# Approximation numbers of composition operators on Hp

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# Approximation numbers of composition operators on $H^p$

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**Abstract.** *We give estimates for the approximation numbers of composition operators on the $H^p$ spaces, $1 \leq p < \infty$.*

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## 1 Introduction

Recently, the study of approximation numbers of composition operators on $H^2$ has been initiated (see [10], [11], [8], [18], [12]), and (upper and lower) estimates have been given. However, most of the techniques used there are specifically Hilbertian (in particular Weyl's inequality; see [10]). Here, we consider the case of composition operators on $H^p$ for $1 \leq p < \infty$. We focus essentially on lower estimates, because the upper ones are similar, with similar proofs, as in the Hilbertian case. We give in Theorem 2.4 a minoration involving the uniform separation constant of finite sequences in the unit disk and the interpolation constant of their images by the symbol. We finish with some upper estimates.

### 1.1 Preliminary

Recall that if $X$ and $Y$ are two Banach spaces of analytic functions on the unit disk $\mathbb{D}$, and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$, one says that $\varphi$ induces a *composition operator* $C_\varphi\colon X \to Y$ if $f \circ \varphi \in Y$ for every $f \in X$; $\varphi$ is then called the *symbol* of the composition operator. One also says that $\varphi$ is a symbol for $X$ and $Y$ if it induces a composition operator $C_\varphi\colon X \to Y$.

For every $a \in \mathbb{D}$, we denote by $e_a \in (H^p)^*$ the evaluation map at $a$, namely:

$$e_a(f) = f(a)\,, \quad f \in H^p. \tag{1.1}$$

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We know that ([22], p. 253):

$$\|e_a\| = \left(\frac{1}{1-|a|^2}\right)^{1/p} \tag{1.2}$$

and the mapping equation

$$C_\varphi^*(e_a) = e_{\varphi(a)} \tag{1.3}$$

still holds.

Throughout this section we denote by $\| \,.\, \|$, without any subscript, the norm in the dual space $(H^p)^*$.

Let us stress that this dual norm of $(H^p)^*$ is, for $1 < p < \infty$, equivalent, but not equal, to the norm $\| \,.\, \|_q$ of $H^q$, and the equivalence constant tends to infinity when $p$ goes to 1 or to $\infty$.

As usual, the notation $A \lesssim B$ means that there is a constant $c$ such that $A \leq c\,B$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

## 1.2 Singular numbers

For an operator $T\colon X \to Y$ between Banach spaces $X$ and $Y$, its *approximation numbers* are defined, for $n \geq 1$, as:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\| \,. \tag{1.4}$$

One has $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq a_{n+1}(T) \geq \cdots$, and (assuming that $Y$ has the Approximation Property), $T$ is compact if and only if $a_n(T) \underset{n\to\infty}{\longrightarrow} 0$.

We will also need other singular numbers (see [2], p. 49).

The *n-th Bernstein number* $b_n(T)$ of $T$, defined as:

$$b_n(T) = \sup_{\substack{E\subseteq X\\ \dim E = n}} \inf_{x\in S_E} \|Tx\| \,, \tag{1.5}$$

where $S_E = \{x \in E\,;\ \|x\| = 1\}$ is the unit sphere of $E$. When these numbers tend to 0, $T$ is said to be superstrictly singular, or finitely strictly singular (see [17]).

The *n-th Gelfand number* of $T$, defined as:

$$c_n(T) = \inf_{\substack{L\subseteq Y\\ \operatorname{codim} L < n}} \|T_{|L}\| \,, \tag{1.6}$$

One always has:

$$a_n(T) \geq c_n(T) \quad \text{and} \quad a_n(T) \geq b_n(T) \,, \tag{1.7}$$

and, when $X$ and $Y$ are Hilbert spaces, one has $a_n(T) = b_n(T) = c_n(T)$ ([16], Theorem 2.1).

## 2 Lower bounds

### 2.1 Sub-geometrical decay

We first show that, as in the Hilbertian case $H^2$ ([10], Theorem 3.1), the approximation numbers of the composition operators on $H^p$ cannot decrease faster than geometrically.

Though we cannot longer appeal to the Hilbertian techniques of [10], Weyl's inequality has the following generalization ([3], Proposition 2).

**Proposition 2.1 (Carl-Triebel)** *Let $T$ be a compact operator on a complex Banach space $E$ and $\big(\lambda_n(T)\big)_{n\geq 1}$ be the sequence of its eigenvalues, indexed such that $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots$. Then, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots, n-1$, one has:*

$$\prod_{j=1}^{n} |\lambda_j(T)| \leq 16^n \|T\|^m a_{m+1}(T)^{n-m} \,. \tag{2.1}$$

(see [1] for an optimal result). Then, we can state:

**Theorem 2.2** *For every non-constant analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, there exist $0 < r \leq 1$ and $c > 0$, depending only on $\varphi$, such that the approximation numbers of the composition operator $C_\varphi\colon H^p \to H^p$ satisfy:*

$$a_n(C_\varphi) \geq c\, r^n \,, \qquad n = 1, 2, \ldots$$

*In particular* $\liminf_{n\to\infty} [a_n(C_\varphi)]^{1/n} \geq r > 0$.

**Proof.** If $C_\varphi$ is not compact, the result is trivial, with $r = 1$; so we assume that $C_\varphi$ is compact.

Before carrying on, we first recall some notation used in [10]. For every $z \in \mathbb{D}$, let

$$\varphi^\sharp(z) = \frac{|\varphi'(z)|\,(1 - |z|^2)}{1 - |\varphi(z)|^2}$$

be the pseudo-hyperbolic derivative of $\varphi$ at $z$, and

$$[\varphi] = \sup_{z\in\mathbb{D}} \varphi^\sharp(z) \,.$$

By the Schwarz-Pick inequality, one has $[\varphi] \leq 1$. Moreover, since $\varphi$ is not constant, one has $[\varphi] > 0$.

We also set, for every operator $T\colon H^p \to H^p$:

$$\beta^-(T) = \liminf_{n\to\infty} [a_n(T)]^{1/n} \,.$$

For every $a \in \mathbb{D}$, we are going to show that $\beta^-(C_\varphi) \geq \big(\varphi^\sharp(a)\big)^2$, which will give $\beta^-(C_\varphi) \geq [\varphi]^2$, by taking the supremum for $a \in \mathbb{D}$, and the stated result, with $0 < r < [\varphi]^2$.

If $\varphi^\sharp(a) = 0$, the result is obvious, so we assume that $\varphi^\sharp(a) > 0$.

We consider the automorphism $\Phi_a$, defined by $\Phi_a(z) = \frac{a-z}{1-\overline{a}z}$, and set

$$\psi_a = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a \, .$$

One has $\psi_a(0) = 0$ and $|\psi_a'(0)| = \varphi^\sharp(a)$.

Since $C_\varphi$ is compact on $H^p$, $C_{\psi_a} = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}$ is also compact on $H^p$. But we know that this is equivalent to say that it is compact on $H^2$. Since $\psi_a(0) = 0$ and $\psi_a'(0) = \varphi^\sharp(a) \neq 0$, we know, by the Eigenfunction Theorem ([19], p. 94), that the eigenvalues of $C_{\psi_a} \colon H^2 \to H^2$ are the numbers $\big(\psi_a'(0)\big)^j$, $j = 0, 1, \ldots$, and have multiplicity one. Moreover, the proof given in [19], § 6.2 shows that the eigenfunctions $\sigma^j$ are not only in $H^2$, but in all $H^q$, $1 \leq q < \infty$. Hence $\lambda_j(C_{\psi_a}) = \big(\psi_a'(0)\big)^{j-1}$. We now use Proposition 2.1, with $2n$ instead of $n$ and $m = n-1$; we get:

$$\begin{aligned} |\psi_a'(0)|^{n(2n-1)} &= \prod_{j=1}^{2n} |\lambda_j(C_{\psi_a})| \leq 16^{2n} \|C_{\psi_a}\|^{n-1} a_n(C_{\psi_a})^{n+1} \\ &\leq 16^{2n} \|C_{\psi_a}\|^n a_n(C_{\psi_a})^n \, , \end{aligned}$$

since $a_n(C_{\psi_a}) \leq \|C_{\psi_a}\|$.

That implies that $\beta^-(C_{\psi_a}) \geq |\psi_a'(0)|^2 = \big(\varphi^\sharp(a)\big)^2$.

Since $C_{\Phi_a}$ and $C_{\Phi_{\varphi(a)}}$ are automorphisms, we have $\beta^-(C_\varphi) = \beta^-(C_{\psi_a})$, hence the result. □

## 2.2 Main result

In this section, we use the fortunate fact that, though the evaluation maps at well-chosen points of $\mathbb{D}$ can no longer be said to constitute a Riesz sequence, they will still constitute an unconditional sequence in $H^p$ with good constants, as we are going to see, which will be sufficient for our purposes.

Recall (see [5], p. 276) that the *interpolation constant* $\kappa_\sigma$ of a finite sequence $\sigma = (z_1, \ldots, z_n)$ of points $z_1, \ldots, z_n \in \mathbb{D}$ is defined by:

$$\kappa_\sigma = \sup_{|a_1|, \ldots, |a_n| \leq 1} \inf\{\|f\|_\infty \, ; \ f \in H^\infty \text{ and } f(z_j) = a_j \, , 1 \leq j \leq n\} \, . \tag{2.2}$$

Then:

**Lemma 2.3** *For every finite sequence $\sigma = (z_1, \ldots, z_n)$ of distinct points $z_1, \ldots, z_n \in \mathbb{D}$, one has:*

$$\kappa_\sigma^{-1} \Big\| \sum_{j=1}^n \lambda_j e_{z_j} \Big\| \leq \Big\| \sum_{j=1}^n \omega_j \lambda_j e_{z_j} \Big\| \leq \kappa_\sigma \Big\| \sum_{j=1}^n \lambda_j e_{z_j} \Big\| \tag{2.3}$$

*for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and all complex numbers numbers $\omega_1, \ldots, \omega_n$ such that $|\omega_1| = \cdots = |\omega_n| = 1$.*

**Proof.** Set $L = \sum_{j=1}^n \lambda_j e_{z_j}$ and $L_\omega = \sum_{j=1}^n \omega_j \lambda_j e_{z_j}$. There exists $h \in H^\infty$ such that $\|h\|_\infty \leq \kappa_\sigma$ and $h(z_j) = \omega_j$ for every $j = 1, \ldots, n$. For every $g \in H^p$, one has $L_\omega(g) = \sum_{j=1}^n \omega_j \lambda_j g(z_j) = \sum_{j=1}^n h(z_j) \lambda_j g(z_j) = L(hg)$; hence:

$$|L_\omega(g)| \leq \|L\| \, \|hg\|_p \leq \|L\| \, \|h\|_\infty \|g\|_p \leq \kappa_\sigma \|L\| \, \|g\|_p$$

and we get $\|L_\omega\| \leq \kappa_\sigma \|L\|$, which is the right-hand side of (2.3). The left-hand side follows, by replacing $\lambda_1, \ldots, \lambda_n$ by $\overline{\omega_1}\lambda_1, \ldots, \overline{\omega_n}\lambda_n$. □

We now prove the following lower estimate.

**Theorem 2.4** *Let $\varphi \colon \mathbb{D} \to \mathbb{D}$ and $C_\varphi \colon H^p \to H^p$, with $1 \leq p < \infty$. Let $u_1, \ldots, u_n \in \mathbb{D}$ such that $v_1 = \varphi(u_1), \ldots, v_n = \varphi(u_n)$ are distinct. Then, for some constant $c_p$ depending only on p, we have:*

$$a_n(C_\varphi) \geq c_p \, \kappa_v^{-1} \Big(1 + \log \frac{1}{\delta_u}\Big)^{-1/\min(p,2)} \inf_{1 \leq j \leq n} \bigg(\frac{1 - |u_j|^2}{1 - |v_j|^2}\bigg)^{1/p}, \tag{2.4}$$

*where $\delta_u$ is the uniform separation constant of the sequence $u = (u_1, \ldots, u_n)$ and $\kappa_v$ the interpolation constant of $v = (v_1, \ldots, v_n)$.*

For the proof, we need to know some precisions on the constant in Carleson's embedding theorem. Recall that the *uniform separation constant* $\delta_\sigma$ of a finite sequence $\sigma = (z_1, \ldots, z_n)$ in the unit disk $\mathbb{D}$, is defined by:

$$\delta_\sigma = \inf_{1 \leq j \leq n} \prod_{k \neq j} \bigg| \frac{z_j - z_k}{1 - \overline{z_j} z_k} \bigg| \, . \tag{2.5}$$

**Lemma 2.5** *Let $\sigma = (z_1, \ldots, z_n)$ be a finite sequence of distinct points in $\mathbb{D}$ with uniform separation constant $\delta_\sigma$. Then:*

$$\sum_{j=1}^n (1 - |z_j|^2) \, |f(z_j)|^p \leq 12 \bigg[1 + \log \frac{1}{\delta_\sigma}\bigg] \, \|f\|_p^p \tag{2.6}$$

*for all $f \in H^p$.*

**Proof.** For $a \in \mathbb{D}$, let $k_a(z) = \frac{\sqrt{1 - |a|^2}}{1 - \bar{a} z}$ be the normalized reproducing kernel. For every positive Borel measure $\mu$ on $\mathbb{D}$, let:

$$\gamma_\mu = \sup_{a \in \operatorname{supp} \mu} \int_{\mathbb{D}} |k_a(z)|^2 \, d\mu(z) \, .$$

The so-called Reproducing Kernel Thesis (see [14], Lecture VII, pp. 151–158) says that there is an absolute positive constant $A_1$ such that:

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \leq A_1 \, \gamma_\mu \, \|f\|_p^p$$

for every $f \in H^p$ (that follows from the case $p = 2$ in writing $f = B h^{2/p}$ where $B$ is a Blaschke product and $h \in H^2$). Actually, one can take $A_1 = 2\,\mathrm{e}$ (see [15], Theorem 0.2). But when $\mu$ is the discrete measure $\sum_{j=1}^{n}(1-|z_j|^2)\,\delta_{z_j}$, it is not difficult to check (see [4], Lemma 1, p. 150, or [6], p. 201) that:

$$\gamma_\mu \leq 1 + 2\,\log\frac{1}{\delta_\sigma}\,\cdot$$

That gives the result since $4\,\mathrm{e} \leq 12$. $\square$

**Proof of Theorem 2.4.** We will actually work with the Bernstein numbers of $C_\varphi^*$. Recall that they are defined in (1.5). That will suffice since $a_n(C_\varphi) \geq a_n(C_\varphi^*)$ (one has equality if $C_\varphi$ is compact: see [7] or [2], pp. 89–91) and $a_n(C_\varphi^*) \geq b_n(C_\varphi^*)$.

Take $u_1, \ldots, u_n \in \mathbb{D}$ such that $v_1 = \varphi(u_1), \ldots, v_n = \varphi(u_n)$ are distinct. The points $u_1, \ldots, u_n$ are then also distinct and the subspace $E = \operatorname{span}\{e_{u_1}, \ldots, e_{u_n}\}$ of $(H^p)^*$ is $n$-dimensional. Let

$$L = \sum_{j=1}^{n} \lambda_j e_{u_j}$$

be in the unit sphere of $E$. We set, for $f \in H^p$ and for $j = 1, \ldots, n$:

$$\Lambda_j = \lambda_j\,\|e_{u_j}\|\,, \quad \text{and} \quad F_j = \|e_{u_j}\|^{-1} f(u_j)\,,$$

and finally:

$$\Lambda = (\Lambda_1, \ldots, \Lambda_n) \quad \text{and} \quad F = (F_1, \ldots, F_n)\,.$$

We will separate three cases.

**Case 1:** $1 < p \leq 2$.

One has $\|C_\varphi^*(L)\| = \big\|\sum_{j=1}^{n} \lambda_j\, e_{v_j}\big\|$. Using Lemma 2.3, we obtain for any choice of complex signs $\omega_1, \ldots, \omega_n$:

$$\|C_\varphi^*(L)\| \geq \kappa_v^{-1}\Big\|\sum_{j=1}^{n} \omega_j \lambda_j e_{v_j}\Big\|\,. \tag{2.7}$$

Let now $q$ be the conjugate exponent of $p$. We know that the space $H^p$ is of type $p$ as a subspace of $L^p$ ([9], p. 169) and therefore its dual $(H^p)^*$ is of cotype $q$ ([9], p. 165), with cotype constant $\leq \tau_p$, the type $p$ constant of $L^p$ (let us note that we might use that $(H^p)^*$ is isomorphic to the subspace $H^q$ of $L^q$, but we have then to introduce the constant of this isomorphism). Hence, by averaging (2.7) over all independent choices of signs and using the cotype $q$ property of $(H^p)^*$, we get:

$$\|C_\varphi^*(L)\| \geq \tau_p^{-1}\,\kappa_v^{-1}\Big(\sum_{j=1}^{n} |\lambda_j|^q \|e_{v_j}\|^q\Big)^{1/q} \geq \tau_p^{-1}\,\kappa_v^{-1}\mu_n\Big(\sum_{j=1}^{n} |\lambda_j|^q \|e_{u_j}\|^q\Big)^{1/q}\,,$$

so that

$$\|C_\varphi^*(L)\| \geq \tau_p^{-1}\,\kappa_v^{-1}\mu_n\,\|\Lambda\|_q\,, \tag{2.8}$$

where:

$$\mu_n = \inf_{1\leq j\leq n} \frac{\|e_{v_j}\|}{\|e_{u_j}\|} = \inf_{1\leq j\leq n} \Big(\frac{1-|u_j|^2}{1-|v_j|^2}\Big)^{1/p}\,.$$

It remains to give a lower bound for $\|\Lambda\|_q$.

But, by Hölder's inequality:

$$|L(f)| = \Big|\sum_{j=1}^n \lambda_j f(u_j)\Big| = \Big|\sum_{j=1}^n \Lambda_j F_j\Big| \leq \|\Lambda\|_q \|F\|_p\,.$$

Since

$$\|F\|_p^p = \sum_{j=1}^n \|e_{u_j}\|^{-p} |f(u_j)|^p = \sum_{j=1}^n (1-|u_j|^2)\,|f(u_j)|^p,$$

Lemma 2.5 gives:

$$|L(f)| \leq \|\Lambda\|_q \Big[12\,\Big(1+\log\frac{1}{\delta_u}\Big)\Big]^{1/p} \|f\|_p.$$

Taking the supremum over all $f$ with $\|f\|_p \leq 1$, we get, taking into account that $\|L\| = 1$:

$$\|\Lambda\|_q \geq \Big[12\,\Big(1+\log\frac{1}{\delta_u}\Big)\Big]^{-1/p}\,. \tag{2.9}$$

By combining (2.8) and (2.9), we get:

$$\|C_\varphi^\ast(L)\| \geq (12)^{-1/p}\,\tau_p^{-1}\,\mu_n\,\kappa_v^{-1}\Big(1+\log\frac{1}{\delta_u}\Big)^{-1/p}\,.$$

Therefore:

$$b_n(C_\varphi^\ast) \geq (12)^{-1/p}\,\tau_p^{-1}\,\mu_n\,\kappa_v^{-1}\Big(1+\log\frac{1}{\delta_u}\Big)^{-1/p}\,.$$

**Case 2:** $2 < p < \infty$.

We follow the same route, but in this case, $H^p$ is of type 2 and hence $(H^p)^\ast$ is of cotype 2. Therefore, we get:

$$\|C_\varphi^\ast(L)\| \geq \tau_2^{-1}\,\kappa_v^{-1}\,\mu_n\,\|\Lambda\|_2 \tag{2.10}$$

and, using Cauchy-Schwarz inequality:

$$\|\Lambda\|_2 \geq \Big[12\Big(1+\log\frac{1}{\delta_u}\Big)\Big]^{-1/2}\,; \tag{2.11}$$

so:

$$\|C_\varphi^\ast(L)\| \geq (12)^{-1/2}\,\tau_2^{-1}\,\mu_n\,\kappa_v^{-1}\Big(1+\log\frac{1}{\delta_u}\Big)^{-1/2}\,. \tag{2.12}$$

**Case 3:** $p = 1$.

In this case $(H^1)^*$ (which is isomorphic to the space $BMOA$) has no finite cotype. But, for each $k = 1, \ldots, n$, one has, using Lemma 2.3:

$$\begin{aligned}|\lambda_k|\,\|e_{v_k}\| &= \frac{1}{2}\bigg\|\Big(\sum_{j\neq k}\lambda_j e_{v_j} + \lambda_k e_{v_k}\Big) - \Big(\sum_{j\neq k}\lambda_j e_{v_j} - \lambda_k e_{v_k}\Big)\bigg\| \\ &\leq \frac{1}{2}\bigg(\Big\|\sum_{j\neq k}\lambda_j e_{v_j} + \lambda_k e_{v_k}\Big\| + \Big\|\sum_{j\neq k}\lambda_j e_{v_j} - \lambda_k e_{v_k}\Big\|\bigg) \\ &\leq \kappa_v\Big\|\sum_{j=1}^{n}\lambda_j e_{v_j}\Big\|;\end{aligned}$$

hence:

$$\|C_\varphi^*(L)\| \geq \kappa_v^{-1}\,\mu_n\,\|\Lambda\|_\infty\,. \tag{2.13}$$

Since $|L(F)| \leq \|\Lambda\|_\infty\|F\|_1$, we get, as above, using Lemma 2.5:

$$\|\Lambda\|_\infty \geq \left[12\left(1+\log\frac{1}{\delta_u}\right)\right]^{-1}, \tag{2.14}$$

and therefore:

$$\|C_\varphi^*(L)\| \geq (12)^{-1}\,\mu_n\,\kappa_v^{-1}\left(1+\log\frac{1}{\delta_u}\right)^{-1} \tag{2.15}$$

and that finishes the proof of Theorem 2.4. □

**Example.** We will now apply this result to lens maps. We refer to [19] or [8] for their definition. For $\theta \in (0,1)$, we denote:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}\,. \tag{2.16}$$

**Proposition 2.6** *Let $\lambda_\theta$ be the lens map of parameter $\theta$ acting on $H^p$, with $1 \leq p < \infty$. Then, for positive constants $a$ and $b$, depending only on $\theta$ and $p$:*

$$a_n(C_{\lambda_\theta}) \geq a\,\mathrm{e}^{-b\sqrt{n}}.$$

Actually, this estimate is valid for polygonal maps as well.

**Proof.** Let $0 < \sigma < 1$ and consider $u_j = 1-\sigma^j$ and $v_j = \lambda_\theta(u_j)$, $1 \leq j \leq n$. We know from [10], Lemma 6.4 and Lemma 6.5, that, for $\alpha = \frac{\pi^2}{2}$ and $\beta = \beta_\theta = \frac{\pi^2}{2^\theta\theta}$:

$$\delta_u \geq \mathrm{e}^{-\alpha/(1-\sigma)} \quad \text{and} \quad \delta_v \geq \mathrm{e}^{-\beta/(1-\sigma)}.$$

But we know that the interpolation constant $\kappa_\sigma$ is related to the uniform separation constant $\delta_\sigma$ by the following inequality ([5] page 278), in which $\Lambda$ is a positive numerical constant:

$$\frac{1}{\delta_\sigma} \leq \kappa_\sigma \leq \frac{\Lambda}{\delta_\sigma}\left(1+\log\frac{1}{\delta_\sigma}\right)\cdot \tag{2.17}$$

Actually, S. A. Vinogradov, E. A. Gorin and S. V. Hruščëv [21] (see [13], p. 505) proved that

$$\kappa_\sigma \leq \frac{2\,\mathrm{e}}{\delta_\sigma}\Big(1 + 2\log\frac{1}{\delta_\sigma}\Big),$$

so we can take $\Lambda \leq 4\,\mathrm{e} \leq 12$.

It follows that

$$\kappa_v^{-1} \geq \frac{1-\sigma}{\Lambda(\beta+1)}\,\mathrm{e}^{-\beta/(1-\sigma)}. \tag{2.18}$$

Setting $\tilde{p} = \min(p, 2)$, we have:

$$\Big(1 + \log\frac{1}{\delta_u}\Big)^{-1/\tilde{p}} \geq \Big(\frac{1-\sigma}{\alpha+1}\Big)^{1/\tilde{p}}. \tag{2.19}$$

We now estimate $\mu_n$.

Since $\lambda_\theta(0) = 0$, Schwarz's lemma says that $|\lambda_\theta(z)| \leq |z|$; hence $\frac{1-|z|^2}{1-|\lambda_\theta(z)|^2} \geq \frac{1-|z|}{1-|\lambda_\theta(z)|}$. But $1 - v_j = 1 - \lambda_\theta(u_j) = \frac{2\sigma^{j\theta}}{(2-\sigma^j)^\theta + \sigma^{j\theta}}$; hence (since $u_j$ and $v_j$ are real):

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \frac{1-u_j}{1-v_j} = \frac{\sigma^j}{2\sigma^{j\theta}}\,[(2-\sigma^j)^\theta + \sigma^{j\theta}].$$

Since the function $f(x) = (2-x)^\theta + x^\theta$ increases on $[0,1]$, one gets:

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \Big(\frac{1}{2}\,\sigma^j\Big)^{1-\theta},$$

and therefore:

$$\mu_n \geq \Big(\frac{1}{2}\,\sigma^n\Big)^{(1-\theta)/p}. \tag{2.20}$$

Applying now Theorem 2.4 and using (2.18), (2.19) and (2.20), we get:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta}\,\mathrm{e}^{-\beta/(1-\sigma)}\,(1-\sigma)^{1/\tilde{p}}\,\sigma^{n(1-\theta)/p}$$

with $\alpha_{p,\theta} = \frac{c_p}{\Lambda(\beta+1)(\alpha+1)^{1/\tilde{p}}2^{(1-\theta)/p}}$.

Taking $\sigma = \mathrm{e}^{-\varepsilon}$ where $0 < \varepsilon < 1$, we get, since $1 - \mathrm{e}^{-\varepsilon} \geq \varepsilon/2$:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta}\,\mathrm{e}^{-2\beta/\varepsilon}\Big(\frac{\varepsilon}{2}\Big)^{1/\tilde{p}}\,\mathrm{e}^{-\varepsilon n(1-\theta)/p}.$$

Optimizing by taking $\varepsilon = \sqrt{\frac{3\beta p}{1-\theta}}\,\frac{1}{\sqrt{n}}$ gives, for $n$ large enough (in order to have $\varepsilon < 1$):

$$a_n(C_{\lambda_\theta}) \geq \alpha'_{p,\theta}\,n^{-1/(2\tilde{p})}\,\mathrm{e}^{-\beta_{p,\theta}\sqrt{n}} \tag{2.21}$$

with $\alpha'_{p,\theta} = \alpha_{p,\theta}\big(\frac{\beta p}{2(1-\theta)}\big)^{1/(2\tilde{p})}$ and $\beta_{p,\theta} = \sqrt{\frac{2\beta(1-\theta)}{p}}$.

We get Theorem 2.6, with $b > \beta_{p,\theta}$. □

Let us note that $\beta_{p,\theta} = \frac{2^{\frac{1-\theta}{2}}\pi}{\sqrt{p}}\sqrt{\frac{1-\theta}{\theta}}$ tends to 0 when $\theta$ goes to 1 and tends to infinity when $\theta$ goes to 0.

## 2.3 A minoration depending on the radial behaviour of $\varphi$

We are using Theorem 2.4 to give, as in [11], Theorem 3.2, a lower bound for $a_n(C_\varphi)$ which depends on the behaviour of $\varphi$ near $\partial\mathbb{D}$.

We recall first (see [11], Section 3) that an analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ is said to be *real* if it takes real values on $]-1,1[$. If $\omega\colon [0,1] \to [0,2]$ is a modulus of continuity (meaning that $\omega$ is continuous, increasing, sub-additive, vanishing at 0, and concave), $\varphi$ is said to be an *$\omega$-radial symbol* if it is real and:

$$1 - \varphi(r) \leq \omega(1 - r)\,, \quad 0 \leq r < 1\,. \tag{2.22}$$

We have the following result.

**Theorem 2.7** *Let $\varphi$ be an $\omega$-radial symbol. Then, for $1 \leq p < \infty$, the approximation numbers of the composition operator $C_\varphi\colon H^p \to H^p$ satisfy:*

$$a_n(C_\varphi) \geq c'_p \sup_{0<\sigma<1} \bigg[ \Big(\frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n}\Big)^{1/p} (1-\sigma)^{1/\max(p^*,2)} \exp\Big(-\frac{5}{1-\sigma}\Big)\bigg], \tag{2.23}$$

*where $c'_p$ is a constant depending only on $p$, $p^*$ is the conjugate exponent of $p$, and $a = 1 - \varphi(0) > 0$.*

**Proof.** As in [11], p. 556, we fix $0 < \sigma < 1$ and define inductively $u_j \in [0,1)$ by $u_0 = 0$ and, using the intermediate value theorem:

$$1 - \varphi(u_{j+1}) = \sigma\,[1 - \varphi(u_j)]\,, \quad \text{with } 1 > u_{j+1} > u_j\,.$$

We set $v_j = \varphi(u_j)$. We have $-1 < v_j < 1$ and $1 - v_n = a\,\sigma^n$. We proved in [11], p. 556, that:

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \frac{1}{2}\,\frac{\omega^{-1}(a\,\sigma^n)}{a\,\sigma^n}\,\cdot \tag{2.24}$$

Moreover, we proved in [11], p. 557, that the uniform separation constant of $v = (v_1, \ldots, v_n)$ is such that:

$$\delta_v \geq \exp\Big(-\frac{5}{1-\sigma}\Big)\,\cdot \tag{2.25}$$

Since $\delta_u \geq \delta_v$, we get, from (2.17), that:

$$\kappa_u \leq 12\,\Big(\frac{6-\sigma}{1-\sigma}\Big)\,\exp\Big(\frac{5}{1-\sigma}\Big) \leq 60\,\Big(\frac{1}{1-\sigma}\Big)\,\exp\Big(\frac{5}{1-\sigma}\Big)\,\cdot \tag{2.26}$$

Using now (2.4) of Theorem 2.4 and combining (2.24), (2.25) and (2.26), we get Theorem 2.7. □

**Example 1: lens maps.** Let us come back to the lens maps $\lambda_\theta$ for testing Theorem 2.7. We have $\omega^{-1}(h) \approx h^{1/\theta}$ (see [8], Lemma 2.5) and $a = 1 - \lambda_\theta(0) = 1$. Setting $K = \frac{1}{10\sqrt{p}} \sqrt{\frac{1-\theta}{\theta}}$ and taking, for $n$ large enough, $\sigma = 1 - \frac{1}{K\sqrt{n}}$, we have, using that $\mathrm{e}^{-s} \leq 1 - \frac{4}{5}s$ for $s > 0$ small enough, $\sigma^n \geq \exp(-\frac{5}{4K}\sqrt{n})$ and hence:

$$a_n(C_{\lambda_\theta}) \geq c_{\theta,p}\, n^{-\frac{1}{2\max(p^*,2)}} \exp\Big[-\frac{5}{\sqrt{p}}\sqrt{\frac{1-\theta}{\theta}}\,\sqrt{n}\Big].$$

Note that the coefficient of $\sqrt{n}$ in the exponential is slightly different of that in (2.21), but of the same order.

**Example 2: cusp map.** We refer to [11], Section 4, for its definition and properties. It is the conformal mapping $\chi$ from $\mathbb{D}$ onto the domain represented on Fig. 1 such that $\chi(1) = 1$, $\chi(-1) = 0$, $\chi(i) = (1+i)/2$ and $\chi(-i) = (1-i)/2$. We proved in [11], Lemma 4.2, that, for $0 \leq r < 1$, one has:

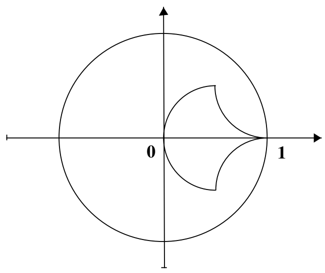


Figure 1: *Cusp map domain*

$$1 - \chi(r) = \frac{1}{1 + \frac{2}{\pi}\log\big[1/2 \arctan\big(\frac{1-r}{1+r}\big)\big]}\,.$$

Since $1 - \frac{2}{\pi}\log 2 > 0$ and $\arctan x \leq x$ for $x \geq 0$, we get that:

$$1 - \chi(r) \leq \frac{\pi}{2}\,\frac{1}{\log\big(\frac{1+r}{1-r}\big)} \leq \frac{\pi}{2}\,\frac{1}{\log\big(\frac{1}{1-r}\big)} \leq 2\,\frac{1}{\log\big(\frac{1}{1-r}\big)}\cdot$$

Hence $\chi$ is an $\omega$-radial symbol with $\omega(x) = 2/\log(1/x)$. Then $\omega^{-1}(h) = \mathrm{e}^{-2/h}$. By choosing $\sigma = 1 - \frac{\log n}{4n}$ in (2.23), we get, using that $\log(1-x) \geq -2x$ for $x > 0$ small enough, that, for $n$ large enough, $\sigma^n \geq 1/\sqrt{n}$; hence:

$$a_n(C_\chi) \geq c''_p\Big(\sqrt{n}\exp\big[-(2\,a)\sqrt{n}\big]\Big)^{1/p}\Big(\frac{\log n}{n}\Big)^{1/\max(p^*,2)}\exp\Big(-\frac{20n}{\log n}\Big)\cdot$$

It follows that, for some constant $C_p > 0$ depending only on $p$, we have:

$$a_n(C_\chi) \geq C_p \exp\Big(-\frac{25n}{\log n}\Big)\cdot \tag{2.27}$$

It has to be stressed that the term in the exponential does not depend on $p$.

**Example 3: Shapiro-Taylor's maps.** These maps $\varsigma_\theta$, for $\theta > 0$, were defined in [20]. Let us recall their definition. For $\varepsilon > 0$, we set $V_\varepsilon = \{z \in \mathbb{C}\,;\ \Re z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define

$$f_\theta(z) = z(-\log z)^\theta, \tag{2.28}$$

for $z \in V_\varepsilon$, where $\log z$ will be the principal determination of the logarithm. Let now $g_\theta$ be the conformal mapping from $\mathbb{D}$ onto $V_\varepsilon$, which maps $\mathbb{T} = \partial\mathbb{D}$ onto $\partial V_\varepsilon$, defined by $g_\theta(z) = \varepsilon\,\varphi_0(z)$, where $\varphi_0$ is the conformal map from $\mathbb{D}$ onto $V_1$, given by:

$$\varphi_0(z) = \frac{\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} - i}{-i\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} + 1}\,\cdot \tag{2.29}$$

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta). \tag{2.30}$$

We saw in [11], p. 560, that $\omega^{-1}(h) = K_\theta\, h\big(\log(1/h)\big)^{-\theta}$. Hence, choosing $\sigma = 1/(\mathrm{e}\,\alpha_\theta^{1/n})$, where $\alpha_\theta = 1 - \varsigma_\theta(0)$, we get that:

$$a_n(C_{\varsigma_\theta}) \geq c_{p,\theta}.\frac{1}{n^{\theta/2p}}\,\cdot \tag{2.31}$$

However, we already remarked in [11], Section 4.2, that, even for $p = 2$, this result is not optimal.

# 3 Upper bound

For upper bounds, there is essentially no change with regard to the case $p = 2$. Hence we essentially only state some results.

We have the following upper bound, which can be obtained with the same proof as in [8].

**Theorem 3.1** *Let $C_\varphi\colon H^p \to H^p$, $1 \leq p < \infty$, a composition operator, and $n \geq 1$. Then, for every Blaschke product $B$ with (strictly) less than $n$ zeros, each counted with its multiplicity, one has:*

$$a_n(C_\varphi) \leq C\sqrt{n}\left(\sup_{\substack{0<h<1\\ \xi\in\mathbb{T}}} \frac{1}{h}\int_{\overline{S(\xi,h)}} |B|^p\, dm_\varphi\right)^{1/p},$$

*where $m_\varphi$ is the pullback measure of $m$, the normalized Lebesgue measure on $\mathbb{T}$, under $\varphi$ and $S(\xi,h) = \mathbb{D} \cap D(\xi,h)$ is the Carleson window of size $h$ centered at $\xi \in \mathbb{T}$.*

**Proof.** We first estimate the Gelfand number $c_n(C_\varphi)$ by restricting to the subspace $BH^p$ which is of codimension $< n$. As in [8], Lemma 2.4:

$$c_n(C_\varphi) \lesssim \left( \sup_{\substack{0<h<1 \\ \xi\in\mathbb{T}}} \frac{1}{h} \int_{\overline{S(\xi,h)}} |B|^p \, dm_\varphi \right)^{1/p} .$$

Now (see [2], Proposition 2.4.3), one has $a_n(C_\varphi) \leq \sqrt{2n}\, c_n(C_\varphi)$, hence the result. $\square$

We can then deduce, with the same proof, the following version of [11], Theorem 2.3.

Recall ([11], Definition 2.2) that a symbol $\varphi \in A(\mathbb{D})$ (i.e. $\varphi \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous and analytic in $\mathbb{D}$) is said to be *globally regular* if $\varphi(\overline{\mathbb{D}}) \cap \partial\mathbb{D} = \{\xi_1, \ldots, \xi_l\}$ and there exists a modulus of continuity $\omega$ (i.e. a continuous, increasing and sub-additive function $\omega \colon [0, A] \to \mathbb{R}^+$, which vanishes at zero, and that we may assume to be concave), such that, writing $E_{\xi_j} = \{t\,;\ \gamma(t) = \xi_j\}$, one has $\mathbb{T} = \bigcup_{j=1}^{l} \big(E_{\xi_j} + [-r_j, r_j]\big)$ for some $r_1, \ldots, r_l > 0$, and for some positive constants $C, c > 0$:

$$|\gamma(t) - \gamma(t_j)| \leq C\big(1 - |\gamma(t)|\big) \tag{3.1}$$

$$c\, \omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)| \tag{3.2}$$

for $j = 1, \ldots, l$, all $t_j \in E_{\xi_j}$ with $|t - t_j| \leq r_j$.

**Theorem 3.2** *Let $\varphi$ be a symbol in $A(\mathbb{D})$ whose image touches $\partial\mathbb{D}$ exactly at the points $\xi_1, \ldots, \xi_l$ and which is globally-regular. Then there are constants $\kappa$, $K$, $L > 0$, depending only on $\varphi$, such that, for every $k \geq 1$:*

$$a_k(C_\varphi) \leq K \left[ \frac{\omega^{-1}(\kappa\, 2^{-N_k})}{\kappa\, 2^{-N_k}} \right]^{1/p} , \tag{3.3}$$

*where $N_k$ is the largest integer such that $lNd_N < k$ and $d_N$ is the integer part of $\big[\log \frac{\kappa\, 2^{-N}}{\omega^{-1}(\kappa\, 2^{-N})} \big/ \log(\chi^{-p})\big] + 1$, with $0 < \chi < 1$ an absolute constant.*

As a corollary, we get for lens maps $\lambda_\theta$ (as well as for polygonal maps), in the same way as Theorem 2.4 in [11], p. 550 (recall that then $\omega(h) \approx h^\theta$), the following upper bound.

**Theorem 3.3** *Let $\varphi = \lambda_\theta$ be the lens map of parameter $\theta$ acting on $H^p$, $1 < p < \infty$. Then, for positive constants $b$ and $c$ depending only on $\theta$ and $p$:*

$$a_n(C_{\lambda_\theta}) \leq c\, \mathrm{e}^{-b\sqrt{n}} .$$

For the cusp map, we also have as in [11], Theorem 4.3 (here, $\omega(h) \approx 1/\log(1/h)$).

**Theorem 3.4** *Let $\varphi = \chi$ be the cusp map. For some positive constants $b$ and $c$ depending only on $p$, one has:*

$$a_n(C_\chi) \leq c\, \mathrm{e}^{-b\, n/\log n} .$$

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