Intersection Homology. General perversities and topological invariance

David Chataur, Martintxo Saralegi-Aranguren, Daniel Tanré

INTERSECTION HOMOLOGY.
GENERAL PERVERSITIES AND TOPOLOGICAL INVARIANCE

DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÉ

ABSTRACT. Topological invariance of the intersection homology of a pseudomanifold without codimension one strata, proven by Goresky and MacPherson, is one of the main features of this homology. This property is true for strata codimension depending perversities with some growth conditions, verifying $\overline{p}(1) = \overline{p}(2) = 0$. King reproves this invariance by associating an intrinsic pseudomanifold X^* to any pseudomanifold X. His proof consists of an isomorphism between the associated intersection homologies $H_*^{\overline{p}}(X) \cong H_*^{\overline{p}}(X^*)$ for any perversity $\overline{p}$ with the same growth conditions verifying $\overline{p}(1) \geq 0$.

In this work, we prove a certain topological invariance within the framework of strata depending perversities, $\overline{p}$, which corresponds to the classical topological invariance if $\overline{p}$ is a GM-perversity. We also extend it to the tame intersection homology, a variation of the intersection homology, particularly suited for "large" perversities, if there is no singular strata on X becoming regular in X^*. In particular, under the above conditions, the intersection homology and the tame intersection homology are invariant under a refinement of the stratification.

The definition of intersection homology relies on the choice of a stratification of the space and it is natural to ask for the impact of such choice. In fact, Goresky and MacPherson proved, in their first work about intersection homology ([10]), that there is no dependence of the stratification, a main property of this theory. More precisely, if $\overline{p}\colon \mathbb{Z}_{>1} \to \mathbb{Z}$ is a classical perversity verifying $\overline{p}(t) \leq \overline{p}(t+1) \leq \overline{p}(t)+1$, $\overline{p}(2) = 0$, then the $\overline{p}$-intersection homology is a topological invariant of a pseudomanifold without codimension one strata. In the sequel, such perversity is called a GM-perversity.

Intersection homology also exists for more general perversities, defined on the family of strata of a pseudomanifold and taking values in $\mathbb{Z}$, cf. [12, Section 1.1] and Remark 2.5. Since these perversities are defined on the set of strata it is natural to conclude that we could not expect any topological invariance in such a general context. In this work, we present some conditions on perversities and stratifications which lead to a topological invariance.

To this end, we consider the method used by King [11] in the framework of locally cone-like spaces introduced by Siebenmann in [17] and called CS sets. Following a process credited to D. Sullivan, King associates to any CS set, X, a new CS set, X^*, with the same underlying topological space but endowed with a different stratification.

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The thrust in the construction of X^* is the elimination of the strata of X that are not topologically singular. For example, let us consider a sphere having a point P as singular stratum. The point P and any other point of the sphere have homeomorphic neighborhoods. So, the point P has no singular character relatively to the other points. In this case, the associated CS set X^* is the sphere with just one (regular) stratum.

Let us come back to the general case. As two CS sets with the same underlying topological space have the same associated CS set, we call X^* an *intrinsic CS set.* In the framework of a GM-perversity $\overline{p}$, the topological invariance means that the $\overline{p}$-intersection homologies of X and X^* are isomorphic.

Let X be a CS set and X^* be the associated intrinsic CS set. In this work, we address the issue of topological invariance from the following point of view: for which general perversities, do we have an isomorphism between the intersection homologies of X and X^*? To that end, we introduce the notion of K-perversity (Definition 6.8) which gives a sufficient condition (see Theorem C-(i)) and covers the case of GM-perversities (Corollary 6.11).

As chain complex giving intersection homology, we use a singular complex made up of filtered simplices, see Definition 3.2. This method is justified in Theorem A for general perversities, the classical case being established in [4, Proposition A.29]. The proof needs the existence of a Mayer-Vietoris sequence (Proposition 4.1), a stratified homotopy invariance (Proposition 3.13) and the determination of the intersection homology of a cone (Proposition 5.2). Theorem A is used in [3, 5].

Lastly, recall that, in the framework of GM-perversities with real coefficients, there exists a particular complex of differential forms verifying an extension of the classic deRham Theorem (a result of Goresky and MacPherson quoted in [1, Proposition 1.2.6]). Intersection homology of a pseudomanifold also verifies Poincaré duality (see [10]). Outside this range of perversities, these two properties disappear.

In [16] (see also [8]) a modification of intersection homology is presented, matching with the usual intersection homology if $\overline{p} \leq \overline{t}$. With this modification, called *tame intersection homology,* the deRham isomorphism ([16]) and the Poincaré Duality ([5, 9]) remain true without restriction on perversities. We present here a filtered version of tame intersection homology and prove in Theorem B that it is isomorphic to the homology introduced in [16] and [8].

The guiding idea for the definition of tame intersection homology is the elimination of the allowable simplices included in the singular part, see Definition 3.6. Such homology cannot be a topological invariant in general since, for any perversity $\overline{p}$ with $\overline{p} > \overline{t}$, it is equal to the relative intersection homology of X and its singular part (see Proposition 5.4 c)). Nevertheless, we prove that, if the CS set X has no singular strata becoming regular in the intrinsic CS set X^*, then the topological invariance remains true for tame intersection homology (cf. Theorem C-(ii)).

In the study of topological invariance with GM-perversities, we consider the *same* perversity on X and X^*. An analogy of this situation is the case of a perversity defined on X^* together with the pull-back perversities on each CS set having X^* as intrinsic CS set. In this situation, our main result implies the topological invariance of tame intersection homology (cf. Corollary 6.14 and Remark 6.15).

The homologies are with coefficients in an abelian group G. In general, we do not explicit them in the proofs.

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1. Stratified spaces.

In this section we recall the notions of stratified spaces, CS sets and pseudomanifolds.

Definition 1.1. A *filtered space* is a Hausdorff space, X, endowed with a filtration by closed subspaces,

$$X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X,$$

with $X_n \backslash X_{n-1} \neq \emptyset$. The *dimension* of X is denoted by $\dim X = n$.

The connected components, S, of $X_i \backslash X_{i-1}$ are the *strata* of X and we write $\dim S = i$ and $\operatorname{codim} S = \dim X - \dim S$. The strata of $X_n \backslash X_{n-1}$ are *regular strata*; the other ones are *singular strata*. The family of non-empty strata is denoted by $\mathcal{S}_X$ (or $\mathcal{S}$ if there is no ambiguity). The subspace $\Sigma_X = X_{n-1}$ is the *singular set*, sometimes also denoted Σ.

Example 1.2. Let X be a filtered space of dimension n.

- An open subset $U \subset X$, endowed with the *induced filtration* given by $U_i = U \cap X_i$, becomes a filtered space.
- If M is a topological manifold, the *product filtration* is defined by $(M \times X)_i = M \times X_i$. The product $M \times X$ becomes a filtered space.
- If X is compact, the open cone $\mathring{c}X = X \times [0,1[/X \times \{0\}$ is endowed with the *conical filtration* defined by $(\mathring{c}X)_i = \mathring{c}X_{i-1}$, $0 \leq i \leq n+1$. By convention, $\mathring{c}\,\emptyset = \{\mathtt{v}\}$, where $\mathtt{v} = [-,0]$ is the apex of the cone.

We enhance the notion of filtered space in order to obtain continuous maps with richer properties.

Definition 1.3. A *stratified space* is a filtered space such that any pair of strata, S and S' with $S \cap \overline{S'} \neq \emptyset$, verifies $S \subset \overline{S'}$.

The family $\mathcal{S}$ is a poset ([4, Section A.2]) or ([6, Section 2.2.1]) relatively to $S \preceq S'$ if $S \subseteq \overline{S}'$. We write $S \prec S'$ if $S \preceq S'$ and $S \neq S'$.

Definition 1.4. Let X be a stratified space. The *depth of* X is the greater integer m for which it exists a chain of strata, $S_0 \prec S_1 \prec \cdots \prec S_m$. We denote $\operatorname{depth} X = m$.

Example 1.5. If X is stratified, each construction of Example 1.2 is a stratified space.

Definition 1.6. A *stratified map,* $f\colon X \to Y$, is a continuous map between stratified spaces such that, for each stratum $S \in \mathcal{S}_X$, there exists a unique stratum $S^f \in \mathcal{S}_Y$ with $f(S) \subset S^f$ and $\operatorname{codim} S^f \leq \operatorname{codim} S$.

Observe that a continuous map $f\colon X \to Y$ is stratified if, and only if, the pull-back of a stratum $S' \in \mathcal{S}_Y$ is empty or a union $f^{-1}(S') = \sqcup_{i\in I} S_i$, with $\operatorname{codim} S' \leq \operatorname{codim} S_i$ for each $i \in I$. Therefore, a stratified map preserves regular strata but the image of a singular stratum can be regular.

Example 1.7. Let X be a stratified space. The canonical projection, $\mathrm{pr}\colon M \times X \to X$, the maps $\iota_t\colon X \to \mathring{\mathtt{c}}X$ with $x \mapsto [x,t]$, $\iota_m\colon X \to M \times X$ with $x \mapsto (m,x)$ and the canonical injection of an open subset $U \hookrightarrow X$, are stratified for the structures described in Example 1.2.

Let us recall some properties of stratified maps from [4, Section A.2].

Proposition 1.8 ([4, Proposition A.23])**.** *A stratified map, $f\colon X \to Y$, induces the order preserving map $(\mathcal{S}_X, \preceq) \to (\mathcal{S}_Y, \preceq)$, defined by $S \mapsto S^f$.*

Let us introduce the notion of homotopy between stratified maps. Here, the product $X \times [0,1]$ is endowed with the product filtration.

Definition 1.9. Two stratified maps $f, g\colon X \to Y$ are *homotopic* if there exists a stratified map, $\varphi\colon X \times [0,1] \to Y$, such that $\varphi(-,0) = f$ and $\varphi(-,1) = g$. Homotopy is an equivalence relation and produces the notion of homotopy equivalence between stratified spaces.

The following notion of locally cone-like stratified space has been introduced by Siebenman, [17].

Definition 1.10. A *CS set* of dimension n is a filtered space,

$$\emptyset \subset X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} \subseteq X_{n-1} \subsetneq X_n = X,$$

such that, for each i, $X_i \backslash X_{i-1}$ is a topological manifold of dimension i or the empty set. Moreover, for each point $x \in X_i \backslash X_{i-1}$, $i \neq n$, there exist

(i) an open neighborhood V of x in X, endowed with the induced filtration,
(ii) an open neighborhood U of x in $X_i \backslash X_{i-1}$,
(iii) a compact filtered space, L, of dimension $n-i-1$, where the open cone, $\mathring{\mathtt{c}}L$, is provided with the *conical filtration*, $(\mathring{\mathtt{c}}L)_j = \mathring{\mathtt{c}}L_{j-1}$.
(iv) a homeomorphism, $\varphi\colon U \times \mathring{\mathtt{c}}L \to V$, such that
  (a) $\varphi(u, \mathtt{v}) = u$, for each $u \in U$,
  (b) $\varphi(U \times \mathring{\mathtt{c}}L_j) = V \cap X_{i+j+1}$, for each $j \in \{0, \ldots, n-i-1\}$.

The pair (V, φ) is a *conical chart* of x and the filtered space L is a *link* of x. The CS set X is called *normal* if its links are connected.

In the above definition, the links are always non-empty sets. Therefore, the open subset $X_n \backslash X_{n-1}$ is dense. Links are not necessarily CS sets but they are always filtered spaces. Note also that the links associated to points living in the same stratum may not be homeomorphic but they always have the same intersection homology, see for example [6, Corollary 5.3.14]. An open subset of the CS set X and the product of X with a topological manifold, endowed with the induced structures of Example 1.2, are CS sets. This is also the case for the cone $\mathring{c}X$ when X is compact.

Proposition 1.11. *A paracompact CS set is metrizable.*

Proof. Let X be a paracompact CS set. From a Smirnov theorem [18], with the hypothesis "paracompact", it is sufficient to prove that X is locally metrizable. We proceed by induction on the depth of X. When depth $(X) = 0$, it is a consequnce of the fact that X is a topological manifold.

Let us suppose that the result is true for any paracompact CS set whose length is smaller than depth (X). Since the studied property is local, we can suppose that X is a conical chart $U \times \mathring{c}L$, with $U = \mathbb{R}^n$. Notice that $U \times \mathring{c}L \backslash (U \times \{\mathtt{v}\}) = U \times L \times]0,1[$ is an open subset of X and therefore a CS set of depth equal to depth $(X) - 1$. Since the three factors of $U \times L \times]0,1[$ are countable intersections of compact subsets, this product is a paracompact CS set and therefore a metrizable space by induction hypothesis.

Hence the product $U \times L \times [1/3, 2/3]$ is metrizable. The homeomorphic space $U \times L \times [0,1]$ is also metrizable. So, the product $[-1,1]^n \times L \times [0,1]$ is a compact metrizable space. Since the image by a continuous map of a compact metrizable space in a Hausdorff space is a metrizable space, we get that $[-1,1]^n \times (L \times [0,1]/L \times \{0\})$ is a metrizable space. We conclude that $]-1,1[^n \times \mathring{c}L$ is metrizable. Thus, the homeomorphic space $U \times \mathring{c}L$ is also metrizable. $\square$

Pseudomanifolds are special cases of CS sets. Their definition varies in the literature; we consider here the original definition of Goresky and MacPherson [10], without restriction on codimension one strata.

Definition 1.12. An *n-dimensional pseudomanifold* (or simply pseudomanifold) is an n-dimensional CS set, for which the link of a point $x \in X_i \backslash X_{i-1}$ is an $(n-i-1)$-dimensional pseudomanifold.

2. Perversities

> The main ingredient in intersection homology is the notion of perversity which "controls" the transverse degree between simplices and strata. Here we recall the various aspects of this notion.

Let us begin with the original definition of perversity [10], called *Goresky-MacPherson perversity,* to make a difference with the more general ones introduced in [12], that we simply call *perversities*.

Definition 2.1. A *loose perversity* [11] is a map $\overline{p} \colon \mathbb{N} \to \mathbb{Z}$, $i \mapsto \overline{p}(i)$, with $\overline{p}(0) = 0$. A *Goresky-MacPherson perversity* is a loose perversity such that

$$\overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1, \quad \text{for each} \quad i \geq 1, \tag{2.1}$$

and $\overline{p}(1) = \overline{p}(2) = 0$. A *King perversity* is a loose perversity verifying only (2.1).

In [11], H. King observed that the condition (2.1) is necessary for the topological invariance of intersection homology and that (2.1) together with $\overline{p}(1) \geq 0$ are sufficient.

This property being the main subject of this work, we particularize these perversities in the previous definition. Observe also that they have already been considered in the literature: the super-perversities of [8] are King perversities verifying $\overline{p}(2) > 0$; they cover the case $\overline{p}(2) = 1$ of [2].

Let us introduce more general perversities. Unlike the previous ones, they are defined on the set of non-empty strata of the filtered space.

Definition 2.2. A *perversity on a filtered space* X, is a map, $\overline{p}\colon \mathcal{S} \to \mathbb{Z}$, taking the value 0 on the regular strata. The pair $(X, \overline{p})$ is a *perverse space*.

When X is a stratified space, or a CS set, we say that $(X, \overline{p})$ is a *perverse stratified space* or a *perverse CS set*.

Let X be a filtered space. A loose perversity in the sense of Definition 2.1 induces a perversity on X by setting $\overline{p}(S) = \overline{p}(\operatorname{codim} S)$.

If $\overline{p}$ and $\overline{q}$ are two perversities on X, we write $\overline{p} \leq \overline{q}$ when $\overline{p}(S) \leq \overline{q}(S)$, for each $S \in \mathcal{S}$. The poset of Goresky-MacPherson perversities possesses a maximal element: the *top perversity* $\overline{t}$, defined by $\overline{t}(i) = i - 2$, if $i \geq 2$. By extension, we shall denote by $\overline{t}$ the perversity on X defined by $\overline{t}(S) = \operatorname{codim} S - 2$, if S is a singular stratum. Given a perversity $\overline{p}$ on X, the *complementary perversity* on X, $D\overline{p}$, is characterized by $D\overline{p} + \overline{p} = \overline{t}$. Mention also the *null perversity*, $\overline{0}$, defined by $\overline{0}(S) = 0$.

Definition 2.3. Let $f\colon X \to Y$ be a stratified map and $\overline{q}$ a perversity on Y. The *pull-back perversity of* $\overline{q}$ *by* f is the perversity $f^*\overline{q}$ on X defined by $(f^*\overline{q})(S) = \overline{q}(S^f)$, for each $S \in \mathcal{S}_X$.

Example 2.4. Let $(X, \overline{p})$ be a perverse stratified space and consider the constructions of Example 1.2.

- Any open subset $U \subset X$ is endowed with the pull-back perversity of $\overline{p}$ by the canonical inclusion $U \hookrightarrow X$.
- The product with a topological manifold, $X \times M$, is endowed with the pull-back perversity of $\overline{p}$ by the canonical projection $X \times M \to X$. We also denote it $\overline{p}$.
- When X is compact, we have $\mathcal{S}_{\mathring{c}X} = \{S \times]0,1[\mid S \in \mathcal{S}_X\} \cup \{\{\mathtt{v}\}\}$. So, a perversity $\overline{q}$ on $\mathring{c}X$ induces a perversity on X defined by $\overline{q}(S) = \overline{q}(S \times]0,1[)$. We also denote it $\overline{q}$.

Remark 2.5. Let us consider the following classical result: free actions of the circle on a topological space, X, are classified by the orbit space, $B = X/S^1$, and the Euler class $e \in H^2(B;\mathbb{Z})$. An extension of this result to the case of no free actions has been developed in [14, 15, 13] in the case of a modeled action (see [14]) and of pseudomanifolds, X and B. It is proven in [14], that the pseudomanifold B and an Euler class $e \in H^2_{\overline{e}}(B;\mathbb{R})$ determine the intersection cohomology of X. The Euler perversity, $\overline{e}$, is not a loose perversity. It takes the values 0, 1 or 2, following the behavior of the circle on the stratum:

- $\overline{e}(S) = 0$ if the stratum S has not fixed points,
- $\overline{e}(S) = 1$ if the stratum S has only fixed points and the circle acts trivially on the link L_S of S,
- $\overline{e}(S) = 2$ on the other strata.

This example shows the interest for more general perversities than the classical ones.

3. Intersection homologies. Theorems A and B

We present two chain complexes computing the intersection homology of Goresky and MacPherson [10]. The first one comes from the work of King [11]. The second one was already present in several works of the second author, cf. [16] for example. We also give a tame version of these homologies.

Let us begin with the notion of allowable simplex.

Definition 3.1. Let $(X, \overline{p})$ be a perverse space. A singular simplex $\sigma\colon \Delta \to X$, is $\overline{p}$-*allowable* if, for each stratum $S \in \mathcal{S}$, the subset $\sigma^{-1}(S)$ is included in the skeleton of Δ of dimension,

$$\dim \Delta - \operatorname{codim} S + \overline{p}(S). \tag{3.1}$$

A singular chain ξ is a $\overline{p}$-*allowable chain* if any simplex with a non-zero coefficient in ξ is $\overline{p}$-allowable. It is a $\overline{p}$-*intersection chain* if ξ and $\partial\xi$ are $\overline{p}$-allowable chains. Denote by $I^{\overline{p}}C_*(X;G)$ the chain complex of $\overline{p}$-intersection chains and by $I^{\overline{p}}H_*(X;G)$ its homology.

Let us observe that the condition (3.1) is satisfied for any regular stratum since $\operatorname{codim} S = \overline{p}(S) = 0$. In [11], King proved that $I^{\overline{p}}H_*(X;G)$ is isomorphic to the original intersection homology of Goresky and MacPherson introduced in [10].

Let us introduce the filtered simplices, more adapted to a simplicial approach of intersection homology.

Definition 3.2. Let X be a filtered space of dimension n. A *filtered simplex* is a singular simplex $\sigma\colon \Delta \to X$ endowed with a filtration $\Delta = \Delta_0 * \Delta_1 * \cdots * \Delta_n$, called σ-*decomposition of* Δ, verifying

$$\sigma^{-1}X_i = \Delta_0 * \Delta_1 * \cdots * \Delta_i,$$

for each $i \in \{0, \ldots, n\}$. Here, $*$ denotes the join of two simplices, with the convention $\emptyset * Y = Y$.

In other words, a singular simplex $\sigma\colon \Delta \to X$ is a filtered simplex if $\sigma^{-1}(X_i)$, $i \in \{0, \ldots, n\}$, is a face of Δ or the emptyset. The dimension of the simplices Δ_i measures the defect of transversality of σ relatively to the strata of X.

Definition 3.3. Let X be a filtered space of dimension n and $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ be a filtered simplex.

(i) The *perverse degree of* σ is the $(n+1)$-tuple, $\|\sigma\| = (\|\sigma\|_0, \ldots, \|\sigma\|_n)$, where $\|\sigma\|_i = \dim \sigma^{-1}X_{n-i} = \dim(\Delta_0 * \cdots * \Delta_{n-i})$, with the convention $\dim \emptyset = -\infty$.

(ii) For each stratum $S \in \mathcal{S}$, the *perverse degree of* σ *along the stratum* S is

$$\|\sigma\|_S = \begin{cases} -\infty, & \text{if } S \cap \sigma(\Delta) = \emptyset, \\ \|\sigma\|_{\operatorname{codim} S}, & \text{if not.} \end{cases}$$

(iii) For each stratum $S \in \mathcal{S}$, the *perverse degree of the chain* $\xi = \sum_{j\in J} \lambda_j \sigma_j$, $\lambda_j \neq 0$, *along the stratum* S is $\|\xi\|_S = \max_{j\in J} \|\sigma_j\|_S$.

The map $\|\xi\|\colon \mathcal{S} \to \mathbb{N} \cup \{-\infty\}$ is defined by $S \mapsto \|\xi\|_S$.

Remark 3.4. Let $\sigma\colon \Delta \to X$ be a filtered simplex and S be a stratum of X. If $\sigma^{-1}(S) \neq \emptyset$, we denote F_S the smallest face of Δ containing $\sigma^{-1}(S)$. Following Definition 3.1, the filtered simplex σ is $\overline{p}$-allowable if, and only if, $\sigma^{-1}(S) = \emptyset$ or $\dim F_S \leq \dim \Delta - \operatorname{codim} S + \overline{p}(S)$, for each $S \in \mathcal{S}$.

Denote $i = \operatorname{codim} S$. We proved in [4, Lemme A.24] that $\sigma^{-1}(S) = \emptyset$ or $\sigma^{-1}(S) = \Delta_0 * \cdots * \Delta_{n-i} \backslash \Delta_0 * \cdots * \Delta_{n-i-1}$. In the second case we have $\dim F_S = \|\sigma\|_i = \|\sigma\|_S$ and therefore we get:

$$\sigma \text{ is } \overline{p}\text{-allowable} \Longleftrightarrow \|\sigma\|_S \leq \dim \Delta - \operatorname{codim} S + \overline{p}(S), \quad \forall S \in \mathcal{S}.$$

Definition 3.5. Let $(X, \overline{p})$ be a perverse space. We denote $C_*^{\overline{p}}(X;G)$ the chain complex of $\overline{p}$-intersection chains made up of filtered simplices and $H_*^{\overline{p}}(X;G)$ its homology.

The next theorem establishes the existence of an isomorphism between the two complexes introduced above, in Definition 3.1 and Definition 3.5.

Theorem A. *Let $(X, \overline{p})$ be a perverse CS set. The canonical inclusion $C_*^{\overline{p}}(X;G) \hookrightarrow I^{\overline{p}}C_*(X;G)$ is a chain map inducing an isomorphism in homology,*

$$H_*^{\overline{p}}(X;G) \cong I^{\overline{p}}H_*(X;G).$$

For Goresky-MacPherson perversities this result is [4, Proposition A.29]. We find in [16, Proposition 2.4.1] a slightly different version using perversities. There, a blow-up of X is used and this process needs extra conditions on the space X. (For example, the blow-up of X exists if X is a Thom-Mather space.) We give in Section 5 a proof of Theorem A, without this restriction.

We introduce now the *tame intersection homology.* When we are dealing with perversities such that $\overline{p} \not\leq \overline{t}$, one needs this variation for having a deRham Theorem [16] or a Poincaré Duality [5, 9]. More precisely, a perversity $\overline{p}$ with $\overline{p} \leq \overline{t}$ has an essential property coming from (3.1): any $\overline{p}$-allowable simplex $\sigma\colon \Delta \to X$ verifies $\sigma(\partial\Delta) \not\subset \Sigma$. But if $\overline{p} \not\leq \overline{t}$ this kind of simplices may exist with the consequence that deRham Theorem and Poincaré Duality fail. Tame intersection homology is defined from the elimination of the allowable simplices included in the singular part. Let us begin with the notion of tame simplex (called non-GM in [6]).

Definition 3.6. Let X be a filtered space. A singular simplex $\sigma\colon \Delta \to X$ is *regular* if $\sigma(\Delta) \not\subset \Sigma$. In the boundary $\partial\sigma = \sum_{i=0}^m (-1)^i \sigma_i$, we keep only the regular simplices $\{\sigma_i\}_{i\in\mathfrak{I}}$ and set

$$\mathfrak{d}\sigma = \sum_{i\in\mathfrak{I}} (-1)^i \sigma_i.$$

Notice that $\mathfrak{d}^2 = 0$. The chain complex $I\mathfrak{C}_*(X;G)$ is the complex generated by the regular simplices of X endowed with the differential $\mathfrak{d}$.

And now come the perversities.

Definition 3.7. Let $(X, \overline{p})$ be a perverse space. A *$\overline{p}$-tame simplex* is a $\overline{p}$-allowable regular simplex. A chain $\xi \in I\mathfrak{C}_*(X;G)$ is a *$\overline{p}$-tame chain* if we can write it as a combination of $\overline{p}$-tame simplices. A $\overline{p}$-tame chain ξ such that $\mathfrak{d}\xi$ is $\overline{p}$-tame is called a *$\overline{p}$-tame intersection chain.* The complex of $\overline{p}$-tame intersection chains is denoted by $I^{\overline{p}}\mathfrak{C}_*(X;G)$. Its homology, $I^{\overline{p}}\mathfrak{H}_*(X;G)$, is the *$\overline{p}$-tame intersection homology* of X.

We adapt this definition to filtered simplices as follows.

Definition 3.8. Let $(X,\overline{p})$ be a perverse space. We denote $\mathfrak{C}^{\overline{p}}_*(X;G)$ the chain complex of $\overline{p}$-tame intersection chains made up of filtered simplices. Its homology is $\mathfrak{H}^{\overline{p}}_*(X;G)$

Remark 3.9. Condition (3.1) implies that the ∂-boundary of a $\overline{p}$-allowable simplex does not live in Σ, when $\overline{p}\leq\overline{t}$. So, in this case, we have $I^{\overline{p}}\mathfrak{C}_*(X;G)=I^{\overline{p}}C_*(X;G)$, $\mathfrak{C}^{\overline{p}}_*(X;G)=C^{\overline{p}}_*(X;G)$, $I^{\overline{p}}\mathfrak{H}_*(X;G)=I^{\overline{p}}H_*(X;G)$ and $\mathfrak{H}^{\overline{p}}_*(X;G)=H^{\overline{p}}_*(X;G)$

This homology is isomorphic to the non-GM intersection homology of Friedman [6].

Theorem B. *Let $(X,\overline{p})$ be a perverse CS set. The canonical inclusion $\mathfrak{C}^{\overline{p}}_*(X;G)\hookrightarrow I^{\overline{p}}\mathfrak{C}_*(X;G)$ is a chain map inducing an isomorphism in homology,*

$$\mathfrak{H}^{\overline{p}}_*(X;G)\cong I^{\overline{p}}\mathfrak{H}_*(X;G).$$

The proof of Theorem B is postpone to Section 5. We end this Section with some properties of stratified maps relatively to $H^{\overline{p}}_*(X;G)$ and $\mathfrak{H}^{\overline{p}}_*(X;G)$. We need the following notion introduced in [16].

Definition 3.10. Let $(X,\overline{p})$ be a perverse CS set. We set:

$$X_{\overline{p}}=\cup\{\overline{S}\mid S\in\mathcal{S}\text{ singular stratum with }\overline{p}(S)>\overline{t}(S)\}.$$

Proposition 3.11. *Let $f\colon(X,\overline{p})\to(Y,\overline{q})$ be a stratified map between two perverse stratified spaces with $f^*D\overline{q}\leq D\overline{p}$. The association $\sigma\mapsto f\circ\sigma$ defines a chain map $f_*\colon C^{\overline{p}}_*(X;G)\to C^{\overline{q}}_*(Y;G)$. Moreover, if $f(X_{\overline{p}})\subset\Sigma_Y$, then it induces also a chain map $f_*\colon\mathfrak{C}^{\overline{p}}_*(X;G)\to\mathfrak{C}^{\overline{q}}_*(Y;G)$.*

Since the map f is stratified, we observe that the condition $\overline{p}\leq f^*\overline{q}$ implies $f^*D\overline{q}\leq D\overline{p}$. We also remark that the condition $f(\Sigma_X)\subset\Sigma_Y$ implies $f(X_{\overline{p}})\subset\Sigma_Y$.

Proof. Let $\sigma\colon\Delta=\Delta_0*\dots*\Delta_n\to X$ be a $\overline{p}$-allowable filtered simplex. We proceed in two steps.

- *The map f_* preserves allowability;* i.e.,

$$\sigma\ \overline{p}\text{-allowable filtered simplex}\Rightarrow f_*(\sigma)\ \overline{q}\text{-allowable filtered simplex.}$$

By definition, the simplex σ verifies

$$\|\sigma\|_S\leq\dim\Delta-\operatorname{codim}S+\overline{p}(S),\tag{3.2}$$

for each stratum $S\in\mathcal{S}_X$ and we have to prove

$$\|f\circ\sigma\|_T\leq\dim\Delta-\operatorname{codim}T+\overline{q}(T)\tag{3.3}$$

for each singular stratum T of Y.

If $(f\circ\sigma)(\Delta)\cap T=\emptyset$ then $\|f\circ\sigma\|_T=-\infty$ and (3.3) is verified. We can suppose $(f\circ\sigma)(\Delta)\cap T\neq\emptyset$. Since f is a stratified map, then $f^{-1}(T)=\sqcup_{i\in I}S_i$, with $\operatorname{codim}T\leq\operatorname{codim}S_i$, for each $i\in I$. The hypothesis $(f\circ\sigma)(\Delta)\cap T\neq\emptyset$ implies the existence of

$\alpha\in I$, with $\sigma(\Delta)\cap S_\alpha\neq\emptyset$, and therefore $S^f_\alpha=T$ (cf. Definition 1.6). (Remark that S_α is a singular stratum.) We get

$$\begin{aligned}
\|f\circ\sigma\|_{\operatorname{codim} T} &\le \|f\circ\sigma\|_{\operatorname{codim} S_\alpha}\le\|\sigma\|_{\operatorname{codim} S_\alpha}\\
&\le \dim\Delta-\operatorname{codim} S_\alpha+\overline{p}(S_\alpha)=\dim\Delta-D\overline{p}(S_\alpha)+2\\
&\le \dim\Delta-f^*D\overline{q}(S_\alpha)+2\le\dim\Delta-D\overline{q}(T)+2\\
&\le \dim\Delta-\operatorname{codim} T+\overline{q}(T),
\end{aligned}$$

where we have used Definition 3.2, [4, Theorem F], the hypothesis of (3.2), the definition of the complementary perversity, the hypothesis $f^*D\overline{q}\le D\overline{p}$ and the definition of the pull-back of a perversity. Inequality (3.3) is proven.

• *The operator f_* is differential.* We have $f_*(\partial\sigma)=\partial f_*(\sigma)$ and it suffices to prove $f_*(\mathfrak{d}\sigma)=\mathfrak{d}f_*(\sigma)$ in the case of a regular simplex $\sigma\colon\Delta=\Delta_0*\cdots*\Delta_n\to X$. It is enough to prove $f_*(\partial\sigma-\mathfrak{d}\sigma)\subset\Sigma_Y$. Let us see that.

Since $f(X_{\overline{p}})\subset\Sigma_Y$, we are reduced to establish $\operatorname{Im}\sigma'\subset X_{\overline{p}}$ for each codimension one non-regular face $\sigma'\colon\Delta'\to X$ of σ. By dimension reasons, we have $\dim\Delta_n=0$ and also there exists $i<n$ with $\Delta_i\neq\emptyset$. We denote j the largest integer i with this property. Let $S\in\mathcal{S}_X$ be the singular stratum with $n-j=\operatorname{codim} S$ and $S\cap\sigma(\Delta)\neq\emptyset$. The $\overline{p}$-allowability of σ gives $\|\sigma\|_{n-j}=\dim(\Delta_0*\cdots*\Delta_j)=\dim\Delta-1\le\dim\Delta-\operatorname{codim} S+\overline{p}(S)$. Hence $\overline{t}(S)=\operatorname{codim} S-2<\overline{p}(S)$ and therefore $S\subset X_{\overline{p}}$, which gives the result. □

Remark 3.12. In the previous proof, the stratum S^f is singular. So, it suffices to verify the inequality $f^*D\overline{q}(S)\le D\overline{p}(S)$ when S^f, and therefore S, are singular strata.

Proposition 3.13. *Let $\varphi\colon(X\times[0,1],\overline{p})\to(Y,\overline{q})$ be a homotopy between two stratified maps $f,\,g\colon(X,\overline{p})\to(Y,\overline{q})$ with $\varphi^*D\overline{q}\le D\overline{p}$. Then f and g induce the same map in homology, $f_*=g_*\colon H^{\overline{p}}_*(X;G)\to H^{\overline{q}}_*(Y;G)$. Moreover, if $\varphi(X_{\overline{p}}\times[0,1])\subset\Sigma_Y$ then we also have $f_*=g_*\colon\mathfrak{H}^{\overline{p}}_*(X;G)\to\mathfrak{H}^{\overline{q}}_*(Y;G)$.*

Proof. Consider the canonical injections, $\iota_0,\,\iota_1\colon X\to X\times[0,1]$, defined by $\iota_k(x)=(x,k)$ for $k=0,\,1$. They are stratified maps and verify $\iota_k(X_{\overline{p}}\times[0,1])\subset\Sigma_X$ for $k=0,\,1$. From Proposition 3.11, we have the homomorphism $\varphi_*\colon H^{\overline{p}}_*(X\times[0,1])\to H^{\overline{q}}_*(Y)$ and also $\varphi_*\colon\mathfrak{H}^{\overline{p}}_*(X\times[0,1])\to\mathfrak{H}^{\overline{q}}_*(Y)$ when $\varphi(X_{\overline{p}}\times[0,1])\subset\Sigma_Y$ Since $f=\varphi\circ\iota_0$ and $g=\varphi\circ\iota_1$ then it suffices to prove $\iota_{0,*}=\iota_{1,*}\colon H^{\overline{p}}_*(X)\to H^{\overline{p}}_*(X\times[0,1])$ and $\iota_{0,*}=\iota_{1,*}\colon\mathfrak{H}^{\overline{p}}_*(X)\to\mathfrak{H}^{\overline{p}}_*(X\times[0,1])$.

Let $\sigma\colon\Delta=\langle e_0,\dots,e_m\rangle\to X$ be a filtered simplex. The vertices of $\Delta\times[0,1]$ are $a_j=(e_j,0)$ and $b_j=(e_j,1)$. We define an $(m+1)$-chain on $\Delta\times[0,1]$ by $P=\sum_{j=0}^m(-1)^j\langle a_0,\dots,a_j,b_j,\dots,b_m\rangle$. This gives a chain homotopy $h\colon C_*(X)\to C_{*+1}(X\times[0,1])$, between $\iota_{0,*}$ and $\iota_{1,*}$, defined by $\sigma\mapsto(\sigma\times\mathrm{id})_*(P)$, where $C_*(-)$ denotes the complex generated by filtered simplices.

Consider the complex $\mathfrak{C}_*(-)$, generated by regular simplices. Since h preserves regular simplices and non-regular simplices, we get the chain homotopy $h\colon\mathfrak{C}_*(X)\to\mathfrak{C}_{*+1}(X\times[0,1])$ between $\iota_{0,*}$ and $\iota_{1,*}$, Now, it remains to prove

$$\sigma\ \overline{p}\text{-allowable filtered simplex}\Rightarrow h(\sigma)\ \overline{p}\text{-allowable filtered simplex.}\tag{3.4}$$

Let $j \in \{0, \dots, m\}$. We denote $\tau_j \colon \nabla = \langle v_0, \dots, v_{m+1} \rangle \to \Delta^m \times [0,1]$, the simplex defined by $(v_0, \dots, v_{m+1}) \mapsto (a_0, \dots, a_j, b_j, \dots, b_m)$. By construction we have $\Delta^m \times [0,1] = \cup_{j=0}^m \tau_j(\nabla)$. For each stratum $S \in \mathcal{S}$, we have

$$\tau_j^{-1}(\sigma \times \mathrm{id})^{-1}(S \times [0,1]) = \tau_j^{-1}(\sigma^{-1}(S) \times [0,1]) \subset \sigma^{-1}(S) \times [0,1],$$

and therefore

$$\|(\sigma \times \mathrm{id}) \circ \tau_j\|_{S \times [0,1]} \le \|\sigma\|_S + 1 \le m - \operatorname{codim} S + \overline{p}(S) + 1.$$

We get

$$\begin{aligned} \|h(\sigma)\|_{S \times [0,1]} &\le \max_j \|(\sigma \times \mathrm{id}) \circ \tau_j\|_{S \times [0,1]} \le m - \operatorname{codim} S + \overline{p}(S) + 1 \\ &= \dim \nabla - \operatorname{codim}(S \times [0,1]) + \overline{p}(S \times [0,1]). \end{aligned}$$

This proves (3.4). ∎

Corollary 3.14. *Let $(X, \overline{p})$ be a perverse space. The inclusions $\iota_z \colon X \hookrightarrow \mathbb{R} \times X$, $x \mapsto (z, x)$, with $z \in \mathbb{R}$, and the canonical projection* $\mathrm{pr} \colon \mathbb{R} \times X \to X$ *induce isomorphisms $H_k^{\overline{p}}(X; G) \cong H_k^{\overline{p}}(\mathbb{R} \times X; G)$ and $\mathfrak{H}_k^{\overline{p}}(X; G) \cong \mathfrak{H}_k^{\overline{p}}(\mathbb{R} \times X; G)$.*

4. Mayer-Vietoris sequence

In this section, we construct Mayer-Vietoris sequences for the intersection homology and the tame intersection homology, $H_*^{\overline{p}}(X; G)$ and $\mathfrak{H}_*^{\overline{p}}(X; G)$. This is a key point in the proofs of Theorems A, B and C.

Proposition 4.1 (Mayer-Vietoris sequence)**.** *Given a perverse space $(X, \overline{p})$ and an open covering $\{U, V\}$ of X, there exist two exact sequences,*

$$\dots \longrightarrow H_i^{\overline{p}}(U \cap V; G) \longrightarrow H_i^{\overline{p}}(U; G) \oplus H_i^{\overline{p}}(V; G) \longrightarrow H_i^{\overline{p}}(X; G) \longrightarrow H_{i-1}^{\overline{p}}(U \cap V; G) \longrightarrow \dots \tag{4.1}$$

and

$$\dots \longrightarrow \mathfrak{H}_i^{\overline{p}}(U \cap V; G) \longrightarrow \mathfrak{H}_i^{\overline{p}}(U; G) \oplus \mathfrak{H}_i^{\overline{p}}(V; G) \longrightarrow \mathfrak{H}_i^{\overline{p}}(X; G) \longrightarrow \mathfrak{H}_{i-1}^{\overline{p}}(U \cap V; G) \longrightarrow \dots \tag{4.2}$$

The connecting map of (4.1) is given by $\delta_h(\xi) = [\partial \xi_U]$, where ξ_U is obtained from the subdivision operator, $\mathrm{sd}^k \xi = \xi_U + \xi_V \in C_*^{\overline{p}}(U) + C_*^{\overline{p}}(V)$ of (4.7). The connecting map of (4.2) is similar from the subdivision operator (4.8).

Before proving this result we need to study the $\overline{p}$-allowable simplices that are not $\overline{p}$-intersection chains. We follow the method used in [4, Proposition A.14.(i)]. The thrust is that the $\overline{p}$-allowability default of the boundary of a $\overline{p}$-allowable simplex is concentrated in only one face.

Definition 4.2. ([4, Definition A.11]) Let $(X, \overline{p})$ be a perverse space and $\sigma \colon \Delta \to X$ be a filtered $\overline{p}$-allowable simplex. For each stratum $S \in \mathcal{S}$ with $\sigma^{-1}(S) \neq \emptyset$, we denote F_S the smallest face of Δ containing $\sigma^{-1}(S)$. *The $\overline{p}$-bad face* of σ is the restriction $\sigma \colon T_{\overline{p}}(\sigma) \to X$ where

$$T_{\overline{p}}(\sigma) = \min\{F_S \mid S \in \mathcal{S}, \dim \Delta > \dim F_S = \dim \Delta - \operatorname{codim} S + \overline{p}(S)\}, \tag{4.3}$$

if this family is not empty. By extension, the face $T_{\overline{p}}(\sigma)$ of Δ is also called a *$\overline{p}$-bad face.*

Since the previous family $\{F_S\}$ mentioned above is totally ordered (cf. [4, Lemma A.24]), the definition of $T_{\overline{p}}(\sigma)$ makes sense. Observe also that if the stratum S is regular then $\overline{p}(S) = 0 = \operatorname{codim} S$ and the inequality of (4.3) does not occur.

Proposition 4.3. *Let $(X, \overline{p})$ be a perverse space and $\sigma\colon \Delta \to X$ a filtered $\overline{p}$-allowable simplex.*

(a) *A codimension one face s of σ is not $\overline{p}$-allowable if, and only if, σ has a $\overline{p}$-bad face included in s.*
(b) *Let $\sigma'\colon \Delta \to X$ be another $\overline{p}$-allowable simplex. If σ and σ' share a codimension one face σ'' which is not $\overline{p}$-allowable, then σ and σ' have the same $\overline{p}$-bad face. Moreover, this face is a face of σ''.*
(c) *The simplex σ is a $\overline{p}$-intersection chain if, and only if, it has no $\overline{p}$-bad face.*

Proof. (a) Let $s\colon \nabla \to X$ be a face of codimension one of σ.

• Let us suppose that the face s is not $\overline{p}$-allowable. So, there exists a stratum $S \in \mathcal{S}$ such that $s^{-1}(S)$ is not included in the $(\dim \nabla - \operatorname{codim} S + \overline{p}(S))$- skeleton of ∇. Since $s^{-1}(S) \neq \emptyset$ then $\sigma^{-1}(S) \neq \emptyset$. Recall that F_S is the smallest face of Δ containing $\sigma^{-1}(S)$. As the simplex σ is $\overline{p}$-allowable, we have

$$\dim(F_S \cap \nabla) \leq \dim F_S \leq \dim \Delta - \operatorname{codim} S + \overline{p}(S).$$

On the other hand, as $s^{-1}(S) \subset F_S \cap \nabla$ and the simplex s is not $\overline{p}$-allowable, we get

$$\dim(F_S \cap \nabla) \geq \dim \nabla + 1 - \operatorname{codim} S + \overline{p}(S) = \dim \Delta - \operatorname{codim} S + \overline{p}(S).$$

We conclude that $\dim(F_S \cap \nabla) = \dim F_S = \dim \Delta - \operatorname{codim} S + \overline{p}(S)$ and $F_S \subset \nabla$. This gives $T_{\overline{p}}(\sigma) \subset \nabla$.

• Reciprocally, let us suppose that the $\overline{p}$-bad face of σ exists and verifies $T_{\overline{p}}(\sigma) \subset \nabla$. So, there exists a stratum $S \in \mathcal{S}$ with

$$F_S \subset \nabla \text{ and } \dim F_S = \dim \Delta - \operatorname{codim} S + \overline{p}(S),$$

where F_S is the smallest face of Δ containing $s^{-1}(S)$. The inclusions $\sigma^{-1}(S) \subset F_S \subset \nabla$ imply $\sigma^{-1}(S) = s^{-1}(S)$. Thus, F_S is also the smallest face of ∇ containing $s^{-1}(S)$. From

$$\dim F_S = \dim \Delta - \operatorname{codim} S + \overline{p}(S) > \dim \nabla - \operatorname{codim} S + \overline{p}(S),$$

we deduce that the simplex s is not $\overline{p}$-allowable.

(b) Consider a stratum $S \in \mathcal{S}$ with $F_S = T_{\overline{p}}(\sigma)$. We have proven that $F_S \subset \nabla$ and $\emptyset \neq \sigma^{-1}(S) = \sigma''^{-1}(S) \subset \sigma'^{-1}(S)$. Let F'_S be the smallest face of Δ containing $\sigma'^{-1}(S)$. It also contains $T_{\overline{p}}(\sigma)$ and we have

$$\dim F'_S \leq \dim \Delta - \operatorname{codim} S + \overline{p}(S) = \dim T_{\overline{p}}(\sigma),$$

since σ' is $\overline{p}$-allowable. We get $T_{\overline{p}}(\sigma) = F'_S$ and $\dim F'_S = \dim \Delta - \operatorname{codim} S + \overline{p}(S)$. By definition of $\overline{p}$-bad faces, we conclude that $T_{\overline{p}}(\sigma') \subset F'_S$ and therefore $T_{\overline{p}}(\sigma') \subset T_{\overline{p}}(\sigma)$. Since the simplices σ and σ' play the same role we have $T_{\overline{p}}(\sigma') = T_{\overline{p}}(\sigma)$. Property (a) gives that $T_{\overline{p}}(\sigma)$ is a face of σ''.

(c) If the simplex σ has a $\overline{p}$-bad face F, then F is included in the boundary of σ. The result comes from (a). ☐

Proof of Proposition 4.1. Let us consider the two short exact sequences

$$0 \longrightarrow C_*^{\overline{p}}(U\cap V) \longrightarrow C_*^{\overline{p}}(U)\oplus C_*^{\overline{p}}(V) \xrightarrow{\varphi} C_*^{\overline{p}}(U)+C_*^{\overline{p}}(V) \longrightarrow 0 \tag{4.4}$$

and

$$0 \longrightarrow \mathfrak{C}_*^{\overline{p}}(U\cap V) \longrightarrow \mathfrak{C}_*^{\overline{p}}(U)\oplus \mathfrak{C}_*^{\overline{p}}(V) \xrightarrow{\psi} \mathfrak{C}_*^{\overline{p}}(U)+\mathfrak{C}_*^{\overline{p}}(V) \longrightarrow 0. \tag{4.5}$$

where the chain maps φ and ψ are defined by $(\alpha,\beta)\mapsto \alpha+\beta$. The existence of the Mayer-Vietoris exact sequences come from the facts that the two inclusions $\operatorname{Im}\varphi \hookrightarrow C_*^{\overline{p}}(X)$ and $\operatorname{Im}\psi \hookrightarrow \mathfrak{C}_*^{\overline{p}}(X)$ induce isomorphisms in homology.

The complex generated by the filtered simplices of X is denoted by $C_*(X)$. We have constructed two operators $\operatorname{sd}: C_*(X)\to C_*(X)$ and $T\colon C_*(X)\to C_{*+1}(X)$ verifying $\operatorname{sd}\partial = \partial \operatorname{sd}$ and $\partial T + T\partial = \operatorname{id} - \operatorname{sd}$ (cf. [4, Lemma A.16]). It is also proven that the chain $\operatorname{sd}(\sigma)$ is composed of simplices of the form $\sigma\circ\zeta$, where $\zeta\in\Delta^{per}(\Delta)$ lives in the sub-complex generated by the linear simplices of Δ verifying $\dim \zeta^{-1}(F)\leq \dim F$, for any face F of Δ. We have also established the inequality

$$||\sigma\circ\zeta||_i \leq ||\sigma||_i. \tag{4.6}$$

We decompose the proof of Proposition 4.1 in three steps.

• *First step: The subdivision operators* $\operatorname{sd}: C_*^{\overline{p}}(X)\to C_*^{\overline{p}}(X)$ *and* $\mathfrak{sd}: \mathfrak{C}_*^{\overline{p}}(X)\to \mathfrak{C}_*^{\overline{p}}(X)$ *are homotopic to the identity.* For the first assertion, it suffices to prove that the operators sd and T preserve the perverse degree, that is, they induce the operators $\operatorname{sd}: C_*^{\overline{p}}(X)\to C_*^{\overline{p}}(X)$ and $: C_*^{\overline{p}}(X)\to C_{*+1}^{\overline{p}}(X)$.

Let us begin with sd. Given a filtered simplex $\sigma\colon\Delta\to X$, we need to prove the inequality $||\sigma\circ\zeta||_S\leq||\sigma||_S$, for each $S\in\mathcal{S}$ and each $\zeta\in\Delta^{per}(\Delta)$. Write $i=\operatorname{codim} S$. This inequality is clear when $\operatorname{Im}(\sigma\circ\zeta)\cap S=\emptyset$. If $\operatorname{Im}(\sigma\circ\zeta)\cap S\neq\emptyset$, then $\operatorname{Im}\sigma\cap S\neq\emptyset$ and we get

$$||\sigma\circ\zeta||_S = ||\sigma\circ\zeta||_i \leq ||\sigma||_i = ||\sigma||_S,$$

where the inequality comes from (4.6). A similar proof works for the operator T.

Let $\sigma\colon\Delta\to X$ be a filtered simplex and $\zeta\in\Delta^{per}(\Delta)$ be a linear simplex. We consider the σ-decomposition $\Delta=\Delta_0*\cdots*\Delta_n$ and the $(\sigma\circ\zeta)$-decomposition $\Delta=\Delta'_0*\cdots*\Delta'_n$. By definition of ζ we have $\Delta'_0*\cdots*\Delta'_{n-1}\subset\Delta_0*\cdots*\Delta_{n-1}$ and therefore $\dim\Delta_n\leq\dim\Delta'_n$. So, if σ is a regular simplex then all the simplices of $\operatorname{sd}(\sigma)$ are also regular simplices. On the other hand, if σ is not a regular simplex, that is $\operatorname{Im}\sigma\subset\Sigma$, then $\sigma\circ\zeta$ neither is. This implies that *sd* induces a chain map $\mathfrak{sd}: \mathfrak{C}_*^{\overline{p}}(X)\to\mathfrak{C}_*^{\overline{p}}(X)$. A similar proof works for the construction of a homotopy operator $\mathfrak{T}\colon \mathfrak{C}_*(X)\to\mathfrak{C}_{*+1}(X)$ from T.

• *Second step. We prove the two following implications:*

$$\xi\in C_*^{\overline{p}}(X) \Rightarrow \operatorname{sd}^k\xi\in C_*^{\overline{p}}(U)+C_*^{\overline{p}}(V) \textit{ for some } k\geq 0, \tag{4.7}$$

and

$$\xi\in \mathfrak{C}_*^{\overline{p}}(X) \Rightarrow \mathfrak{sd}^k\xi\in \mathfrak{C}_*^{\overline{p}}(U)+\mathfrak{C}_*^{\overline{p}}(V) \textit{ for some } k\geq 0. \tag{4.8}$$

The *canonical decomposition* of a $\overline{p}$-allowable chain ξ is $\xi=\xi_0+\sum_{\tau\in I_\xi}\xi_\tau$ where:

- ξ_0 is the chain containing the simplices of ξ without $\overline{p}$-bad faces.
- I_ξ is the family of the $\overline{p}$-bad faces of the simplices of ξ.
- ξ_τ is the chain containing the simplices of ξ having τ as $\overline{p}$-bad face.

The boundaries $\partial\xi_0$ and $\mathfrak{d}\xi_0$ are $\overline{p}$-allowable chains. A non-$\overline{p}$-allowable face (resp. regular non-$\overline{p}$-allowable face) s of a simplex σ of ξ_τ contains necessarily τ. When $\partial\xi$ (resp. $\mathfrak{d}\xi$) is a $\overline{p}$-allowable chain then s does not appear in $\partial\xi$ (resp. $\mathfrak{d}\xi$). So, there exists another simplex σ' in ξ having s as a face. Since s contains τ then τ is the bad face of σ'. We conclude that $\partial\xi_\tau$ (resp. $\mathfrak{d}\xi_\tau$) is also a $\overline{p}$-allowable chain. (These facts come from Proposition 4.3.) For each $\tau \in I_\xi$, we have proven

$$\left(\xi \in C_*^{\overline{p}}(X) \Rightarrow \xi_0, \xi_\tau \in C_*^{\overline{p}}(X)\right) \text{ and } \left(\xi \in \mathfrak{C}_*^{\overline{p}}(X) \Rightarrow \xi_0, \xi_\tau \in \mathfrak{C}_*^{\overline{p}}(X)\right).$$

The usual subdivision argument gives the existence of an integer $k \geq 1$ such that the canonical decomposition of $\mathrm{sd}^k\xi$ verifies the following properties.

- Each simplex of $(\mathrm{sd}^k\xi)_0$ lives in U or in V.
- For each $\tau \in I_{\mathrm{sd}^k\xi}$ the chain $(\mathrm{sd}^k\xi)_\tau$ lives in U or in V.

This gives (4.7) and (4.8)

• *Third step. The two inclusions* $\iota\colon \mathrm{Im}\,\varphi \hookrightarrow C_*^{\overline{p}}(X)$ *and* $\kappa\colon \mathrm{Im}\,\psi \hookrightarrow \mathfrak{C}_*^{\overline{p}}(X)$ *induce isomorphisms in homology.*

Let $[\xi] \in H_*^{\overline{p}}(X)$. The first step implies $[\xi] = [\mathrm{sd}^i\xi]$, for each $i \geq 0$. From the second step, we get an integer $k \geq 0$ with $\mathrm{sd}^k\xi \in \mathrm{Im}\,\varphi$. This gives the surjectivity of ι_*.

In order to prove the injectivity of ι_*, we consider $[\alpha] \in H_*(\mathrm{Im}\,\varphi)$ and $\xi \in C_{*+1}^{\overline{p}}(X)$ with $\alpha = \partial\xi$. The second step gives an integer $k \geq 0$ with $\mathrm{sd}^k\xi \in \mathrm{Im}\,\varphi$, and therefore we have $[\alpha] = [\mathrm{sd}^k\alpha] = [\mathrm{sd}^k(\partial\xi)] = [\partial(\mathrm{sd}^k\xi)]$ and $[\alpha] = 0$ in $H_*(\mathrm{Im}\,\varphi)$.

The same arguments work for κ_* and the connecting map is computed in the usual way. □

Excision property also exists in this context.

Definition 4.4. Let $(X, \overline{p})$ be a perverse space and $U \subset X$ an open subset. The *relative chains complexes* are the quotients $C_*^{\overline{p}}(X, U; G) = C_*^{\overline{p}}(X; G)/C_*^{\overline{p}}(U; G)$ and $\mathfrak{C}_*^{\overline{p}}(X, U; G) = \mathfrak{C}_*^{\overline{p}}(X; G)/\mathfrak{C}_*^{\overline{p}}(U; G)$. The *relative homologies*, $H_*^{\overline{p}}(X, U; G)$ and $\mathfrak{H}_*^{\overline{p}}(X, U; G)$, are the homologies of these two complexes, respectively.

These relative homologies give long exact sequences:

$$\cdots \to H_i^{\overline{p}}(U; G) \to H_i^{\overline{p}}(X; G) \to H_i^{\overline{p}}(X, U; G) \to H_i^{\overline{p}}(U; G) \to \ldots \tag{4.9}$$

and

$$\cdots \to \mathfrak{H}_i^{\overline{p}}(U; G) \to \mathfrak{H}_i^{\overline{p}}(X; G) \to \mathfrak{H}_i^{\overline{p}}(X, U; G) \to \mathfrak{H}_i^{\overline{p}}(U; G) \to \ldots \tag{4.10}$$

Proceeding as in the classical case ([19, Chapter 4, Section 6, Corollary 5]), we get an excision property.

Corollary 4.5. *Let* $(X, \overline{p})$ *be a perverse space. If* F *is a closed subset of* X *and* U *is an open subset of* X *with* $F \subset U$, *then the natural inclusion* $(X\backslash F, U\backslash F) \hookrightarrow (X, U)$ *induces the isomorphisms* $H_i^{\overline{p}}(X\backslash F, U\backslash F; G) \cong H_i^{\overline{p}}(, U; G)$ *and* $H_i^{\overline{p}}(X\backslash F, U\backslash F; G) \cong H_i^{\overline{p}}(, U; G)$.

The principle of $\mathfrak{U}$-small chains also exists in this context.

Corollary 4.6. *Let $(X,\overline{p})$ be a perverse space endowed with an open covering $\mathcal{U}$. Denote $C_*^{\overline{p},\mathcal{U}}(X;G)$ (resp. $\mathfrak{C}_*^{\overline{p},\mathcal{U}}(X;G)$) the sub-complex of $C_*^{\overline{p}}(X;G)$ (resp. $\mathfrak{C}_*^{\overline{p}}(X;G)$) made up of chains whose support is included in an element of $\mathcal{U}$. Then, the canonical inclusions, $C_*^{\overline{p},\mathcal{U}}(X;G) \hookrightarrow C_*^{\overline{p}}(X;G)$ and $\mathfrak{C}_*^{\overline{p},\mathcal{U}}(X;G) \hookrightarrow \mathfrak{C}_*^{\overline{p}}(X;G)$), induce isomorphisms in homology.*

Proof. By construction, for any element $\xi \in C_*^{\overline{p}}(X)$ (resp. $\mathfrak{C}_*^{\overline{p}}(X)$) there exists an integer m with $\mathrm{sd}^m\xi \in C_*^{\overline{p},\mathcal{U}}(X)$ (resp. $\mathfrak{sd}^m\xi \in \mathfrak{C}_*^{\overline{p},\mathcal{U}}(X)$). Moreover $\mathrm{sd}^m\xi$ and $\mathfrak{sd}^m\xi$ are cycles if ξ is one. The argument used in the third step of the proof of Proposition 4.1 gives the injectivity and the surjectivity of the two canonical inclusions. □

The following result is used in the proof of Proposition 6.16.

Corollary 4.7. *For any perverse space $(X,\overline{p})$ and sphere S^ℓ, there are isomorphisms, $H_k^{\overline{p}}(S^\ell \times X;G) \cong H_k^{\overline{p}}(X;G) \oplus H_{k-\ell}^{\overline{p}}(X;G)$ and $\mathfrak{H}_k^{\overline{p}}(S^\ell \times X;G) \cong \mathfrak{H}_k^{\overline{p}}(X;G) \oplus \mathfrak{H}_{k-\ell}^{\overline{p}}(X;G)$.*

Proof. With an induction on ℓ, it is a consequence of Propositions 3.14 and 4.1, relatively to the decomposition $\mathbb{S}^\ell = \mathbb{S}^\ell \backslash \{\mathrm{North}\} \cup \mathbb{S}^\ell \backslash \{\mathrm{South}\}$. □

5. Proofs of Theorems A and B

The method of proof is a variant of [11, Theorem 10], [16, Lemma 1.4.1]. We choose the formulation presented by Friedman in [6, Section 5.1].

Theorem 5.1. ([6, Theorem 5.1.1]) *Let $\mathcal{F}_X$ be the category whose objects are (stratified homeomorphic to) open subsets of a given CS set X and whose morphisms are stratified homeomorphisms and inclusions. Let $\mathcal{A}b_*$ be the category of graded abelian groups. Let F^*, $G^*\colon \mathcal{F}_X \to \mathcal{A}b$ be two functors and $\Phi\colon F_* \to G_*$ be a natural transformation satisfying the conditions listed below.*

- *(i) F^* and G^* admit exact Mayer-Vietoris sequences and the natural transformation Φ induces a commutative diagram between these sequences.*
- *(ii) If $\{U_\alpha\}$ is an increasing collection of open subsets of X and $\Phi\colon F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each α, then $\Phi\colon F^*(\cup_\alpha U_\alpha) \to G^*(\cup_\alpha U_\alpha)$ is an isomorphism.*
- *(iii) If L is a compact filtered space such that X has an open subset which is stratified homeomorphic to $\mathbb{R}^i \times \mathring{c}L$ and, if $\Phi\colon F^*(\mathbb{R}^i \times (\mathring{c}L\backslash\{\mathtt{v}\})) \to G^*(\mathbb{R}^i \times (\mathring{c}L\backslash\{\mathtt{v}\}))$ is an isomorphism, then so is $\Phi\colon F^*(\mathbb{R}^i \times \mathring{c}L) \to G^*(\mathbb{R}^i \times \mathring{c}L)$. Here, $\mathtt{v}$ is the apex of the cone $\mathring{c}L$.*
- *(iv) If U is an open subset of X contained within a single stratum and homeomorphic to an Euclidean space, then $\Phi\colon F^*(U) \to G^*(U)$ is an isomorphism.*

Then $\Phi\colon F^*(X) \to G^*(X)$ is an isomorphism.

Before using Theorem 5.1, we need to compute the (tame) intersection homology of a cone. Let us emphasize the difference between the two homologies.

Proposition 5.2 (Homology of a cone)**.** *Let X be a compact filtered space of dimension n. Consider a perversity $\overline{p}$ on the cone $\mathring{c}X$. We have:*

$$H_k^{\overline{p}}(\mathring{c}X;G) \cong \begin{cases} H_k^{\overline{p}}(X;G) & \text{if } k < n - \overline{p}(\{\mathtt{v}\}), \\ 0 & \text{if } 0 \neq k \geq n - \overline{p}(\{\mathtt{v}\}), \\ G & \text{if } 0 = k \geq n - \overline{p}(\{\mathtt{v}\}), \end{cases}$$

and

$$\mathfrak{H}_k^{\overline{p}}(\mathring{\tt c}X;G) \cong \begin{cases} \mathfrak{H}_k^{\overline{p}}(X;G) & \text{if } k < n - \overline{p}(\{\mathtt{v}\}),\\ 0 & \text{if } k \geq n - \overline{p}(\{\mathtt{v}\}).\end{cases}$$

In the two cases, the isomorphism at the top is induced by the map $\iota_{\mathring{\tt c}X}\colon X \to \mathring{\tt c}X$, $x \mapsto [x,1/2]$. *The other non-zero determination comes from the canonical inclusion* $C_k^{\overline{p}}(\mathring{\tt c}X;G) \to C_k(\mathring{\tt c}X;G)$.

Proof. We proceed in several steps.

• *In the low degrees.* Let $k = \dim \Delta \leq n - \overline{p}(\{\mathtt{v}\})$ and $\sigma\colon \Delta \to \mathring{\tt c}X$ be a $\overline{p}$-allowable filtered simplex. The allowability condition $\|\sigma\|_{\{\mathtt{v}\}} \leq k-(n+1)+\overline{p}(\{\mathtt{v}\}) < 0$, implies $\sigma^{-1}(\{\mathtt{v}\}) = \emptyset$. We therefore have $C^{\overline{p}}_{\leq n-\overline{p}(\mathtt{v})}(\mathring{\tt c}X) = C^{\overline{p}}_{\leq n-\overline{p}(\mathtt{v})}(X\times]0,1[)$ and $\mathfrak{C}^{\overline{p}}_{\leq n-\overline{p}(\mathtt{v})}(\mathring{\tt c}X) = \mathfrak{C}^{\overline{p}}_{\leq n-\overline{p}(\mathtt{v})}(X\times]0,1[)$. Thus, Corollary 3.14 gives the requested isomorphisms.

• *Construction of a cone operator.* If $[x,t] \in \mathring{\tt c}X$, we define $s\cdot[x,t] = [x,st]$, for each $s \in [0,1]$. Let $\sigma\colon \Delta \to \mathring{\tt c}X$ be a $\overline{p}$-allowable filtered simplex. We define the cone $c\sigma\colon \{\mathtt{v}\} * \Delta \to \mathring{\tt c}X$ by $c\sigma(sx+(1-s)\mathtt{v}) = s\cdot\sigma(x)$, for each $s\in[0,1]$ and $x \in \Delta$.

If $\dim\Delta = k \geq n-\overline{p}(\{\mathtt{v}\})$, we prove that $c\sigma$ is $\overline{p}$-allowable. For that, let us observe that the cone $\mathring{\tt c}X$ has strata of two types: the products $S\times]0,1[$ with $S \in \mathcal{S}_X$ and the apex $\{\mathtt{v}\}$ of the cone.

(i) For a stratum $S\times]0,1[$, we have

$$(c\sigma)^{-1}(S\times]0,1[) = \{sa+(1-s)\mathtt{v} \mid s\cdot\sigma(a) \in S\times]0,1[\} = \sigma^{-1}(S\times]0,1[)\times]0,1[.$$

If $(c\sigma)^{-1}(S\times]0,1[) \neq \emptyset$, then $\sigma^{-1}(S\times]0,1[)\neq\emptyset$ and therefore

$$\begin{aligned} ||c\sigma||_{S\times]0,1[} &= 1+||\sigma||_{S\times]0,1[}\\ &\leq 1+k-\operatorname{codim}(S\times]0,1[)+\overline{p}(S\times]0,1[)\\ &\leq \dim(\{\mathtt{v}\}*\Delta)-\operatorname{codim}(S\times]0,1[)+\overline{p}(S\times]0,1[).\end{aligned}$$

(ii) For the apex, we have $(c\sigma)^{-1}(\{\mathtt{v}\}) = \{\mathtt{v}\} * \sigma^{-1}(\{\mathtt{v}\})$ which implies

$$||c\sigma||_{\{\mathtt{v}\}} = \begin{cases} 0 & \text{if } \sigma^{-1}(\{\mathtt{v}\}) = \emptyset,\\ 1+||\sigma||_{\{\mathtt{v}\}} & \text{if } \sigma^{-1}(\{\mathtt{v}\})\neq\emptyset.\end{cases}$$

If $\sigma^{-1}(\{\mathtt{v}\})\neq\emptyset$, we get $\|c\sigma\|_{\{\mathtt{v}\}} = 1+\|\sigma\|_{\{\mathtt{v}\}} \leq \dim(\{\mathtt{v}\}*\Delta)-\operatorname{codim}\{\mathtt{v}\}+\overline{p}(\{\mathtt{v}\})$.
If $\sigma^{-1}(\{\mathtt{v}\}) = \emptyset$, the $\overline{p}$-allowability condition relatively to $\{\mathtt{v}\}$ becomes

$$0 = \|c\sigma\|_{\{\mathtt{v}\}} \leq \dim(\mathtt{v}*\Delta)-\operatorname{codim}\{\mathtt{v}\}+\overline{p}(\{\mathtt{v}\}) = (k+1)-(n+1)+\overline{p}(\{\mathtt{v}\}),$$

which is exactly, $k \geq n-\overline{p}(\{\mathtt{v}\})$.

We have proven that $c\sigma$ is a $\overline{p}$-allowable filtered simplex, as required.

• *Let* $0 \neq k \geq n-\overline{p}(\{\mathtt{v}\})$. Given a cycle $\xi \in C_k^{\overline{p}}(X)$ (resp. $\xi' \in \mathfrak{C}_k^{\overline{p}}(X)$), the equality $\partial c\xi = \xi$ (resp. $\mathfrak{d} c\xi' = \xi'$) gives the result.

• *Let* $0 = k \geq n-\overline{p}(\{\mathtt{v}\})$. Given a simplex $\sigma \in C_0^{\overline{p}}(X)$ (resp. $\sigma' \in \mathfrak{C}_0^{\overline{p}}(X)$), the equality $\partial c\sigma = \sigma - \mathtt{v}$ (resp. $\mathfrak{d} c\sigma' = \sigma'$) gives the result. □

Proposition 5.2, Corollary 3.14 and the long exact sequence of a pair imply the following result.

Corollary 5.3. *With the hypotheses and the notations of Proposition 5.2, we have*

$$H_k^{\overline{p}}(\mathring{\mathtt{c}}X, \mathring{\mathtt{c}}X\backslash\{\mathtt{v}\}; G) = \begin{cases} \widetilde{H}_{k-1}^{\overline{p}}(\mathring{\mathtt{c}}X; G) & \text{if } k \geq n+1-\overline{p}(\{\mathtt{v}\}), \\ 0 & \text{if } k \leq n-\overline{p}(\{\mathtt{v}\}), \end{cases}$$

where $\widetilde{H}$ is the reduced homology, and

$$\mathfrak{H}_k^{\overline{p}}(\mathring{\mathtt{c}}X, \mathring{\mathtt{c}}X\backslash\{\mathtt{v}\}; G) = \begin{cases} \mathfrak{H}_{k-1}^{\overline{p}}(\mathring{\mathtt{c}}X; G) & \text{if } k \geq n+1-\overline{p}(\{\mathtt{v}\}), \\ 0 & \text{if } k \leq n-\overline{p}(\{\mathtt{v}\}). \end{cases}$$

Proof of Theorem A. We verify the conditions of Theorem 5.1 for the natural transformation $\Phi\colon H_*^{\overline{p}}(U) \to I^{\overline{p}}H_*(U)$ induced by the canonical inclusion $C_*^{\overline{p}}(U) \hookrightarrow I^{\overline{p}}C_*(U)$.

(a) The Mayer-Vietoris exact sequences have been constructed in Proposition 4.1 for the complex $C_*^{\overline{p}}(X)$ and in [6, Theorem 4.4.19] for the complex $I^{\overline{p}}C_*(X)$.

(b) This a classical argument for homology theories with compact supports.

(c) Let L be a compact filtered space such that the natural inclusion induces the isomorphism

$$\Phi_{(\mathbb{R}^i\times(\mathring{\mathtt{c}}L\backslash\{\mathtt{v}\})}\colon H_*^{\overline{p}}(\mathbb{R}^i\times(\mathring{\mathtt{c}}L\backslash\{\mathtt{v}\})) \xrightarrow{\cong} I^{\overline{p}}H_*(\mathbb{R}^i\times(\mathring{\mathtt{c}}L\backslash\{\mathtt{v}\})).$$

Since $\mathbb{R}^i\times]0,1[\times L = \mathbb{R}^i\times(\mathring{\mathtt{c}}L\backslash\{\mathtt{v}\})$, we get the isomorphism

$$\Phi_{\mathbb{R}^i\times]0,1[\times L}\colon H_*^{\overline{p}}(\mathbb{R}^i\times]0,1[\times L) \xrightarrow{\cong} I^{\overline{p}}H_*(\mathbb{R}^i\times]0,1[\times L).$$

Let us consider the following commutative diagram

$$\begin{tikzcd}
H_*^{\overline{p}}(\mathbb{R}^i\times]0,1[\times L) \arrow[r, "\Phi_{\mathbb{R}^i\times]0,1[\times L}"] \arrow[d, "\mathrm{pr}_*"'] & I^{\overline{p}}H_*(\mathbb{R}^i\times]0,1[\times L) \arrow[d, "\mathrm{pr}_*"] \\
H_*^{\overline{p}}(L) \arrow[r, "\Phi_L"] \arrow[d, "(\iota_{\mathring{\mathtt{c}}L})_*"'] & I^{\overline{p}}H_*(L) \arrow[d, "(\iota_{\mathring{\mathtt{c}}L})_*"] \\
H_*^{\overline{p}}(\mathring{\mathtt{c}}L) \arrow[r, "\Phi_{\mathring{\mathtt{c}}L}"] & I^{\overline{p}}H_*(\mathring{\mathtt{c}}L).
\end{tikzcd}$$

From Corollary 3.14 and [6, Example 4.1.13.], we know that the two maps pr_*, induced by the canonical projections, are isomorphisms. We conclude that Φ_L is an isomorphism.

If $* < n-\overline{p}(\{\mathtt{v}\})$ then Proposition 5.2 and [6, Theorem 4.2.1] imply that the two maps $(\iota_{\mathring{\mathtt{c}}L})_*$ are isomorphisms. So, $\Phi_{\mathring{\mathtt{c}}L}$ is an isomorphism in these degrees.

When $* \geq n-\overline{p}(\{\mathtt{v}\})$, the map $\Phi_{\mathring{\mathtt{c}}L}$ is directly an isomorphism (cf. Proposition 5.2 and [6, Section 5.4]).

(d) The map $\Phi\colon H_*^{\overline{p}}(U;G) \to I^{\overline{p}}H_*(U;G)$ becomes the identity $G \to G$. □

Proof of Theorem B. The same proof works, changing [6, Theorem 4.4.19], [6, Example 4.1.13] and [6, Theorem 4.2.1] by [6, Theorem 6.3.12], [6, Corollary 6.3.8] and [6, Theorem 6.2.13] respectively . □

We end this Section with concrete computations for some particular perversities.

Proposition 5.4. *For any perverse CS set* $(X, \overline{p})$*, we have the following equalities.*

a) $H_*^{\overline{t}}(X;G) = \mathfrak{H}_*^{\overline{t}}(X;G) = H_*(X;G)$*, if* X *is normal,*
b) $H_*^{\overline{p}}(X;G) = H_*(X;G)$*, if* $\overline{p} > \overline{t}$*,*
c) $\mathfrak{H}_*^{\overline{p}}(X;G) = H_*(X,\Sigma;G)$*, if* $\overline{p} > \overline{t}$*,*
d) $H_*^{\overline{p}}(X;G) = \mathfrak{H}_*^{\overline{p}}(X;G) = H_*(X\backslash\Sigma;G)$*, if* $\overline{p} < \overline{0}$*.*

Proof. The argument is similar in the four cases, by applying Theorem 5.1. We only detail the proof of property a). In fact, the only relevant point is the item (iii).

Let $U \subset X$ be an open subset. We use Remark 3.9 and consider the natural transformation $\Phi_U \colon H_*^{\overline{t}}(U) \to H_*(U)$, induced by the canonical inclusion $C_*^{\overline{t}}(U) \hookrightarrow C_*(U)$. In order to verify Property (iii) of Theorem 5.1, we consider a compact stratified space L and the commutative diagram

$$\begin{array}{ccc} H_*^{\overline{t}}(L) & \xrightarrow{\Phi_L} & H_*(L) \\ {\scriptstyle (\iota_{\mathring{c}L})_*}\downarrow & & \downarrow{\scriptstyle (\iota_{\mathring{c}L})_*} \\ H_*^{\overline{t}}(\mathring{c}L) & \xrightarrow{\Phi_{\mathring{c}L}} & H_*(\mathring{c}L). \end{array}$$

where Φ_L is an isomorphism. We have to prove that $\Phi_{\mathring{c}L}$ is an isomorphism. Since $n - \overline{t}(\mathtt{v}) = n - (n-1) = 1$, Proposition 5.2 implies that $H_*^{\overline{t}}(\mathring{c}L) = H_0^{\overline{t}}(\mathring{c}L) = H_0^{\overline{t}}(L)$. From an induction on the depth of X, we get $H_0^{\overline{t}}(L) = H_0(L)$. Since the space X is normal, then L is connected and $\Phi_{\mathring{c}L}$ is an isomorphism. □

Notice that the item c) implies that tame intersection homology is not a topological invariant.

6. Topological invariance. Theorem C

> In this section, we present the construction of the intrinsic CS set associated to a CS set. In Theorem C, we use it to prove a topological invariance of intersection homology for general perversities verifying the conditions of Definition 6.8. A consequence is the topological invariance for general perversities obtained as pull-back's of a perversity defined on the intrinsic CS set. The situation of tame intersection homology is also described.

The intrinsic CS set associated to a CS set is defined by King in [11], crediting the idea to Sullivan. A careful study of this notion has been done by Friedman in [6, Section 2.10].

Definition 6.1. Two points x_0, x_1 of a topological space X are *equivalent* if there exists a homeomorphism $(U_0, x_0) \cong (U_1, x_1)$ between two neighborhoods of x_0 and x_1. This equivalence relation is denoted by $x_0 \sim x_1$.

Two points belonging to the same stratum of a CS set are equivalent. So the equivalence classes are union of strata. The construction of the intrinsic CS set is as follows.

Proposition 6.2. [6, Section 2.10] *Let* X *be a CS set. We denote by* X_i^* *the union of equivalence classes, relatively to* $\sim$*, made up of strata of* X *whose dimension is lower* *than or equal to i. Then, the topological space X endowed with this filtration is a CS set. Moreover, this filtration does not depend on the initial CS structure considered on X.*

Definition 6.3. With the notations of Proposition 6.2, the space X^* is called the *intrinsic CS set* associated to the CS set X and its filtration is the *intrinsic filtration* of X. The identity map $\nu\colon X \to X^*$ is the *intrinsic aggregation.* For sake of simplicity, we denote $\mathcal{S}^* = \mathcal{S}_{X^*}$.

We need to relate the strata of X and X^* in order to compare their homologies. This is the goal of the following definition.

Definition 6.4. Let X be a CS set with intrinsic aggregation $\nu\colon X \to X^*$. A stratum S of X is a *source of the stratum* $T \in \mathcal{S}^*$ if $\nu(S) \subset T$ and $\dim S = \dim T$.

Proposition 6.5. *Let X be a CS set with intrinsic aggregation $\nu\colon X \to X^*$. Then, the following properties hold.*

(a) *The intrinsic aggregation is a stratified map.*
(b) *The union of the sources of a given stratum $T \in \mathcal{S}^*$ is a dense subset of T.*
(c) *The intrinsic filtration of an open subset $U \subset X$ is $U_i^* = X_i^* \cap U$. Let $\iota\colon U \to X$, $j\colon U^* \to X^*$ be the canonical injections and $\nu^U\colon U \to U^*$ the intrinsic aggregation of U. Then, $\nu \circ \iota = j \circ \nu^U$.*

Proof. (a) Let $\nu(x) \in X_i^* \backslash X_{i-1}^*$ be a point of the stratum $T \in \mathcal{S}^*$. There exists a stratum $S \in \mathcal{S}$, with $\dim S = i$, such that $x \sim y$ for each $y \in S$. (If not, from the definition of the filtration of X^*, we would have $x \in X_{i-1}^*$.) Since the strata of X^* are $\sim$-saturated, then $\nu(S) \subset T$.

(b) The set $\nu^{-1}(T) = \cup_{\nu(S)\subset T} S$ is a topological manifold of the same dimension than T. Moreover, the family of strata, $\{S \mid \nu(S) \subset T\}$, is a locally finite family of submanifolds of T. By dimension reasons, the union of the sources of T, $\{S \mid \nu(S) \subset T \text{ and } \dim S = \dim T\}$, is dense in T.

(c) This is true by definition, see [6, Lemma 2.2.10] for more details. ☐

Definition 6.6. Let X be a CS set. Two strata $S, S' \in \mathcal{S}$ are *equivalent* if there exist $x_0 \in S$ and $x_1 \in S'$ with $x_0 \sim x_1$. This equivalence relation is denoted by $S \sim S'$.

Remark 6.7. The original topological invariance of the intersection homology was established by Goresky and MacPherson in [10]. King proposed a different approach in [11] by proving that the intrinsic aggregation $\nu\colon\ X \to X^*$ induces an isomorphism in intersection homology (see also [6, Section 5.5.3]). King did it with slight more general perversities than GM-perversities. He was using loose perversities $\overline{p}$ verifying

(i) $\overline{p} \geq \overline{0}$,
(ii) $\overline{p}(k) \leq \overline{p}(k+1) \leq \overline{p}(k) + 1$ if $k \geq 1$.

We call them *King positive perversities.* If we write them by using strata, the conditions (i) and (ii) are equivalent to

(i') $\overline{p}(S) \geq 0$ if $S \in \mathcal{S}$,
(ii') $\operatorname{codim} S' \leq \operatorname{codim} S \Longrightarrow \overline{p}(S') \leq \overline{p}(S) \leq \overline{p}(S') + 1$ for each singular strata S, S'.

Our next step is to weaken as much as possible these two conditions in order to obtain the broadest family of loose perversities where topological invariance holds. These are the K-perversities we define below.

Observe also that, in the case of a loose perversity, the same perversity is defined on X and X^*. At the opposite, a perversity is defined on the only space X. Thus, the first point is "how can we construct a perversity on X^* from a perversity on X?" This the heuristic justification of the condition (C) in the next definition.

Definition 6.8. A *K-perversity on a CS set* X is a perversity $\overline{p}$ verifying the following properties.

(A) If $S \sim R$ are two strata of $\mathcal{S}$, with S singular and R regular, then $0 \leq \overline{p}(S)$.

(B) If S' is singular and a source stratum then
$$S \preceq S' \text{ and } S \sim S' \Longrightarrow \overline{p}(S') \leq \overline{p}(S) \text{ and } D\overline{p}(S') \leq D\overline{p}(S).$$

(C) If $S, S' \in \mathcal{S}$ are two source strata of the same stratum then $\overline{p}(S) = \overline{p}(S')$.

Moreover, a K^**-perversity* is a K-perversity verifying the following condition.

(D) If $S \sim R$ are two strata of $\mathcal{S}$, with S singular and R regular, then $0 \leq D\overline{p}(S)$.

Remark 6.9. Let's list some basic properties related to this definition.

- A positive King perversity is a K-perversity.
- Condition (D) is satisfied if there is no singular stratum of X becoming regular in X^*.
- A perversity $\overline{p}$ is a K*-perversity if, and only if, $\overline{p}$ and $D\overline{p}$ are K-perversities. In fact, conditions (A) and (D) imply $0 \leq \overline{p}(S) \leq \operatorname{codim} S - 2$, when $S \sim R$ are two strata of $\mathcal{S}$ with S singular and R regular. In this case, we have $\operatorname{codim} S \neq 1$.
- Any perversity $\overline{p}$ on the intrinsic CS set X^* is a K-perversity. The pull-back perversity $\nu^*\overline{p}$ is also a K-perversity.

We show how to construct a perversity on X^* from a K-perversity on X.

Proposition 6.10. *Let X be a CS set with intrinsic aggregation $\nu \colon X \to X^*$ and $\overline{p}$ be a K-perversity on X. Then the following properties are satisfied.*

(a) *The map $\nu_*\overline{p}$ given by*
$$\nu_*\overline{p}(T) = \overline{p}(S),$$
where $S \in \mathcal{S}_X$ is a source of the stratum $T \in \mathcal{S}_{X^*}$, is a perversity on X^*.

(b) *Let $U \subset X$ be an open subset. Denote by $\iota \colon U \to X$, $j \colon U^* \to X^*$ the canonical injections and $\nu^U \colon U \to U^*$ the intrinsic aggregation of U. Then, we have*
$$\nu_*^U \, \iota^* \, \overline{p} = j^* \, \nu_* \, \overline{p}.$$

(c) *The intrinsic aggregation induces a chain map $\nu_* \colon C_*^{\overline{p}}(X;G) \to C_*^{\nu_*\overline{p}}(X^*;G)$. Moreover, if $\overline{p}$ is a K^*-perversity, then it induces a chain map $\nu_* \colon \mathfrak{C}_*^{\overline{p}}(X;G) \to \mathfrak{C}_*^{\nu_*\overline{p}}(X^*;G)$.*

Proof. (a) The equality $\nu_*\overline{p}(T) = \overline{p}(S)$ makes sense since any stratum $T \in \mathcal{S}^*$ has a source, see Proposition 6.5(b). This definition does not depend on the choice of S as condition (C) implies.

(b) Any stratum of U^* is a connected component, ${}_{cc}(T \cap U^*)$, of the intersection of U with a stratum T of X^*. A source of ${}_{cc}(T \cap U^*)$ is a connected component, ${}_{cc}(S \cap U)$, of the intersection of U with a source S of T. So, from item (a) and Definition 2.3, we get
$$\left\{ \begin{array}{l} (j^*\nu_*\overline{p})({}_{cc}(T \cap U^*)) = \nu_*\overline{p}(T) = \overline{p}(S), \\ (\nu_*^U \iota^*\overline{p})({}_{cc}(T \cap U^*)) = \iota^*\overline{p}({}_{cc}(S \cap U)) = \overline{p}(S), \end{array} \right.$$

and $\nu^U_* \, \iota^* \, \overline{p} = j^* \, \nu_* \, \overline{p}$.

(c) Since ν is a stratified map (cf. Proposition 6.5(a)), we can apply Proposition 3.11 (see also Remarks 6.9 and 3.12). Let $T \in \mathcal{S}^*$ be a singular stratum and $S \in \mathcal{S}_X$ be singular and a source stratum of T (cf. Proposition 6.5(b)). From Definition 2.3 and the definition of $\nu_*\overline{p}$, we get

$$\nu^* D\nu_*\overline{p}(S) = D\nu_*\overline{p}(T) = \operatorname{codim} T - 2 - \nu_*\overline{p}(T) \leq \operatorname{codim} S - 2 - \overline{p}(S) = D\overline{p}(S).$$

□

Theorem C. *Let $(X, \overline{p})$ be a perverse CS set with a K-perversity. Then, the intrinsic aggregation $\nu \colon X \to X^*$ verifies the following properties.*

(i) The map ν induces an isomorphism

$$H^{\overline{p}}_*(X; G) \cong H^{\nu_*\overline{p}}_*(X^*; G).$$

(ii) Moreover, if $\overline{p}$ is a K-perversity, then the map ν also induces an isomorphism*

$$\mathfrak{H}^{\overline{p}}_*(X; G) \cong \mathfrak{H}^{\nu_*\overline{p}}_*(X^*; G).$$

This result contains the topological invariance of Goresky and MacPherson, [10, Section 4.1] and the more general version of King [11], (see also [6, Theorems 5.5.1]). Let us see that.

Corollary 6.11. *Let X be a CS set endowed with a King positive perversity, $\overline{p}$. The intrinsic aggregation $\nu \colon X \to X^*$ induces the isomorphism $H^{\overline{p}}_*(X; G) \cong H^{\overline{p}}_*(X^*; G)$. Moreover, if there is no codimension one stratum and $\overline{p}$ is a King perversity with $Dp(1) \geq 0$, then ν induces the isomorphism $\mathfrak{H}^{\overline{p}}_*(X; G) \cong \mathfrak{H}^{\overline{p}}_*(X^*; G)$.*

Proof. From Remark 6.9, the perversity $\overline{p}$ is a K-perversity in the first case and a K*-perversity in the second case. Thus Theorem C gives the result if we prove $\nu_*\overline{p} = \overline{p}$. Let $T \in \mathcal{S}^*$ and S be a source stratum of T. Then, we have, $\nu_*\overline{p}(\operatorname{codim} T) = \nu_*\overline{p}(T) = \overline{p}(S) = \overline{p}(\operatorname{codim} S) = \overline{p}(\operatorname{codim} T)$. □

Remark 6.12. The main theorem of [7] is also a consequence of Theorem C. In this paper, Friedman works with King perversity such that $\overline{p}(2) \geq 0$ and proves the topological invariance of intersection homology under restratifications that fix the singular set.

The topological invariance of the (tame) intersection homology has no precise meaning when we are working with strata-depending perversities. But the refinement invariance is a natural question. It has been treated by G. Valette in the PL framework [20]. We answer this question here for CS sets.

Definition 6.13. A CS set $(X, \mathcal{S})$ is a *refinement* of the CS set $(X, \mathcal{T})$, if any stratum of $\mathcal{T}$ is a union of strata of $\mathcal{S}$. We denote it $(X, \mathcal{S}) \triangleleft (X, \mathcal{T})$ and observe that this relation is equivalent to the fact that the identity map, $\mathrm{id} \colon (X, \mathcal{S}) \to (X, \mathcal{T})$, is stratified.

In particular, any CS set is a refinement of its associated intrinsic CS set.

Corollary 6.14. *Let $(X, \mathcal{S}) \triangleleft (X, \mathcal{T})$ be a refinement between two CS sets. For any K-perversity $\overline{p}$ on $(X, \mathcal{T})$, the identity induces the isomorphism*

$$H^{\mathrm{id}^*\overline{p}}_*(X, \mathcal{S}) \cong H^{\overline{p}}_*(X, \mathcal{T}). \tag{6.1}$$

Moreover, if $\overline{p}$ is a K^*-perversity and there is no codimension one stratum $S \in \mathcal{S}$ becoming regular in $\mathcal{T}$, then the identity map induces an isomorphism

$$\mathfrak{H}_*^{\mathrm{id}^*\overline{p}}(X,\mathcal{S}) \cong \mathfrak{H}_*^{\overline{p}}(X,\mathcal{T}). \tag{6.2}$$

Remark 6.15. In particular, for any perversity $\overline{p}$ on X^*, we have $H_*^{\nu^*\overline{p}}(X;G) \cong H_*^{\overline{p}}(X^*;G)$ and $\mathfrak{H}_*^{\nu^*\overline{p}}(X;G) \cong \mathfrak{H}_*^{\overline{p}}(X^*;G)$, with the previous property on regular strata. Let us also observe that the perversities $\overline{q}$ on X that are obtained as pull-back's of a perversity defined on X^* are characterized by

$$S \sim S' \Rightarrow \overline{q}(S) = \overline{q}(S').$$

Proof of Corollary 6.14. Both CS sets have the same intrinsic space since it does not depend on the stratification. Thus, the two intrinsic aggregations $\nu\colon (X,\mathcal{S}) \to (X^*,\mathcal{S}^*)$ and $\nu'\colon (X,\mathcal{T}) \to (X^*,\mathcal{T}^*) = (X^*,\mathcal{S}^*)$ verify $\nu' \circ \mathrm{id} = \nu$. Theorem C gives the first assertion (6.1) if we prove that

(a) the perversity $\mathrm{id}^*\overline{p}$ is a K-perversity,
(b) and $\nu'_*\overline{p} = \nu_*\mathrm{id}^*\overline{p}$.

Let $S, Q \in \mathcal{S}$. (Recall Definition 6.6 and the notation S^f from Definition 1.6.) We have

$$S \sim Q \Longleftrightarrow S^\nu = Q^\nu \Longleftrightarrow (S^{\mathrm{id}})^{\nu'} = (Q^{\mathrm{id}})^{\nu'} \Longleftrightarrow S^{\mathrm{id}} \sim Q^{\mathrm{id}}, \text{ and}$$

$$S \preceq Q \Longleftrightarrow S \subset \overline{Q} \Longrightarrow S^{\mathrm{id}} \subset \overline{Q^{\mathrm{id}}} \Longleftrightarrow S^{\mathrm{id}} \preceq Q^{\mathrm{id}}.$$

Consider a stratum $T \in \mathcal{S}^*$ and a source stratum $S \in \mathcal{S}$. We prove that S^{id} is a source stratum of T. We have $T = S^\nu$ and $\dim S = \dim T$. This gives $T = (S^{\mathrm{id}})^{\nu'}$. Moreover, since ν' and id are stratified maps, we have $\operatorname{codim} T \leq \operatorname{codim} S^{\mathrm{id}} \leq \operatorname{codim} S = \operatorname{codim} T$. We deduce $\dim T = \dim S^{\mathrm{id}}$.

To get (6.1), we verify the properties (A), (B) and (C) of Definition 6.8 for $\mathrm{id}^*\overline{p}$.

(A) Let $S, R \in \mathcal{S}$ be two strata with $S \sim R$ and R regular. If S^{id} is regular, we have $\mathrm{id}^*\overline{p}(S) = \overline{p}(S^{\mathrm{id}}) = 0$. Suppose now that S^{id} is singular. Since the map id is a stratified map, then R^{id} is a regular stratum. We know that $S^{\mathrm{id}} \sim R^{\mathrm{id}}$. Thus, Property (A) for the perversity $\overline{p}$ gives: $\mathrm{id}^*\overline{p}(S) = \overline{p}(S^{\mathrm{id}}) \geq 0$.

(B) Let $S, S' \in \mathcal{S}$ where S' is singular and a source stratum of $T \in \mathcal{S}^*$. We suppose $S \preceq S'$ and $S \sim S'$. From previous properties, we deduce that S'^{id} is singular and a source stratum of T, with $S^{\mathrm{id}} \preceq S'^{\mathrm{id}}$ and $S^{\mathrm{id}} \sim S'^{\mathrm{id}}$. Thus, Property (B) for the perversity $\overline{p}$ gives $\overline{p}(S'^{\mathrm{id}}) \leq \overline{p}(S^{\mathrm{id}})$ and $D\overline{p}(S'^{\mathrm{id}}) \leq D\overline{p}(S^{\mathrm{id}})$ which is equivalent to

$$\mathrm{id}^*\overline{p}(S') \leq \mathrm{id}^*\overline{p}(S) \text{ and } \mathrm{id}^* D\overline{p}(S') \leq \mathrm{id}^* D\overline{p}(S).$$

It remains to prove that $D\mathrm{id}^*\overline{p}(S') \leq D\mathrm{id}^*\overline{p}(S)$. Let us notice that $\dim S' = \dim T = \dim S'^{\mathrm{id}}$. Since id is a stratified map, we have

$$\begin{aligned} D\mathrm{id}^*\overline{p}(S') &= \operatorname{codim} S' - 2 - \mathrm{id}^*\overline{p}(S') = \operatorname{codim} S'^{\mathrm{id}} - 2 - \overline{p}(S'^{\mathrm{id}}) \\ &= D\overline{p}(S'^{\mathrm{id}}) \leq D\overline{p}(S^{\mathrm{id}}) = \operatorname{codim}(S^{\mathrm{id}}) - \overline{p}(S^{\mathrm{id}}) \\ &\leq \operatorname{codim} S - \mathrm{id}^*\overline{p}(S) = D\mathrm{id}^*\overline{p}(S). \end{aligned}$$

(C) If $S, S' \in \mathcal{T}$ are two source strata of a stratum $T \in \mathcal{S}^*$, then $S^{\mathrm{id}}, S'^{\mathrm{id}} \in \mathcal{S}$ are two source strata of T. Thus using Property (C) for $\overline{p}$ gives

$$\mathrm{id}^*\overline{p}(S) = \overline{p}(S^{\mathrm{id}}) = \overline{p}(S'^{\mathrm{id}}) = \mathrm{id}^*\overline{p}(S').$$

To get (6.2), we have to verify Property (D) of Definition 6.8. Let $S, R \in \mathcal{S}$ be two strata with $S \sim R$, S singular and R regular.

If S^{id} is regular, we have $D\mathrm{id}^*\overline{p}(S) = \overline{t}(S) - \overline{p}(S^{\mathrm{id}}) = \overline{t}(S)$. Since S is singular and S^{id} is regular, we have $\operatorname{codim} S \geq 2$ by hypothesis. Thus, $D\mathrm{id}^*\overline{p}(S) \geq 0$. Suppose now that S^{id} is singular. Since the map id is a stratified map, then R^{id} is a regular stratum and $\operatorname{codim} S^{\mathrm{id}} \leq \operatorname{codim} S$. We know that $S^{\mathrm{id}} \sim R^{\mathrm{id}}$ and $S^{\mathrm{id}} \preceq R^{\mathrm{id}}$. Thus, Property (D) for the perversity $\overline{p}$ implies, $D\mathrm{id}^*\overline{p}(S) = \operatorname{codim} S - \overline{p}(S^{\mathrm{id}}) \geq \operatorname{codim} S^{\mathrm{id}} - \overline{p}(S^{\mathrm{id}}) = D\overline{p}(S^{\mathrm{id}}) \geq 0$. ☐

The proof of Theorem C uses Theorem 5.1. For doing that, we need to make explicit the behavior of a conical neighborhood.

Proposition 6.16. *Let $(X, \overline{p})$ be a perverse CS set where $\overline{p}$ is a* K*-perversity. Consider a stratum $S \in \mathcal{S}$ and (U, φ) a canonical chart of a point $x \in S$. If $\nu \colon X \to X^*$ is the intrinsic aggregation, then the following implication is verified*

$$\nu_* \colon H_*^{\overline{p}}(U \backslash S; G) \xrightarrow{\cong} H_*^{\nu_*\overline{p}}((U \backslash S)^*; G) \Longrightarrow \nu_* \colon H_*^{\overline{p}}(U; G) \xrightarrow{\cong} H_*^{\nu_*\overline{p}}(U^*; G).$$

Moreover, if $\overline{p}$ is a K^*-perversity, we have also

$$\nu_* \colon \mathfrak{H}_*^{\overline{p}}(U \backslash S; G) \xrightarrow{\cong} \mathfrak{H}_*^{\nu_*\overline{p}}((U \backslash S)^*; G) \Longrightarrow \nu_* \colon \mathfrak{H}_*^{\overline{p}}(U; G) \xrightarrow{\cong} \mathfrak{H}_*^{\nu_*\overline{p}}(U^*; G).$$

Proof. From Proposition 6.10, we know that the map ν_* is well defined. We prove the first statement. Let us begin by studying the CS set structures. Without loss of generality, we can suppose $U = \mathbb{R}^k \times \mathring{c}W$, where W is a compact filtered space, $S \cap U = \mathbb{R}^k \times \{\mathtt{w}\}$ and $\mathtt{w}$ is the apex of the cone $\mathring{c}W$. Following [11, Lemma 2 and Proposition 1], there exists a homeomorphism of stratified spaces

$$h \colon (\mathbb{R}^k \times \mathring{c}W)^* \xrightarrow{\cong} \mathbb{R}^m \times \mathring{c}L, \tag{6.3}$$

where L is a compact filtered space (maybe empty) and $m \geq k$. Moreover, the map h verifies,

$$h(\mathbb{R}^k \times \{\mathtt{w}\}) \subset \mathbb{R}^m \times \{\mathtt{v}\} \text{ and } h^{-1}(\mathbb{R}^m \times \{\mathtt{v}\}) = \mathbb{R}^k \times \mathring{c}A, \tag{6.4}$$

where A is an $(m-k-1)$-dimensional sphere and $\mathtt{v}$ the apex of the cone $\mathring{c}L$. Using these notations, the hypothesis and the conclusion of the first statement become

$$h \colon H_*^{\overline{p}}(R^k \times \mathring{c}W \backslash (\mathbb{R}^k \times \{\mathtt{w}\})) \xrightarrow{\cong} H_*^{\nu_*\overline{p}}(\mathbb{R}^m \times \mathring{c}L \backslash h(\mathbb{R}^k \times \{\mathtt{w}\})) \tag{6.5}$$

and

$$h \colon H_*^{\overline{p}}(\mathbb{R}^k \times \mathring{c}W) \xrightarrow{\cong} H_*^{\nu_*\overline{p}}(\mathbb{R}^m \times \mathring{c}L). \tag{6.6}$$

We write $s = \dim W$ and $t = \dim L$. The homeomorphism (6.3) implies $k + s = m + t$ and therefore $s \geq t$ since $m \geq k$.

• *First case. We suppose* $t = -1$. The result is immediate if $s = t = -1$, that is $W = L = \emptyset$. So, we can suppose $s \geq 0$ and $t = -1$, that is $L = \emptyset$ and $W \neq \emptyset$. Under these hypotheses, we have

$$H_*^{\overline{p}}(\mathbb{R}^k \times \mathring{c}W) \cong_{(1)} H_*^{\overline{p}}(\mathbb{R}^k \times \mathring{c}A) \cong_{(2)} H_*^{\overline{p}}(\mathring{c}A) \cong_{(3)} G \cong_{(4)} H_*^{\overline{p}}(\mathbb{R}^m \times \mathring{c}L).$$

Here, $\cong_{(1)}$ is (6.4), $\cong_{(2)}$ comes from Corollary 3.14, $\cong_{(3)}$ from Proposition 5.2 and (A). The last one, $\cong_{(4)}$, is the particular case $t=-1$.

So, for the rest of the proof, we can suppose $t\geq 0$ and $s\geq 0$, that is $W\neq\emptyset$ and $L\neq\emptyset$.

Proposition 6.5 gives a source stratum S_m of $\mathbb{R}^m\times\{\mathtt{v}\}$ verifying: S_m is singular, $\mathbb{R}^k\times\{\mathtt{w}\}\preceq S_m$ and $\mathbb{R}^k\times\{\mathtt{w}\}\sim S_m$. We also have $\nu_*\overline{p}(\mathbb{R}^m\times\{\mathtt{v}\})=\overline{p}(S_m)$. Since $\overline{p}$ is a K-perversity, we deduce

$$\begin{aligned}\overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})-\nu_*\overline{p}(\mathbb{R}^m\times\{\mathtt{v}\}) &= \\ \overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})-\overline{p}(S_m)\leq_{(B)_{\overline{p}}}\operatorname{codim}(\mathbb{R}^k\times\{\mathtt{w}\})-\operatorname{codim}S_m &= \\ \operatorname{codim}(\mathbb{R}^k\times\{\mathtt{w}\})-\operatorname{codim}(\mathbb{R}^k\times\{\mathtt{v}\})=\dim W-\dim L &= s-t.\end{aligned}\tag{6.7}$$

We continue the proof of (6.6) by distinguishing several steps.

• *Second case. We suppose $0\neq i\geq s-\overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})$.*
Corollary 3.14 and Proposition 5.2 imply $H_i^{\overline{p}}(\mathbb{R}^k\times\mathring{\mathtt{c}}W)=0$ and we need to prove $H_i^{\nu_*\overline{p}}(\mathbb{R}^m\times\mathring{\mathtt{c}}L)=0$. We apply (6.7) and obtain

$$i\geq s-\overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})\geq t-\nu_*\overline{p}(\mathbb{R}^m\times\{\mathtt{v}\})=t-\nu_*\overline{p}(\{\mathtt{v}\}).$$

Corollary 3.14 and Proposition 5.2 give $H_i^{\nu_*\overline{p}}(\mathbb{R}^m\times\mathring{\mathtt{c}}L)=0$.

• *Third case. We suppose $0=i\geq s-\overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})$.*
Corollary 3.14 and Proposition 5.2 imply $H_0^{\overline{p}}(\mathbb{R}^k\times\mathring{\mathtt{c}}W;G)=G$. This group is generated by any point of the regular stratum of $\mathbb{R}^k\times\mathring{\mathtt{c}}W$. Let us notice that ν sends this point on a regular stratum of $\mathbb{R}^m\times\mathring{\mathtt{c}}L$. We need to prove that $H_0^{\nu_*\overline{p}}(\mathbb{R}^m\times\mathring{\mathtt{c}}L;G)=G$. We apply (6.7) and we obtain

$$s\leq\overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})\leq\nu_*\overline{p}(\mathbb{R}^m\times\{\mathtt{v}\})+s-t.$$

We get $t\leq\nu_*\overline{p}(\mathbb{R}^m\times\{\mathtt{v}\})$. Thus, Corollary 3.14 and Proposition 5.2 give the result.

• *Fourth case. We suppose $i<s-\overline{p}(\mathbb{R}^k\times\{\mathtt{w}\})$.*
We have the following isomorphisms,

$$\begin{aligned}H_i^{\nu_*\overline{p}}(\mathbb{R}^m\times\mathring{\mathtt{c}}L\backslash h(\mathbb{R}^k\times\{\mathtt{w}\})) &\cong_{(1)} H_i^{\overline{p}}(\mathbb{R}^k\times\mathring{\mathtt{c}}W\backslash(\mathbb{R}^k\times\{\mathtt{w}\}))\\ &\cong H_i^{\overline{p}}(\mathbb{R}^k\times(\mathring{\mathtt{c}}W\backslash\{\mathtt{w}\}))\\ &\cong H_i^{\overline{p}}(\mathbb{R}^k\times]0,1[\times W)\\ &\cong_{(2)} H_i^{\overline{p}}(W),\end{aligned}\tag{6.8}$$

where $\cong_{(1)}$ is the hypothesis (6.5) and $\cong_{(2)}$ is Corollary 3.14. Consider $h(\mathbb{R}^k\times\{\mathtt{w}\})=B\times\{\mathtt{v}\}\subset\mathbb{R}^m\times\{\mathtt{v}\}$ with B a closed subset. Using the excision property (cf. Corollary 4.5) we get the following isomorphisms

$$\begin{aligned} H_i^{\nu_*\overline{p}}(\mathbb{R}^m \times \mathring{c}L\backslash h(\mathbb{R}^k \times \{\mathtt{w}\}), \mathbb{R}^m \times \mathring{c}L\backslash \mathbb{R}^m \times \{\mathtt{v}\}) \cong \\ H_i^{\nu_*\overline{p}}((\mathbb{R}^m \times \mathring{c}L)\backslash(B \times \{\mathtt{v}\}), \mathbb{R}^m \times (\mathring{c}L\backslash\{\mathtt{v}\})) \cong_{(1)} \\ H_i^{\nu_*\overline{p}}((\mathbb{R}^m\backslash B) \times \mathring{c}L, (\mathbb{R}^m\backslash B) \times (\mathring{c}L\backslash\{\mathtt{v}\})) \cong \\ H_i^{\nu_*\overline{p}}((\mathbb{R}^m\backslash B) \times (\mathring{c}L, \mathring{c}L\backslash\{\mathtt{v}\})) \cong_{(2)} \\ H_i^{\nu_*\overline{p}}(\mathbb{R}^{k+1} \times A \times (\mathring{c}L, \mathring{c}L\backslash\{\mathtt{v}\})) \cong_{(3)} \\ H_i^{\nu_*\overline{p}}(\mathring{c}L, \mathring{c}L\backslash\{\mathtt{v}\}) \oplus H_{i-m+1+k}^{\nu_*\overline{p}}(\mathring{c}L, \mathring{c}L\backslash\{\mathtt{v}\}), \end{aligned} \tag{6.9}$$

where $\cong_{(1)}$ is the excision of $B \times (\mathring{c}L\backslash\{\mathtt{v}\})$, $\cong_{(2)}$ comes from (6.4) and $\cong_{(3)}$ from Corollaires 4.7 and 3.14. Since $i < s - \overline{p}(\mathbb{R}^k \times \{\mathtt{w}\})$, we have

$$i - m + 1 + k < s - \overline{p}(\mathbb{R}^k \times \{\mathtt{w}\}) - m + 1 + k = t - \overline{p}(\mathbb{R}^k \times \{\mathtt{w}\}) + 1.$$

Proposition 6.5 gives a source stratum S_m of $\mathbb{R}^m \times \{\mathtt{v}\}$ verifying $\mathbb{R}^k \times \{\mathtt{w}\} \preceq S_m$ and $\mathbb{R}^k \times \{\mathtt{w}\} \sim S_m$. We also have $\nu_*\overline{p}(\mathbb{R}^m \times \{\mathtt{v}\}) = \overline{p}(S_m)$. Since $\overline{p}$ is a K-perversity, then

$$0 \leq \overline{p}(\mathbb{R}^k \times \{\mathtt{w}\}) - \overline{p}(S_m) = \overline{p}(\mathbb{R}^k \times \{\mathtt{w}\}) - \nu_*\overline{p}(\mathbb{R}^m \times \{\mathtt{v}\}).$$

We get $i - m + 1 + k \leq t - \nu_*\overline{p}(\mathbb{R}^m \times \{\mathtt{v}\})$. Since $L \neq \emptyset$, using Corollary 5.3, we conclude that the second term of the direct sum (6.9) vanishes. We have proven

$$H_i^{\nu_*\overline{p}}(\mathbb{R}^m \times \mathring{c}L\backslash h(\mathbb{R}^k \times \{\mathtt{w}\}), \mathbb{R}^m \times \mathring{c}L\backslash \mathbb{R}^m \times \{\mathtt{v}\}) \cong H_i^{\nu_*\overline{p}}(\mathring{c}L, \mathring{c}L\backslash\{\mathtt{v}\}).$$

Finally, using this isomorphism, the long exact sequence of a pair (4.9) and (6.8), we conclude $H_i^{\overline{p}}(W) \cong H_i^{\nu_*\overline{p}}(\mathring{c}L)$ which gives, using Corollary 3.14 and Proposition 5.2,

$$H_i^{\nu_*\overline{p}}(\mathbb{R}^m \times \mathring{c}L) \cong H_i^{\nu_*\overline{p}}(\mathring{c}L) \cong H_i^{\overline{p}}(W) \cong H_i^{\overline{p}}(\mathring{c}W) \cong H_i^{\overline{p}}(\mathbb{R}^k \times \mathring{c}W).$$

This implies the first statement.

The proof of the second statement follows the same process, except for the intersection homology of a cone in degree 0. We distinguish the same cases.

- *First case.* The same proof works, by replacing (A) by (D).
- *Second case.* This is the same proof.
- *Third case.* Since $0 = i \geq s - \overline{p}(\mathbb{R}^k \times \{\mathtt{w}\})$ and $0 = i \geq t - \nu_*\overline{p}(\mathbb{R}^m \times \{\mathtt{v}\})$, we get $\mathfrak{H}_0^{\overline{p}}(W) = 0 = \mathfrak{H}_0^{\nu_*\overline{p}}(L)$ from Proposition 5.2.
- *Fourth case.* This is the same proof. □

Proof of Theorem C. The proof is reduced to the verification of the hypotheses of Theorem 5.1, where $\Phi_U \colon H_*^{\overline{p}}(U) \to H_*^{\nu_*\overline{p}}(U^*)$ (resp. $\Phi_U \colon \mathfrak{H}_*^{\overline{p}}(U) \to \mathfrak{H}_*^{\nu_*\overline{p}}(U^*)$) is the natural transformation induced by the intrinsic aggregation $\nu^U : U \to U^*$, see Proposition 6.5(c) and Proposition 6.10. First, conditions (ii) and (iv) of Theorem 5.1 are direct. Condition (i) comes from Proposition 4.1 and condition (iii) is exactly Proposition 6.16. □

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Lafma, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens Cedex 1, France
E-mail address: `David.Chataur@u-picardie.fr`

Laboratoire de Mathématiques de Lens EA 2462, Fédération CNRS Nord-Pas-de-Calais FR2956, Université d'Artois, 62307 Lens Cedex, France
E-mail address: `martin.saraleguiaranguren@univ-artois.fr`

Département de Mathématiques, UMR 8524, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France
E-mail address: `Daniel.Tanre@univ-lille1.fr`