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Abstract. *We give general estimates for the approximation numbers of composition operators on the Hardy space on the ball B_d and the polydisk $\mathbb{D}^d$.*

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1 Introduction

This work is an attempt to investigate approximation numbers of composition operators on the Hardy space $H^2(\Omega)$ where Ω is an open subset of $\mathbb{C}^d$, i.e. when we work with d complex variables instead of one. In fact, we will essentially consider the two cases when $\Omega = B_d$ is the unit ball of $\mathbb{C}^d$ endowed with its usual hermitian norm $\|z\| = \big(\sum_{j=1}^{d} |z_j|^2\big)^{1/2}$ and $\Omega = \mathbb{D}^d$ is the unit ball of $\mathbb{C}^d$ endowed with the sup-norm $\|z\|_\infty = \sup_{j=1}^{d} |z_j|$, that is when Ω is the unit polydisk of $\mathbb{C}^d$. In order to treat these two cases jointly, we will work in the setting of bounded symmetric domains.

An interesting feature is that the rate of decay of approximation numbers *highly depends on d*, becoming slower and slower as d increases, which might lead to think that no compact composition operators exist for truly infinite-dimensional symbols. We will see in the forthcoming paper [17] that this is not the case.

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2 Notations and background

A *bounded symmetric domain* of $\mathbb{C}^d$ is an open convex and circled subset Ω of $\mathbb{C}^d$ such that for every point $a \in \Omega$, there is an involutive bi-holomorphic map $u\colon \Omega \to \Omega$ such that a is an isolated fixed point of σ (equivalently, $u(a) = a$ and $u'(a) = -id$ (see [21], Proposition 3.1.1). É. Cartan showed that every bounded symmetric domain of $\mathbb{C}^d$ is homogeneous, i.e. the group of automorphisms of Ω acts transitively on Ω: for every $a, b \in \Omega$, there is an automorphism u of Ω such that $u(a) = b$ (see [21], p. 250). The unit ball B_d and the polydisk $\mathbb{D}^d$ are examples of bounded symmetric domains.

The Shilov boundary S_Ω of such a domain Ω is the smallest closed set $F \subseteq \partial\Omega$ such that $\sup_{z\in\overline{\Omega}} |f(z)| = \sup_{z\in F} |f(z)|$ for every function f holomorphic in some neighborhood of $\overline{\Omega}$. For example, the Shilov boundary of the bidisk is $S_{\mathbb{D}^2} = \{(z_1, z_2) \in \mathbb{C}^2\,;\ |z_1| = |z_2| = 1\}$, whereas, its usual boundary $\partial\mathbb{D}^2$ is $\{(z_1, z_2) \in \mathbb{C}^2\,;\ |z_1|, |z_2| \leq 1 \text{ and } |z_1| = 1 \text{ or } |z_2| = 1\}$; for the unit ball B_d, the Shilov boundary is equal to the usual boundary $\mathbb{S}^{d-1}$ ([7], § 4.1). Equivalently (see [7], Theorem 4.2), S_Ω is the set of the extreme points of the convex set $\overline{\Omega}$.

If σ is the unique probability measure on S_Ω invariant by the automorphisms u of Ω such that $u(0) = 0$, the *Hardy space* $H^2(\Omega)$ is the space of all complex-valued holomorphic functions f on Ω such that:

$$\|f\|_{H^2(\Omega)} := \left(\sup_{0<r<1} \int_{S_\Omega} |f(r\xi)|^2\, d\sigma(\xi)\right)^{1/2}$$

(see [11]). It is a Hilbert space (see [10]).

A Schur map, associated with Ω, will be a *non-constant* analytic self-map of Ω into itself. It will be called *truly d-dimensional* if the differential $\varphi'(a)\colon \mathbb{C}^d \to \mathbb{C}^d$ is an invertible linear map for at least one point $a \in \Omega$. Then, by the implicit function Theorem, $\varphi(\Omega)$ has non-void interior. We say that the Schur map φ is a *symbol* if it defines a *bounded* composition operator $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$ by $C_\varphi(f) = f \circ \varphi$.

Let us recall that if any Schur function generates a bounded composition operator on $H^2(\mathbb{D}^d)$ when $d = 1$, this is no longer the case as soon as $d \geq 2$, as shown for example by the Schur map $\varphi(z_1, z_2) = (z_1, z_1)$. Indeed, if say $d = 2$, taking $f(z) = (z_1 + z_2)^n$, we see that

$$\|f\|_2^2 = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}\,,$$

while:

$$\|C_\varphi f\|_2 = \|(2z_1)^n\|_2 = 2^n\,.$$

The same phenomenon occurs on $H^2(B_d)$ ([18]; see also [4] and [5]).

If H is a Hilbert space and $T\colon H \to H$ is a bounded linear operator, the *approximation numbers* of T are defined, for $n \geq 1$ by:

$$a_n(T) = \inf_{\operatorname{rank} R<n} \|T - R\|\,. \tag{2.1}$$

One has $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq a_{n+1}(T) \geq \cdots$, and T is compact if and only if $a_n(T) \underset{n\to\infty}{\longrightarrow} 0$.

The approximation numbers have (obviously) the following ideal property: for every bounded linear operators $S, U \colon H \to H$, one has:

$$a_n(STU) \leq \|S\| \, \|U\| \, a_n(T) \, , \qquad n = 1, 2 \ldots .$$

For an operator $T \colon H^2(\Omega) \to H^2(\Omega)$ with approximation numbers $a_n(T) = a_n$, we will introduce the non-negative numbers $0 \leq \gamma_d^-(T) \leq \gamma_d^+(T) \leq \infty$ defined by:

$$\gamma_d^-(T) = \liminf_{n\to\infty} \frac{\log 1/a_n}{n^{1/d}} \quad \text{and} \quad \gamma_d^+(T) = \limsup_{n\to\infty} \frac{\log 1/a_n}{n^{1/d}} \, \cdot \tag{2.2}$$

The relevance of those parameters to the decay of approximation numbers is indicated by the following obvious facts, in which $0 < c \leq C < \infty$ denote constants independent of n:

$$\gamma_d^-(T) > 0 \quad \Longleftrightarrow \quad a_n \leq C \, \mathrm{e}^{-c\, n^{1/d}} \, , \quad n = 1, 2, \ldots \tag{2.3}$$

$$\gamma_d^+(T) < \infty \quad \Longleftrightarrow \quad a_n \geq c \, \mathrm{e}^{-C n^{1/d}} \, , \quad n = 1, 2, \ldots . \tag{2.4}$$

So, the positivity of $\gamma_d^-(T)$ indicates that a_n is "small" and the finiteness of $\gamma_d^+(T)$ indicates that a_n is "big".

As usual, the notation $A \lesssim B$ means that there is a constant c such that $A \leq c\, B$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

3 Lower bound

The next theorem shows that the approximation numbers of composition operators cannot be very small. We have already seen that in the one-dimensional case in [14]. The important fact here is that this lower bound depend highly of the dimension.

Theorem 3.1 *Let Ω be a bounded symmetric domain of $\mathbb{C}^d$ and $\varphi \colon \Omega \to \Omega$ be a truly d-dimensional Schur map inducing a compact composition operator $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$. Then, for some constants $0 < c \leq C < \infty$, independent of n, we have:*

$$a_n(C_\varphi) \geq c \, \mathrm{e}^{-C n^{1/d}} , \qquad \forall n \geq 1,$$

that is

$$\gamma_d^+(C_\varphi) < \infty \, .$$

For proving that, we shall use the following results, the first of which is due to D. Clahane [6], Theorem 2.1 (and to B. MacCluer [18] in the particular case of the unit ball B_d).

Theorem 3.2 (D. Clahane) *Let Ω be a bounded symmetric domain of $\mathbb{C}^d$ and $\varphi\colon \Omega \to \Omega$ be a holomorphic map inducing a compact composition operator $C_\varphi\colon H^2(\Omega) \to H^2(\Omega)$. Then φ has a unique fixed point $z_0 \in \Omega$ and the spectrum of C_φ consists of 0, and all possible products of eigenvalues of the derivative $\varphi'(z_0)$.*

When φ is truly d-dimensional, 0 cannot be an eigenvalue of C_φ since if $f \circ \varphi = 0$, then f vanishes on $\varphi(\Omega)$ which have a non-void interior, and hence $f \equiv 0$. Note that 1 is an eigenvalue, by taking the product of zero eigenvalue of $\varphi'(z_0)$.

In fact, in our case, we will not need the existence of z_0, for we will force 0 to be a fixed point by a harmless change of the symbol φ.

Lemma 3.3 *Let H be a complex Hilbert space and $T\colon H \to H$ be a compact operator with eigenvalues $\lambda_1, \ldots, \lambda_n, \ldots$, written in non-increasing order and with singular values a_n, $n = 1, 2, \ldots$. Then:*

$$|\lambda_{2n}|^2 \leq a_1\, a_n\,. \tag{3.1}$$

Indeed, it suffices to apply an immediate consequence of Weyl's inequalities, namely $|\lambda_n| \leq (a_1 \cdots a_n)^{1/n}$, with n changed into $2n$, and square to get

$$|\lambda_{2n}|^2 \leq (a_1 \cdots a_{2n})^{1/n} \leq (a_1^n\, a_n^n)^{1/n} = a_1\, a_n\,.$$

Lemma 3.4 *Let N_p be the number of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq p$. Then, as p goes to infinity:*

$$N_p \sim \frac{p^d}{d!}\,. \tag{3.2}$$

Proof. Let n_k be the number of multi-indices $(\alpha_1, \ldots, \alpha_d, \alpha_{d+1})$ such that $\alpha_1 + \cdots + \alpha_d + \alpha_{d+1} = k$. We have (see [13], page 498), classically, for $|t| < 1$:

$$\sum_{p=0}^{\infty} n_p t^p = \Big(\sum_{\alpha_1=0}^{\infty} t^{\alpha_1}\Big) \cdots \Big(\sum_{\alpha_{d+1}}^{\infty} t^{\alpha_{d+1}}\Big) = \Big(\sum_{k=0}^{\infty} t^k\Big)^{d+1} = \frac{1}{(1-t)^{d+1}}\,;$$

hence

$$n_p = \binom{d+p}{p}.$$

But $N_p = n_p$, and hence:

$$N_p = \frac{(d+1)\cdots(d+p)}{p!} = \frac{(d+p)!}{p!\,d!} \sim \frac{p^d}{d!}\,,$$

by Stirling's formula for example. $\square$

Claim 3.5 *We may assume that $\varphi(0) = 0$ and $\varphi'(0)$ is invertible.*

Proof. Since φ is truly d-dimensional, there exists $a \in \Omega$ such that $\varphi'(a)$ is invertible. Since Ω is homogeneous, there exist two automorphisms Φ_a and $\Phi_{\varphi(a)}$ of Ω such that $\Phi_a(0) = a$ and $\Phi_{\varphi(a)}[\varphi(a)] = 0$. Set $\psi = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a$. Then $\psi(0) = 0$. Now, every analytic automorphism Φ of Ω induces a bounded composition operator on $H^2(\Omega)$ and $C_\Phi^{-1} = C_{\Phi^{-1}}$ ([6], Theorem 3.1); hence we can write $C_\psi = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}$ and it follows that C_ψ, as C_φ, is compact. The ideal property of approximation numbers implies that, for $n = 1, 2, \ldots$, one has:

$$\big(\|C_{\Phi_a}\| \, \|C_{\Phi_{\varphi(a)}}\|\big)^{-1} a_n(C_\varphi) \leq a_n(C_\psi) \leq \|C_{\Phi_a}\| \, \|C_{\Phi_{\varphi(a)}}\| \, a_n(C_\varphi) \,,$$

so $\gamma_d^-(C_\psi) = \gamma_d^-(C_\varphi)$. Moreover, using the chain rule, we see that $\psi'(0)$ is invertible, since $\varphi'(a)$ is. $\square$

Proof of Theorem 3.1. Let $\mu_1, \ldots, \mu_d$ be the eigenvalues of $\varphi'(0)$ and set $\min_{1 \leq j \leq d} |\mu_j| = \mathrm{e}^{-A} > 0$. By Theorem 3.2, the eigenvalues $\lambda_1, \ldots, \lambda_n, \ldots$ of C_φ are the numbers $\mu_1^{\alpha_1} \cdots \mu_d^{\alpha_d}$ rearranged in non-increasing order. By definition, we have $\lambda_{N_p} = \prod_{j=1}^d \mu_j^{\alpha_j}$ for some d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $|\alpha| \leq p$. Therefore, $|\lambda_{N_p}| \geq \mathrm{e}^{-A|\alpha|} \geq \mathrm{e}^{-Ap}$. If $M_p = [N_p]/2$ where $[\,.\,]$ stands for the integer part, equation (3.1) gives:

$$\mathrm{e}^{-2Ap} \leq |\lambda_{N_p}|^2 \leq |\lambda_{2M_p}|^2 \leq a_1 \, a_{M_p} .$$

Since $M_p \sim C_d \, p^d$ in view of Lemma 3.4, inverting this relation and using the monotonicity of the a_n's clearly gives the claimed result. $\square$

4 An alternative approach for the polydisk and the unit ball

The previous proof of Theorem 3.1 is essentially a "functional analysis" one. It is interesting to give a proof using complex analysis tools instead of functional analysis ones. Moreover, this approach will be useful for the example in Section 6.

In the general case, we are not be able to do that, and we only do it for the polydisk. The same approach works for the unit ball, by using results of B. Berndtsson in [2]. To save notation, we will give the proof in the case $d = 2$ but it clearly works in any dimension d. We will make use of the following theorem of P. Beurling ([9] p. 285), in which the word *interpolation sequence* refers to the space H^∞ of bounded analytic functions on Ω ($\Omega = \mathbb{D}$ or $\mathbb{D}^2$), the interpolation constant M_S of the sequence $S = (s_j)$ being the smallest number M such that, for any sequence (w_j) of data satisfying $\sup |w_j| \leq 1$, there exists $f \in H^\infty(\Omega)$ such that $f(s_j) = w_j$ and $\|f\|_\infty \leq M$.

Theorem 4.1 (P. Beurling) *Let (z_j) be an interpolating sequence in the unit disk $\mathbb{D}$, with interpolation constant M. Then, there exist analytic functions f_j, $j \geq 1$, on $\mathbb{D}$ such that:*

$$f_j(z_k) = \delta_{j,k} \qquad \text{and} \qquad \sum_{j=1}^{\infty} |f_j(z)| \leq M\,, \quad \forall z \in \mathbb{D}\,.$$

As a consequence, if $A = (a_j)$ and $B = (b_k)$ are interpolation sequences of $\mathbb{D}$ with respective interpolation constants M_A and M_B, their "cartesian product" $(p_{j,k})_{j,k} = \big((a_j, b_k)\big)_{j,k}$ is an interpolation sequence, with respect to $H^\infty(\mathbb{D}^2)$, with interpolation constant $\leq M_A\, M_B$.

The consequence was observed in the paper [3]. Indeed, if (f_j) and (g_k) are P. Beurling's functions associated to A and B respectively, any sequence $(w_{j,k})$ with $\sup_{j,k} |w_{j,k}| \leq 1$ can be interpolated by the bounded analytic function

$$f(z,w) = \sum_{j,k\geq 1} w_{j,k} f_j(z)\, g_k(w)$$

which satisfies $\|f\|_\infty \leq M_A\, M_B$.

Alternatively, in the sequel, we might use the result of [3] on the sufficiency of Carleson's condition on products of Gleason distances in the case of several variables. But we will stick to the previous approach. We now make use of the following lemma of [15] which was enunciated in the one-dimensional case, but whose proof works word for word in our new setting; indeed, the space of multipliers of $H^2(\mathbb{D}^2)$ is (isometrically) $H^\infty(\mathbb{D}^2)$ and then one shows that the unconditionality constant of the sequence $(K_{s_j})_{1\leq j\leq n}$ of reproducing kernels associated to a finite sequence $S = (s_j)_{1\leq j\leq n}$ is less than M_u (see also [14]). Also note that he reproducing kernel of $H^2(\mathbb{D}^2)$ is now, for $a = (a_1, a_2) \in \mathbb{D}^2$:

$$K_a(z_1, z_2) = \frac{1}{(1 - \overline{a_1} z_1)(1 - \overline{a_2} z_2)}\,,$$

with $\|K_a\|^2 = [(1 - |a_1|^2)(1 - |a_2|^2)]^{-1}$.

Lemma 4.2 *Let $\varphi\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a symbol inducing a compact composition operator $C_\varphi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$. Let $u = (u_1, \ldots, u_N)$ be a finite sequence of distinct points of $\mathbb{D}^2$ with interpolation constant M_u and let $v_j = \varphi(u_j)$, $1 \leq j \leq N$. Let M_v be the interpolation constant of $v = (v_1, \ldots, v_N)$. Then, setting:*

$$\mu_N^2 = \inf_{1\leq j\leq N} \frac{|K_{v_j}|^2}{|K_{u_j}|^2} = \inf_{1\leq j\leq N} \frac{(1 - |u_{j,1}|^2)(1 - |u_{j,2}|^2)}{(1 - |v_{j,1}|^2)(1 - |v_{j,2}|^2)}\,,$$

with $u_j = (u_{j,1}, u_{j,2})$ and $v_j = (v_{j,1}, v_{j,2})$, one has:

$$a_N(C_\varphi) \geq c'\, \mu_N\, M_u^{-1}\, M_v^{-1} \geq c'\, \mu_N\, M_v^{-2}\,. \tag{4.1}$$

The last inequality $M_u \leq M_v$ is proved as follows: let $\sup|w_j| \leq 1$ and choose $f \in H^\infty$ such that $f(v_j) = w_j$ and $\|f\|_\infty \leq M_v$; then $g = f \circ \varphi \in H^\infty$ and satisfies $\|g\|_\infty \leq M_v$ and $g(u_j) = f(v_j) = w_j$. $\square$

It remains to choose u and v and to estimate the parameters of the lemma.

As in the first proof, we may assume that $\varphi(0) = 0$ and that the differential $\varphi'(0)$ is invertible.

Since $\varphi'(0)$ is invertible, the set $\varphi(\mathbb{D}^2)$ contains a closed polydisk of radius $0 < r < 1$ with center 0. We then take for v the sequence $v_{j,k} = (r\omega^j, r\omega^k)$ where ω is a primitive nth-root of unity, e.g. $\omega = \mathrm{e}^{2i\pi/n}$. We have $v = A \times A$ where $A = (r\omega, r\omega^2, \ldots, r\omega^n)$ so that the sequence v has length $N = n^2$. We know ([9], p. 284) that $M_A = r^{1-n}$, so that Theorem 4.1 gives us $M_v \leq r^{2-2n}$. We now write $v_j = \varphi(u_j)$ with $|u_j| \leq r$, which is always possible by decreasing r if necessary (this r can be ridiculously small, but remains positive). Finally,

$$\frac{\|K_{v_j}\|^2}{\|K_{u_j}\|^2} \geq (1 - |u_{j,1}|^2)(1 - |u_{j,2}|^2) \geq (1 - r^2)^2 \,.$$

Collecting all those estimates and using (4.1), we obtain:

$$a_{n^2}(C_\varphi) \geq (1 - r^2)^2 \, r^{4n-4} \geq c \, r^{4n} \,.$$

Interpolating an arbitrary integer m between two consecutive squares, we clearly obtain Theorem 3.1 for $\mathbb{D}^2$ (note that in dimension d a factor $(1 - r^2)^d$ instead of $(1 - r^2)^2$ shows up). $\square$

5 An upper bound

Though the result of this section is undoubtedly true in the general setting of bounded symmetric domains, we are not familiar enough with complex analysis in several variables to work it out. Therefore, we will assume in this section that:

$$\Omega = B_{l_1} \times \cdots \times B_{l_N} \,, \quad \text{with } l_1 + \cdots + l_N = d \tag{5.1}$$

is the product of N unit balls. That covers the case of the unit ball of $\mathbb{C}^d$ ($N = 1$) and the case of the polydisk of $\mathbb{C}^d$ ($N = d$ and $l_1 = \cdots = l_N = 1$). To save notations, we will assume in the sequel with $N = 2$.

A point $z = (z_j)_{1 \leq j \leq d} \in \Omega$ is of the form $z = (u, v)$ with $u = (u_j)_{1 \leq j \leq l_1}$, $v = (v_j)_{l_1 < j \leq d}$ and $\sum_{j=1}^{l_1} |u_j|^2 < 1$, $\sum_{j=l_1+1}^{d} |v_j|^2 < 1$. We see that Ω is the unit ball of $\mathbb{C}^d$ equipped with the following norm:

$$|||z||| = \max \left[\Big(\sum_{j=1}^{l_1} |u_j|^2 \Big)^{1/2}, \Big(\sum_{j=l_1+1}^{d} |v_j|^2 \Big)^{1/2} \right], \tag{5.2}$$

where $z = (u, v)$ with $u \in \mathbb{C}^{l_1}$ and $v \in \mathbb{C}^{l_2}$.

The Shilov boundary of Ω is $S_\Omega = S_{l_1} \times S_{l_2}$ and the normalized invariant measure on S_Ω is $\sigma = \sigma_{l_1} \otimes \sigma_{l_2}$ where σ_{l_1} and σ_{l_2} denote respectively the area measure on the hermitian spheres S_{l_1} and S_{l_2}.

The following is in Rudin ([19] p. 16).

Lemma 5.1 *The monomials e_α, with $e_\alpha(z) = z^\alpha$, form an orthogonal basis of $H^2(\Omega)$. Moreover if $\alpha = (\beta, \gamma)$ with $\beta = (\alpha_1, \ldots, \alpha_{l_1})$ and $\gamma = (\alpha_{l_1+1}, \ldots, \alpha_d)$, then writing $z = (u, v)$ we have:*

$$\|e_\alpha\|^2 = \int_{S_{l_1} \times S_{l_2}} |u^\beta|^2 \, |v^\gamma|^2 \, d\sigma_{l_1}(u) \, d\sigma_{l_2}(v) = \frac{(l_1 - 1)! \, \beta!}{(l_1 - 1 + |\beta|)!} \, \frac{(l_2 - 1)! \, \gamma!}{(l_2 - 1 + |\gamma|)!} \, \cdot$$

Therefore, if $f = \sum_\alpha c_\alpha \, e_\alpha \in H^2(\Omega)$, one has:

$$\|f\|^2 = \sum_\alpha |c_\alpha|^2 \frac{(l_1 - 1)! \, \beta!}{(l_1 - 1 + |\beta|)!} \, \frac{(l_2 - 1)! \, \gamma!}{(l_2 - 1 + |\gamma|)!} \, \cdot$$

We can now state the main result of that section, in which we set $\|\varphi\|_\infty := \sup_{z \in \Omega} |||\varphi(z)|||$.

Theorem 5.2 *Let $\Omega = B_{l_1} \times B_{l_2}$, $d = l_1 + l_2$, and $\varphi \colon \Omega \to \Omega$ be a truly d-dimensional Schur map, inducing a compact composition operator $C_\varphi \colon H^2(\Omega) \to H^2(\Omega)$. Then, if $\|\varphi\|_\infty < 1$, one has $\gamma_d^-(C_\varphi) > 0$, that is there exist some constants $0 < c \leq C < \infty$, independent of n, such that:*

$$a_n(C_\varphi) \leq C \, \mathrm{e}^{-c n^{1/d}} \, , \qquad n = 1, 2, \ldots . \tag{5.3}$$

Proof. Let us set $r = \|\varphi\|_\infty < 1$. Let $f = \sum c_\alpha \, e_\alpha \in H^2(\Omega)$ with

$$c_\alpha = \widehat{f}(\alpha) \text{ and } \|f\|^2 = \sum_\alpha |c_\alpha|^2 \|e_\alpha\|^2 \leq 1 \, . \tag{5.4}$$

Then $C_\varphi f = \sum c_\alpha \varphi^\alpha$.

We approximate C_φ by the N_n-rank operator R defined by

$$Rf = \sum_{|\alpha| \leq n} c_\alpha \varphi^\alpha$$

and we set $g = C_\varphi(f) - R(f)$ as well as $\alpha = (\beta, \gamma)$ and $z = (u, v)$. We begin with observing that $\frac{(l_1 - 1 + p)!}{(l_1 - 1)! p!} \leq (p + 1)^{l_1 - 1}$ and $\frac{(l_2 - 1 + q)!}{(l_2 - 1)! q!} \leq (q + 1)^{l_2 - 1}$. Since $|c_\alpha| \leq \|e_\alpha\|^{-1}$, we get by Lemma 5.1 and the multinomial formula:

$$\sum_{|\beta| = p} \frac{p!}{\beta!} \, |\varphi^\beta(u)|^2 = \Bigg(\sum_{j=1}^{l_1} |\varphi_j(u)|^2 \Bigg)^p \tag{5.5}$$

and a similar formula with $|\gamma| = q$ that, setting $p + q = N$:

$$\sum_{\substack{|\beta|=p \\ |\gamma|=q}} \|e_\alpha\|^{-2} |\varphi^\alpha(z)|^2 = \sum_{\substack{|\beta|=p \\ |\gamma|=q}} \frac{(l_1 - 1 + p)!}{\beta!(l_1 - 1)!} \frac{(l_2 - 1 + q)!}{\gamma!(l_2 - 1)!} |\varphi^\beta(u)|^2 \, |\varphi^\gamma(v)|^2$$

$$\leq (p+1)^{l_1 - 1} (q+1)^{l_2 - 1} \bigg(\sum_{j=1}^{l_1} |\varphi_j(u)|^2 \bigg)^p \bigg(\sum_{j=l_1+1}^{d} |\varphi_j(v)|^2 \bigg)^q$$

$$\leq (p+1)^{l_1 - 1} (q+1)^{l_2 - 1} \, r^{2p} \, r^{2q} \leq (N+1)^{l_1 + l_2 - 2} \, r^{2N} \, .$$

We thus have for $z \in \Omega$ the pointwise estimate (where we used (5.4) and the Cauchy-Schwarz inequality):

$$|g(z)|^2 \leq \sum_{|\alpha|>n} \|e_\alpha\|^{-2} |\varphi^\alpha(z)|^2 \leq \sum_{N>n} \sum_{p+q=N} (N+1)^{d-2} r^{2N} \leq C_d \, n^d \, r^{2n}$$

for all $z \in \Omega$. This now implies $\|(C_\varphi - R)f\|_{H^2} = \|g\|_{H^2} \leq C'_d \, n^{d/2} \, r^n$. Hence:

$$\|C_\varphi - R\| \leq C'_d \, n^{d/2} \, r^n \, .$$

Therefore:

$$a_{N_n + 1} \leq C'_d \, n^{d/2} \, r^n \, .$$

Since $N_n \sim n^d$, we get, with $r < \rho < 1$:

$$a_{n^d} \lesssim \rho^n \, .$$

We end the proof by interpolation between two indices of the form n^d. $\square$

6 An example

For $0 < \theta < 1$, the lens map λ_θ of parameter θ is defined by:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta} \tag{6.1}$$

(see [20] or [12]).

Let $\lambda_1 = \lambda_{\theta_1}, \ldots, \lambda_d = \lambda_{\theta_d}$ be lens maps of parameters $0 < \theta_1, \ldots, \theta_d < 1$. We define a multi-lens map φ on the polydisk $\mathbb{D}^d$ as:

$$\varphi(z_1, \ldots, z_d) = \big(\lambda_1(z_1), \ldots, \lambda_d(z_d) \big) \, , \tag{6.2}$$

for $(z_1, \ldots, z_d) \in \mathbb{D}^d$. We write it $\varphi = \lambda_1 \otimes \cdots \otimes \lambda_d$.

Since we may replace $\theta_1, \ldots, \theta_d$ by $\max_k \theta_k$ or by $\inf_k \theta_k$ without changing the results, we will assume in the sequel that $\theta_1 = \cdots = \theta_d = \theta$, and we will say that the multi-lens map $\varphi = \varphi_\theta$ has parameter θ.

Theorem 6.1 *Let φ be a multi-lens map with parameter θ. Then, for positive constants a, b, a', b' depending only on θ and d, one has:*

$$a' \,\mathrm{e}^{-b' n^{1/(2d)}} \leq a_n(C_\varphi) \leq a \,\mathrm{e}^{-b\, n^{1/(2d+1)}} \tag{6.3}$$

In particular, $\gamma_d^-(C_\varphi) = 0$ even though C_φ is all Schatten classes.

The exponent $1/(2d+1)$ in the upper estimate should certainly be $1/(2d)$, but our method does not give it.

Proof. 1) Let us first show that C_φ is Hilbert-Schmidt (and hence compact). We know by [20], § 2.3, that each composition operator C_{λ_k} is Hilbert-Schmidt. Since $(e_\alpha)_\alpha$ is an orthonormal basis of $H^2(\mathbb{D}^d)$, one has:

$$\begin{aligned}
\|C_\varphi\|_{HS}^2 &= \sum_\alpha \|C_\varphi(e_\alpha)\|_{H^2(D^d)}^2 = \sum_\alpha \|\varphi^\alpha\|_{H^2(D^d)}^2 \\
&= \sum_\alpha \|\lambda_1^{\alpha_1} \otimes \cdots \otimes \lambda_d^{\alpha_d}\|_{H^2(D^d)}^2 \\
&= \sum_\alpha \|\lambda_1^{\alpha_1}\|_{H^2(\mathbb{D})} \cdots \|\lambda_d^{\alpha_d}\|_{H^2(\mathbb{D})}^2\,, \quad \text{by Fubini's Theorem} \\
&= \prod_{k=1}^d \sum_{\alpha_k=0}^\infty \|\lambda_k^{\alpha_k}\|_{H^2(\mathbb{D})}^2 = \prod_{k=1}^d \sum_{\alpha_k=0}^\infty \|C_{\lambda_k}(e_{\alpha_k})\|_{H^2(\mathbb{D})}^2 \\
&= \prod_{k=1}^d \|C_{\lambda_k}\|_{HS}^2 < +\infty\,;
\end{aligned}$$

hence C_φ is Hilbert-Schmidt. Since $\|C_{\lambda_k}\|_{HS} \leq \frac{K}{1-\theta}$ for some constant K (see [12], Lemma 2.2), one gets:

$$\|C_\varphi\|_{HS} \leq \Big(\frac{K}{1-\theta}\Big)^d\,.$$

Since the approximation numbers are non-increasing, one has:

$$n\,[a_n(C_\varphi)]^2 \leq \sum_{l=1}^n [a_l(C_\varphi]^2 \leq \sum_{l=1}^\infty [a_l(C_\varphi]^2 = \|C_\varphi\|_{HS}^2\,;$$

hence:

$$a_n(C_\varphi) \lesssim \frac{1}{\sqrt{n}\,(1-\theta)^d}\,. \tag{6.4}$$

As in [12], § 2, this inequality improves itself, by the semi-group property of the lens maps: $\lambda_\theta \circ \lambda_{\theta'} = \lambda_{\theta\theta'}$. Indeed, multi-lens maps have the same property:

$$\varphi_\theta \circ \varphi_{\theta'} = \varphi_{\theta\theta'}\,,$$

and hence, for $0 < \tau < 1$ and $k = 1, 2, \ldots$:

$$C_{\varphi_\tau^k} = [C_{\varphi_\tau}]^k\,.$$

Now, the approximation numbers satisfy the sub-multiplicative property: $a_{m+n-1}(ST) \leq a_m(S)\, a_n(T)$. Since $a_{m+n}(ST) \leq a_{m+n-1}(ST)$, this implies that $a_{kn}(T) \leq [a_n(T)]^k$ for $n, k \geq 1$.

For $k \geq 1$ to be choosen later, let $\tau = \theta^{1/k}$. We get, using (6.4) with τ instead of θ:

$$a_{kn}(C_{\varphi_\theta}) = a_{kn}(C_{\varphi_\tau}^k) \leq [a_n(C_{\varphi_\tau})]^k \lesssim \bigg(\frac{1}{\sqrt{n}\,(1-\tau)^d}\bigg)^k \leq \bigg(\frac{k^d}{\sqrt{n}\,(1-\theta)^d}\bigg)^k .$$

since $1 - \theta = 1 - \tau^k \leq k(1-\tau)$.

Choosing now for k the integer part of $\delta n^{1/(2d)}$, where $\delta > 0$ is small enough (namely $\delta < 1 - \theta$), we get that:

$$a_{kn}(C_{\varphi_\theta}) \lesssim \mathrm{e}^{-b_1 k} \lesssim \mathrm{e}^{-b_2 n^{1/(2d)}} .$$

Changing notation, we fall on, for every $N \geq 1$:

$$a_N(C_{\varphi_\theta}) \lesssim \mathrm{e}^{-b\, N^{1/(2d+1)}} .$$

This implies that, for all $p > 0$, $\sum_{N=1}^\infty [a_N(C_{\varphi_\theta})]^p < \infty$ i.e. C_{φ_θ} is in all Schatten classes S_p.

2) To prove the lower bound, we will use Theorem 4.1 and Lemma 4.2.

Let $\sigma > 0$ and, for $1 \leq j_k \leq N$, $1 \leq k \leq d$:

$$u_{j_1, \ldots, j_d} = (1 - \mathrm{e}^{-j_1 \sigma}, \ldots, 1 - \mathrm{e}^{-j_d \sigma}) .$$

Let:

$$v_{j_1, \ldots, j_d} = \varphi(u_{j_1, \ldots, j_d}) = \big(\lambda_1(1 - \mathrm{e}^{-j_1\sigma}), \ldots, \lambda_d(1 - \mathrm{e}^{-j_d \sigma})\big) .$$

By (4.1), one has, with $N = n^d$:

$$a_N(C_\varphi) \geq c' \mu_N M_v^{-2} . \tag{6.5}$$

Actually, if

$$\mu_{k,N} = \inf_{1 \leq j_k \leq N} \frac{1 - |1 - \mathrm{e}^{-j_k \sigma}|^2}{1 - |\lambda_k(1 - \mathrm{e}^{-j_k\sigma})|^2} ,$$

one has:

$$a_N(C_\varphi) \geq c' \prod_{1 \leq k \leq d} \mu_{k,N} M_v^{-2} .$$

On the other hand, if $M_{k,v}$ is the interpolation constant of the sequence

$$\big(\lambda_k(1 - \mathrm{e}^{-\sigma}), \ldots, \lambda_k(1 - \mathrm{e}^{-N\sigma})\big) ,$$

of points of $\mathbb{D}$, one has $M_v \leq M_{1,v} \cdots M_{d,v}$, by Theorem 4.1; hence:

$$a_N(C_\varphi) \geq c' \prod_{1 \leq k \leq d} \mu_{k,N} M_{k,v}^{-2} .$$

But we proved in [16] (see the proof of Proposition 2.6 there) that:

$$\mu_{k,N} M_{k,v}^{-2} \gtrsim \mathrm{e}^{-\beta\sqrt{n}}$$

for some constant $\beta > 0$ depending only on θ. We get hence:

$$a_N(C_\varphi) \gtrsim \mathrm{e}^{-\beta d\,\sqrt{n}}\,.$$

Since $N = n^d$, we get, by interpolation, that, for every $N \geq 1$:

$$a_N(C_\varphi) \gtrsim \mathrm{e}^{-\beta d\,N^{1/(2d)}}\,,$$

and that ends the proof of Theorem 6.1. □

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