)-homology already introduced in [4, Section 11].

Let $\Delta = [e_0, \ldots, e_r, \ldots, e_m]$ be an euclidean simplex. The classical cap-product

$$- \cap \Delta \colon N^*(\Delta) \to N_{m-*}(\Delta)$$

is defined by

$$\mathbf{1}_F \cap \Delta = \begin{cases} [e_r, \ldots, e_m] & \text{if } F = [e_0, \ldots, e_r], \\ 0 & \text{otherwise.} \end{cases}$$

We extend it to filtered simplices $\Delta = \Delta_0 * \cdots * \Delta_n$ as follows. First we set $\widetilde{\Delta} = \mathtt{c}\Delta_0 \times \cdots \times \mathtt{c}\Delta_{n-1} \times \Delta_n$. If $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n} \in \widetilde{N}^*(\Delta)$, we define:

$$\begin{aligned} \mathbf{1}_{(F,\varepsilon)} \cap \widetilde{\Delta} &= (-1)^{\nu(F,\varepsilon,\Delta)} (\mathbf{1}_{(F_0,\varepsilon_0)} \cap \mathtt{c}\Delta_0) \otimes \cdots \otimes (\mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \cap \mathtt{c}\Delta_{n-1}) \otimes (\mathbf{1}_{F_n} \cap \Delta_n), \\ &\in \widetilde{N}_*(\Delta) := N_*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N_*(\mathtt{c}\Delta_{n-1}) \otimes N_*(\Delta_n), \end{aligned} \tag{5}$$

where $\nu(F,\varepsilon,\Delta) = \sum_{j=0}^{n-1} (\dim \Delta_j + 1) \left(\sum_{i=j+1}^n |(F_i,\varepsilon_i)|\right)$, with the convention $\varepsilon_n = 0$.

We define now a morphism, $\mu_*^\Delta \colon \widetilde{N}_*(\Delta) \to N_*(\Delta)$, by describing it on the elements $(F,\varepsilon) = (F_0,\varepsilon_0) \otimes \cdots \otimes (F_{n-1},\varepsilon_{n-1}) \otimes F_n$. Let ℓ be the smallest integer, j, such that $\varepsilon_j = 0$. We set

$$\mu_*^\Delta(F,\varepsilon) = \begin{cases} F_0 * \cdots * F_\ell & \text{if } \dim(F,\varepsilon) = \dim(F_0 * \cdots * F_\ell), \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The application $\mu_*^\Delta \colon \widetilde{N}_*(\Delta) \to N_*(\Delta)$ is a chain map, [4, Proposition 11.10] which allows the next local and global definitions of the cap-product.

Definition 2.1. Let $\Delta = \Delta_0 * \cdots * \Delta_n$ be a regular filtered simplex of dimension m. The *cap-product* $- \cap \Delta \colon \widetilde{N}^*(\Delta) \to N_{m-*}(\Delta)$ is defined by

$$\omega \cap \Delta = \mu_*^\Delta(\omega \cap \widetilde{\Delta}).$$

Let X be a pseudomanifold, $\omega \in \widetilde{N}^*(X)$ and $\sigma \colon \Delta_\sigma \to X$ be a filtered simplex. We define $\omega \cap \sigma \in N_*(X)$ by

$$\omega \cap \sigma = \begin{cases} \sigma_*(\omega_\sigma \cap \Delta_\sigma) & \text{if } \sigma \text{ is regular}, \\ 0 & \text{otherwise.} \end{cases}$$

This definition is extended to any chain by linearity and this cap-product satisfies the next properties.

Proposition 2.2 ([4, Proposition 11.16]). *Let X be an n-dimensional pseudomanifold and let $\overline{p}$, $\overline{q}$ be two loose perversities. The cap-product defines a chain map,*

$$-\cap -\colon \widetilde{N}^k_{\overline{p}}(X;R)\otimes C^{\overline{q}}_n(X;R)\to C^{\overline{p}+\overline{q}}_{n-k}(X;R).$$

3. Proofs of Theorems A and B

To define the morphism $\mathcal{M}^*_{\overline{p}}$ of (2), we first consider the morphism dual of μ^Δ_* denoted $\mu^*_\Delta\colon N^*(\Delta)\to \widetilde{N}^*(\Delta)$. Let

$$\widetilde{N}^*_{\overline{0}}(\Delta)=\left\{\omega\in\widetilde{N}^*(\Delta)\mid \|\omega\|_\ell\leqslant 0 \text{ and } \|d\omega\|_\ell\leqslant 0 \text{ for all } \ell\in\{1,\dots,n\}\right\}.$$

The next result is in the spirit of a theorem of Verona, see [15].

Proposition 3.1. *Let $\Delta=\Delta_0*\cdots*\Delta_n$ be a regular filtered simplex. Then the chain map,*

$$\mu^*_\Delta\colon N^*(\Delta)\to\widetilde{N}^*_{\overline{0}}(\Delta)\subset\widetilde{N}^*(\Delta),$$

is an isomorphism.

Proof. Let $G=G_0*\cdots*G_s$ be a face of Δ with $s\leqslant n$ and $G_s\neq\emptyset$. By definition of μ^*_Δ, we have

$$\mu^*_\Delta(\mathbf{1}_G)=\sum_{(a_{s+1},\dots,a_n)}\mathbf{1}_{(G_0,1)}\otimes\cdots\otimes\mathbf{1}_{(G_{s-1},1)}\otimes\mathbf{1}_{(G_s,0)}\otimes\mathbf{1}_{[a_{s+1}]}\otimes\cdots\otimes\mathbf{1}_{[a_n]},$$

where the a_i's run over the vertices of $\mathtt{c}\Delta_i$ if $i\in\{1,\dots,n-1\}$ and a_n over the vertices of Δ_n. From Definition 1.2, we observe $\|\mu^*_\Delta(\mathbf{1}_G)\|_\ell\leqslant 0$ for any $\ell\in\{1,\dots,n\}$ and the injectivity of μ^*_Δ.

Consider now a cochain of $\omega\in\widetilde{N}^*_{\overline{0}}(\Delta)$. Since $\|\omega\|_\ell\leqslant 0$ for each $\ell\in\{1,\dots,n\}$, the cochain ω is a linear combination of elements $\mathbf{1}_{(G_0,1)}\otimes\cdots\otimes\mathbf{1}_{(G_{s-1},1)}\otimes\mathbf{1}_{(G_s,0)}\otimes\mathbf{1}_{[a_{s+1}]}\otimes\cdots\otimes\mathbf{1}_{[a_n]}$ as above. Since $d\mathbf{1}_{[a_i]}$ is of degree 1, the condition $\|d\omega\|_\ell\leqslant 0$ for any $\ell\in\{1,\dots,n\}$ implies that this linear combination contains all the vertices of $\mathtt{c}\Delta_{s+1}\times\cdots\times\mathtt{c}\Delta_{n-1}\times\Delta_n$. Therefore, ω is in the image of μ^*_Δ and we get the surjectivity. $\square$

The next result connects the classical cup-product to the intersection cap-product defined above.

Proposition 3.2. *Let $\Delta=\Delta_0*\cdots*\Delta_n$ be a regular filtered simplex. For each cochain $\omega\in\widetilde{N}^*(\Delta)$, we have*

$$\mu^\Delta_*(\mu^*_\Delta(\omega)\cap\widetilde{\Delta})=\omega\cap\Delta.$$

Proof. The result is clear for $n=0$. In the general case, we set $\nabla=\Delta_1*\cdots*\Delta_n$ and observe that μ^Δ_* can be decomposed as

$$N_*(\mathtt{c}\Delta_0)\otimes\widetilde{N}_*(\nabla)\xrightarrow{\mathrm{id}\otimes\mu^\nabla_*}N_*(\mathtt{c}\Delta_0)\otimes N_*(\nabla)\xrightarrow{\mu^{\Delta_0*\nabla}_*}N_*(\Delta).$$

Thus, by using an induction, it is sufficient to prove the result for $\Delta=\Delta_0*\Delta_1$. Let $\mathbf{1}_{(F_0,\varepsilon_0)}\otimes\mathbf{1}_{F_1}\in N^*(\mathtt{c}\Delta_0)\otimes N^*(\Delta_1)$. If the cap-product with Δ is not equal to zero, we, observe from Definition 2.1, that $(\mathbf{1}_{(F_0,\varepsilon_0)}\otimes\mathbf{1}_{F_1})\cap\Delta)=G_0*G_1$, where $F_1\cup G_1=\Delta_1$, $G_0=\emptyset$ if $\varepsilon_0=1$ and $F_0\cup G_0=\Delta_0$ if $\varepsilon_0=0$.

• If $F_1 \neq \emptyset$, we have:

$$\begin{array}{rcl}\mu_*^\Delta(\mu_\Delta^*(\mathbf{1}_{F_0 * F_1}) \cap (\mathtt{c}\Delta_0 \times \Delta_1)) & = & \mu_*^\Delta((\mathbf{1}_{(F_0,1)} \otimes \mathbf{1}_{F_1}) \cap (\mathtt{c}\Delta_0 \times \Delta_1)) \\ & = & (-1)^{|F_1||\mathtt{c}\Delta_0|}\mu_*^\Delta((\mathbf{1}_{(F_0,1)} \cap \mathtt{c}\Delta_0) \otimes (\mathbf{1}_{F_1} \cap \Delta_1)) \\ & = & (-1)^{|F_1||\mathtt{c}\Delta_0|} \left\{ \begin{array}{ll} \mu_*^\Delta(\mathtt{v} \otimes G_1) & \text{if } \Delta_0 = \emptyset, \\ 0 & \text{otherwise,} \end{array} \right. \\ & = & (-1)^{|F_1||\mathtt{c}\Delta_0|} \left\{ \begin{array}{ll} G_1 & \text{if } \Delta_0 = \emptyset, \\ 0 & \text{otherwise,} \end{array} \right. \\ & = & \mathbf{1}_{F_0 * F_1} \cap (\Delta_0 * \Delta_1). \end{array}$$

• If $F_1 = \emptyset$, the previous computation becomes:

$$\begin{array}{rcl}\mu_*^\Delta(\mu_\Delta^*(\mathbf{1}_{F_0}) \cap (\mathtt{c}\Delta_0 \times \Delta_1)) & = & \mu_*^\Delta\left((\sum_{[a_i] \subset \Delta_1} \mathbf{1}_{(F_0,0)} \otimes \mathbf{1}_{[a_i]}) \cap (\mathtt{c}\Delta_0 \times \Delta_1) \right) \\ & = & \mu_*^\Delta\left(\sum_{[a_i] \subset \Delta_1} (\mathbf{1}_{(F_0,0)} \cap \mathtt{c}\Delta_0) \otimes (\mathbf{1}_{[a_i]} \cap \Delta_1) \right) \\ & = & \mu_*^\Delta((G_0,1) \otimes \Delta_1) = G_0 * \Delta_1 = \mathbf{1}_{F_0} \cap (\Delta_0 * \Delta_1). \end{array}$$

□

Let us recall that $C^*(X)$ is the complex of singular cochains with coefficients in R.

Proposition 3.3. *Let X be a normal compact pseudomanifold. Then, the operator $\mathcal{M}_{\overline{0}}\colon C^*(X) \to \widetilde{N}^*_{\overline{0}}(X)$, defined by $\mathcal{M}_{\overline{0}}(\omega)_\sigma = \mu_\Delta^*(\omega_\sigma)$, for any regular filtered simplex $\sigma\colon \Delta \to X$, is a chain map which induces an isomorphism*

$$\mathcal{M}^*_{\overline{0}}\colon H^*(X) \xrightarrow{\cong} H^*_{\mathrm{TW},\overline{0}}(X).$$

This result remains true for normal CS-sets, with the same proof. We do not introduce this notion here, see [14] for its definition.

We denote by $\mathcal{M}_{\overline{p}}\colon C^*(X) \to \widetilde{N}^*_{\overline{p}}(X)$ the composition of $\mathcal{M}_{\overline{0}}$ with the canonical inclusion of complexes and by $\mathcal{M}^*_{\overline{p}}\colon H^*(X) \to H^*_{\mathrm{TW},\overline{p}}(X)$ the induced morphism.

Proof. The maps μ_*^Δ being compatible with restrictions, the map $\mathcal{M}_{\overline{0}}$ is well defined. Its compatibility with the differentials is a consequence of [4, Proposition 11.10] and its behaviour with perversities a consequence of Proposition 3.1. For proving the isomorphism, we use a method similar to the argument used by H. King in [11], see also [7, Section 5.1]. We have to check the hypotheses of [4, Proposition 8.1].

• The first one is the existence of Mayer-Vietoris sequences. This is clear for $C^*(-)$ and has been proved in [4, Théorème A] for $\widetilde{N}^*_{\overline{0}}(-)$.

• The second and the fourth hypotheses are straightforward.

• The third one consists in the computation of the intersection cohomology of a cone. Let L be a compact pseudomanifold. It is well known that $H^*(\mathbb{R}^i \times \mathring{\mathtt{c}}L) = R$. From [4, Propositions 6.1 and 7.1], we have

$$H^*_{\mathrm{TW},\overline{0}}(\mathbb{R}^i \times \mathring{\mathtt{c}}L) = H^0_{\mathrm{TW},\overline{0}}(L) =_{(1)} H^0(L) =_{(2)} R,$$

where the equality (1) is the hypothesis and (2) a consequence of the normality of X. □

Proof of Theorem A. Let $\overline{p}$ be a perversity. We consider the following diagram.

$$\begin{tikzcd}
H^*(X) \arrow[r, "-\cap[X]"] \arrow[d, "\mathcal{M}^*_{\overline{0}}"'] \arrow[dd, bend right=60, "\mathcal{M}^*_{\overline{p}}"'] & H_{n-*}(X) \\
H^*_{\mathrm{TW},\overline{0}}(X) \arrow[d, "\alpha_{\overline{0},\overline{p}}"'] & H^{\overline{t}}_{n-*}(X) \arrow[u, "\beta^{\overline{t}}"'] \\
H^*_{\mathrm{TW},\overline{p}}(X) \arrow[r, "-\cap[X]"] & H^{\overline{p}}_{n-*}(X), \arrow[u, "\beta^{\overline{p},\overline{t}}"']
\end{tikzcd} \tag{7}$$

where $\alpha_{\overline{0},\overline{p}}$, $\beta^{\overline{p},\overline{t}}$ and $\beta^{\overline{t}}$ are induced by the natural inclusions of complexes. The bottom isomorphism comes from [4, Théorème D]. It remains to check the commutativity of this diagram. Consider a cochain $\omega \in C^*(X)$ and a regular simplex $\sigma\colon \Delta \to X$. (Observe also that $\overline{p}$-allowable simplexes are always regular if $\overline{p} \leqslant \overline{t}$.) We have

$$\mathcal{M}^*_{\overline{0}}(\omega) \cap \sigma =_{(1)} \sigma_* \mu^{\Delta}_* \left(\mu^*_{\Delta}(\omega_\sigma) \cap \widetilde{\Delta} \right) =_{(2)} \sigma_*(\omega_\sigma \cap \Delta) =_{(3)} \omega \cap \sigma, \tag{8}$$

where (1) and (3) come from the definitions and (2) from Proposition 3.2. □

Proof of Theorem B. By specifying the diagram (7) to the case $\overline{p} = \overline{0}$, we get the next commutative diagram.

$$\begin{tikzcd}
H^*(X) \arrow[r, "-\cap[X]"] \arrow[d, "\mathcal{M}^*_{\overline{0}}"', "\cong"] & H_{n-*}(X) \\
H^*_{\mathrm{TW},\overline{0}}(X) \arrow[d, equal] & H^{\overline{t}}_{n-*}(X) \arrow[u, "\cong", "\beta^{\overline{t}}"'] \\
H^*_{\mathrm{TW},\overline{0}}(X) \arrow[r, "-\cap[X]", "\cong"'] & H^{\overline{0}}_{n-*}(X). \arrow[u, "\beta^{\overline{0},\overline{t}}"']
\end{tikzcd}$$

From Proposition 3.3, [5, Proposition 5.5] and [4, Théorème D], we get that $\mathcal{M}^*_{\overline{0}}$, $\beta^{\overline{t}}$ and the bottom map are isomorphisms. This ends the proof. □

4. Remarks and comments

Cup-product. In [2], [4], we define from the local structure on the Euclidean simplices, a cup-product in intersection cohomology, induced by a chain map

$$\cup\colon \widetilde{N}^{k_1}_{\mathrm{TW},\overline{p}_1}(X) \otimes \widetilde{N}^{k_2}_{\mathrm{TW},\overline{p}_2}(X) \to \widetilde{N}^{k_1+k_2}_{\mathrm{TW},\overline{p}_1+\overline{p}_2}(X). \tag{9}$$

By construction, the isomorphism $\mathcal{M}^*_{\overline{0}}$ of Proposition 3.3 preserves this cup-product. (Details will be given in [1]). Moreover, the classical properties of cup and cap-product can be extended to the intersection setting, as for instance ([4, Proposition 11.16]) the equality,

$$(\omega \cup \eta) \cap \xi = \omega \cap (\eta \cap \xi), \tag{10}$$

for any cochains ω, η and chain ξ. If X is a compact oriented n-dimensional pseudomanifold, the Poincaré duality induces (see [6, Section VIII.13]) an intersection product on the intersection homology, defined by the commutativity of the next diagram.

$$\begin{tikzcd} H^{k_1}_{\mathrm{TW},\overline{p}_1}(X)\otimes H^{k_2}_{\mathrm{TW},\overline{p}_2} \arrow[r, "\cup"] \arrow[d, "-\cap[X]\otimes-\cap[X]"', "\cong"] & H^{k_1+k_2}_{\mathrm{TW},\overline{p}_1+\overline{p}_2}(X) \arrow[d, "\cong"', "-\cap[X]"] \\ H^{\overline{p}_1}_{n-k_1}(X)\otimes H^{\overline{p}_2}_{n-k_2}(X) \arrow[r, "\pitchfork"] & H^{\overline{p}_1+\overline{p}_2}_{n-k_1-k_2}(X) \end{tikzcd} \tag{11}$$

Let $[\omega],[\eta]\in H^*(X)$ and $\alpha^{\overline{p}}=(-\cap[X])\circ\mathcal{M}^*_{\overline{p}}$. From the compatibility of $\mathcal{M}^*_{\overline{p}}$ with the cup-product and from (10), we deduce

$$\alpha^{\overline{p}}([\omega]\cup[\eta])=\alpha^{\overline{p}}([\omega])\pitchfork\alpha^{\overline{p}}([\eta]), \tag{12}$$

which is the analogue of the decomposition established in [9] in the PL case.

A second intersection cohomology. In this work, we have used a cohomology theory obtained by a simplicial blow-up, arising from the complex $\widetilde{N}^*_{\overline{p}}(X)$ and denoted $H^*_{\mathrm{TW},\overline{p}}(X)$. Alternatively we could also choose (see [8]) the dual complex of the intersection chains. We denote

$$C^*_{\mathrm{GM},\overline{p}}(X;R)=\hom(C^{\overline{p}}_*(X;R),R) \text{ and } H^*_{\mathrm{GM},\overline{p}}(X;R) \text{ its cohomology.} \tag{13}$$

In [4, Theorem 4], we have proved

$$H^*_{\mathrm{TW},\overline{p}}(X;R)\cong H^*_{\mathrm{GM},D\overline{p}}(X;R),$$

if R is a field, or more generally with an hypothesis on the torsion of the links, introduced in [10]. But, in the general case, these two cohomologies may differ. A natural question is the existence of a factorization of $\alpha^{\overline{p}}$ as above but in which the cohomology $H^*_{\mathrm{GM},D\overline{p}}$ is substituted to $H^*_{\mathrm{TW},\overline{p}}$. This question arises in [7, Section 8.2.6] together with a nice development on this point. The next example shows that such factorization does not occur in general.

Example 4.1. Consider the 4-dimensional compact oriented pseudomanifold $X=\Sigma\mathbb{RP}^3$ and set $R=\mathbb{Z}$. The map $\alpha^{\overline{0}}\colon H^*(X;\mathbb{Z})\to H^{\overline{0}}_{4-*}(X;\mathbb{Z})$ is an isomorphism, see [9, 4.3]. The complementary perversity of $\overline{0}$ is the (constant) perversity $D\overline{p}=\overline{2}$. The previous question can be expressed as a factorization of $\alpha^{\overline{0}}$ as

$$H^*(X;\mathbb{Z})\longrightarrow H^*_{\mathrm{GM},\overline{2}}(X;\mathbb{Z})\longrightarrow H^{\overline{0}}_{4-*}(X;\mathbb{Z}). \tag{14}$$

We determine easily (see [7, 7.4], [9, 6.1]) that, in the case $*=3$, the previous line becomes

$$H^3(X;\mathbb{Z})=\mathbb{Z}_2\longrightarrow H^3_{\mathrm{GM},\overline{2}}(X;\mathbb{Z})=0\longrightarrow H^{\overline{0}}_1(X;\mathbb{Z})=\mathbb{Z}_2.$$

Therefore a factorization like (14) does not exist. (The TW-cohomology of this example can be determined from [3, Proposition 5.1].)

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