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Two remarks on composition operators on the Dirichlet space

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Abstract. We show that the decay of approximation numbers of compact composition operators on the Dirichlet space $D$ can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on $D$ of self-maps of the disk all of whose powers are norm-bounded in $D$.

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1 Introduction

Recall that if $\varphi$ is an analytic self-map of $\mathbb{D}$, a so-called Schur function, the composition operator $C_\varphi$ associated to $\varphi$ is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$ 

The Littlewood subordination principle ([4], p. 30) tells us that $C_\varphi$ maps the Hardy space $H^2$ to itself for every Schur function $\varphi$. Also recall that if $H$ is a Hilbert space and $T: H \to H$ a bounded linear operator, the $n$-th approximation number $a_n(T)$ of $T$ is defined as

$$a_n(T) = \inf \{ \|T - R\| ; \text{rank } R < n\}, \quad n = 1, 2, \ldots.$$ 

In [12], working on that Hardy space $H^2$ (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators $C_\varphi$, and proved among other facts the following:

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Theorem 1.1 Let \((\varepsilon_n)_{n \geq 1}\) be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator \(C_\varphi\) on \(H^2\) such that
\[
\liminf_{n \to \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.
\]
As a consequence, there are composition operators on \(H^2\) which are compact but in no Schatten class.

The last item had been previously proved by Carroll and Cowen ([3]), the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of \(D\) generates a bounded composition operator on \(D\). When this is the case, we will say that \(\varphi\) is a symbol (understanding “of \(D\)”). Note that every symbol is necessarily in \(D\).

In [11], we have performed a similar study on that Dirichlet space \(D\), and established several results on approximation numbers in that new setting, in particular the existence of symbols \(\varphi\) for which \(C_\varphi\) is compact without being in any Schatten class \(S_p\). But we have not been able in [11] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

Theorem 1.2 For every sequence \((\varepsilon_n)_{n \geq 1}\) of positive numbers tending to 0, there exists a compact composition operator \(C_\varphi\) on the Dirichlet space \(D\) such that
\[
\liminf_{n \to \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.
\]

Turning now to the question of necessary or sufficient conditions for a Schur function \(\varphi\) to be a symbol, we can observe that, since \((z^n/\sqrt{n})_{n \geq 1}\) is an orthonormal sequence in \(D\) and since formally \(C_\varphi(z^n) = \varphi^n\), a necessary condition is as follows:

\[
(1.2) \quad \varphi \text{ is a symbol} \implies \|\varphi^n\|_D = O(\sqrt{n}).
\]

It is worth noting that, for any Schur function, one has:
\[
\varphi \in D \implies \|\varphi^n\|_D = O(n)
\]
(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer \(n \geq 1\):
\[
\|\varphi^n\|^2_D = |\varphi(0)|^{2n} + \int_D n^2 |\varphi(z)|^{2(n-1)}|\varphi'(z)|^2 \, dA(z)
\leq |\varphi(0)|^{2n} + \int_D n^2 |\varphi'(z)|^2 \, dA(z) \leq n^2 \|\varphi\|^2_D,
\]
giving the result.
Now, the following sufficient condition was given in [5]:

\[(1.3) \quad \|\phi^n\|_D = O(1) \quad \Rightarrow \quad \phi \text{ is a symbol.}\]

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

**Theorem 1.3** Let \((M_n)_{n \geq 1}\) be an arbitrary sequence of positive numbers tending to \(\infty\). Then, there exists a Schur function \(\phi \in D\) such that:

1) \(\|\phi^n\|_D = O(M_n)\) as \(n \to \infty\);
2) \(\phi\) is not a symbol on \(D\).

The organization of that paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

## 2 Notation and background.

We denote by \(D\) the open unit disk of the complex plane and by \(A\) the normalized area measure \(dx \, dy/\pi\) of \(D\). The unit circle is denoted by \(T = \partial D\).

The notation \(A \lesssim B\) indicates that \(A \leq c \, B\) for some positive constant \(c\).

A Schur function is an analytic self-map of \(D\) and the associated composition operator is defined, formally, by \(C_\phi(f) = f \circ \phi\). The operator \(C_\phi\) maps the space \(\text{Hol}(D)\) of holomorphic functions on \(D\) into itself.

The Dirichlet space \(D\) is the space of analytic functions \(f: D \to \mathbb{C}\) such that:

\[(2.1) \quad \|f\|^2_D := |f(0)|^2 + \int_D |f'(z)|^2 \, dA(z) < +\infty.\]

If \(f(z) = \sum_{n=0}^{\infty} c_n z^n\), one has:

\[(2.2) \quad \|f\|^2_D = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.\]

Then \(\|\cdot\|_D\) is a norm on \(D\), making \(D\) a Hilbert space, and \(\|\cdot\|_{H^2} \lesssim \|\cdot\|_D\). For further information on the Dirichlet space, the reader may see [1] or [16].

The Bergman space \(B\) is the space of analytic functions \(f: D \to \mathbb{C}\) such that:

\[(3.1) \quad \|f\|^2_B := \int_D |f(z)|^2 \, dA(z) < +\infty.\]

If \(f(z) = \sum_{n=0}^{\infty} c_n z^n\), one has \(\|f\|^2_B = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}\). If \(f \in D\), one has by definition:

\[
\|f\|^2_D = \|f\|^2_B + |f(0)|^2.
\]
Recall that, whereas every Schur function $\varphi$ generates a bounded composition operator $C_\varphi$ on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [14], Proposition 3.12, for instance).

We denote by $b_n(T)$ the $n$-th Bernstein number of the operator $T: H \to H$, namely:

$$b_n(T) = \sup_{\dim E = n} \left( \inf_{f \in S_E} \|Tx\| \right)$$

where $S_E$ denotes the unit sphere of $E$. It is easy to see ([11]) that

$$b_n(T) = a_n(T) \quad \text{for all } n \geq 1.$$

(recall that the approximation numbers are defined in (1.1)).

If $\varphi$ is a Schur function, let

$$n_\varphi(w) = \# \{ z \in D : \varphi(z) = w \} \geq 0$$

be the associated counting function. If $f \in D$ and $g = f \circ \varphi$, the change of variable formula provides us with the useful following equation ([17], [11]):

$$\int_D |g'(z)|^2 dA(z) = \int_D |f'(w)|^2 n_\varphi(w) dA(w)$$

(the integrals might be infinite). In those terms, a necessary and sufficient condition for $\varphi$ to be a symbol is as follows ([17], Theorem 1). Let:

$$\rho_\varphi(h) = \sup_{\xi \in T} \int_{S(\xi, h)} n_\varphi dA$$

where $S(\xi, h) = D \cap D(\xi, h)$ is the Carleson window centered at $\xi$ and of size $h$.

Then $\varphi$ is a symbol if and only if:

$$\sup_{0 < h < 1} \frac{1}{h^2} \rho_\varphi(h) < \infty.$$

This is not difficult to prove. In view of (2.5), the boundedness of $C_\varphi$ amounts to the existence of a constant $C$ such that:

$$\int_D |f'(w)|^2 n_\varphi(w) dA(w) \leq C \int_D |f'(z)|^2 dA(z), \quad \forall f \in D.$$

Since $f' = h$ runs over $\mathcal{B}$ as $f$ runs over $D$, and with equal norms, the above condition reads:

$$\int_D |h(w)|^2 n_\varphi(w) dA(w) \leq C \int_D |h(z)|^2 dA(z), \quad \forall h \in \mathcal{B}.$$

This exactly means that the measure $n_\varphi dA$ is a Carleson measure for $\mathcal{B}$. Such measures have been characterized in [7] and that characterization gives (2.7).
But this condition is very abstract and difficult to test, and sometimes more “concrete” sufficient conditions are desirable. In [11], we proved that, even if the Schur function extends continuously to \( D \), no Lipschitz condition of order \( \alpha \), \( 0 < \alpha < 1 \), on \( \varphi \) is sufficient for ensuring that \( \varphi \) is a symbol. It is worth noting that the limiting case \( \alpha = 1 \), so restrictive it is, guarantees the result.

**Proposition 2.1** Suppose that the Schur function \( \varphi \) is in the analytic Lipschitz class on the unit disk, i.e. satisfies:

\[
|\varphi(z) - \varphi(w)| \leq C|z - w|, \quad \forall z, w \in D.
\]

Then \( C \varphi \) is bounded on \( D \).

**Proof.** Let \( f \in D \); one has:

\[
\left\| C \varphi(f) \right\|_B^2 = |f(\varphi(0))|^2 + \int_D |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z)
\]

\[
\leq |f(\varphi(0))|^2 + \left\| \varphi' \right\|^2_\infty \int_D |f'(\varphi(z))|^2 dA(z).
\]

This integral is nothing but \( \|C \varphi(f')\|_\mathcal{B}^2 \) and hence, since \( C \varphi \) is bounded on the Bergman space \( \mathcal{B} \), we have, for some constant \( K_1 \):

\[
\int_D |f'(\varphi(z))|^2 dA(z) \leq K_1^2 \|f'\|_\mathcal{B}^2 \leq K^2 \|f\|_D^2.
\]

On the other hand,

\[
|f(\varphi(0))| \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{H^2} \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_D,
\]

and we get

\[
\left\| C \varphi(f) \right\|_D^2 \leq K^2 \|f\|_D^2,
\]

with \( K^2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1} \). \( \square \)

**3 Proof of Theorem 1.2**

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

**Theorem 3.1** For every sequence \((\varepsilon_n)\) of positive numbers with limit 0, there exists a compact composition operator \( C \varphi \) on \( D \) such that

\[
\liminf_{n \to \infty} \frac{a_n(C \varphi)}{\varepsilon_n} > 0.
\]

Before entering really in the proof, we may remark that, without loss of generality, by replacing \( \varepsilon_n \) with \( \inf(2^{-8}, \sup_{k \geq n} \varepsilon_k) \), we can, and do, assume that \( (\varepsilon_n) \) decreases and \( \varepsilon_1 \leq 2^{-8} \).

Moreover, we can assume that \( (\varepsilon_n) \) decreases “slowly”, as said in the following lemma.
Let \( (\varepsilon_i) \) be a decreasing sequence with limit zero and let \( 0 < \rho < 1 \). Then, there exists another sequence \( (\hat{\varepsilon}_i) \), decreasing with limit zero, such that \( \hat{\varepsilon}_i \geq \varepsilon_i \) and \( \hat{\varepsilon}_{i+1} \geq \rho \hat{\varepsilon}_i \), for every \( i \geq 1 \).

**Proof.** We define inductively \( \hat{\varepsilon}_i \) by \( \hat{\varepsilon}_1 = \varepsilon_1 \) and 
\[
\hat{\varepsilon}_{i+1} = \max(\rho \hat{\varepsilon}_i, \varepsilon_{i+1}).
\]

It is seen by induction that \( \hat{\varepsilon}_i \geq \varepsilon_i \) and that \( \hat{\varepsilon}_i \) decreases to a limit \( a \geq 0 \). If \( \hat{\varepsilon}_i = \varepsilon_i \) for infinitely many indices \( i \), we have \( a = 0 \). In the opposite case, \( \hat{\varepsilon}_{i+1} = \rho \hat{\varepsilon}_i \) from some index \( i_0 \) onwards, and again \( a = 0 \) since \( \rho < 1 \).

We will take \( \rho = 1/2 \) and assume for the sequel that \( \varepsilon_{i+1} \geq \varepsilon_i / 2 \).

**Proof of Theorem 3.1.** We first construct a subdomain \( \Omega = \Omega_\theta \) of \( \mathbb{D} \) defined by a cuspidal inequality:

\[
\Omega = \{ z = x + iy \in \mathbb{D} \mid |y| < \theta(1-x), \ 0 < x < 1 \},
\]

where \( \theta: [0,1] \rightarrow [0,1] \) is a continuous increasing function such that 
\[
\theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1 - x.
\]

Note that since \( 1 - x \leq \sqrt{1 - x^2} \), the condition \( |y| < \theta(1-x) \) implies that \( z = x + iy \in \mathbb{D} \). Note also that \( 1 \in \overline{\Omega} \) and that \( \Omega \) is a Jordan domain.

We introduce a parameter \( \delta \) with \( \varepsilon_1 \leq \delta \leq 1 - \varepsilon_1 \). We put:

\[
\theta(\delta^j) = \varepsilon_j \delta^j
\]

and we extend \( \theta \) to an increasing continuous function from \( (0,1) \) into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

\[
\theta(h) \leq h \quad \text{and} \quad \theta(h) = o(h) \text{ as } h \rightarrow 0. \tag{3.4}
\]

Indeed, if \( \delta^{j+1} \leq h < \delta^j \), we have \( \theta(h)/h \leq \theta(\delta^j)/\delta^{j+1} = \varepsilon_j/\delta \), which is \( \leq \varepsilon_1/\delta \leq 1 \) and which tends to 0 with \( h \).

We define now \( \varphi = \varphi_\theta: \mathbb{D} \rightarrow \overline{\Omega} \) as a continuous map which is a Riemann map from \( \mathbb{D} \) onto \( \Omega \), and with \( \varphi(1) = 1 \) (a cusp-type map). Since \( \varphi \) is univalent, one has \( n_\varphi = 1 \), and since \( \Omega \) is bounded, \( \varphi \) defines a symbol on \( \mathcal{D} \), by (2.7). Moreover, (3.4) implies that \( A[\{\xi \mid \xi \in \mathbb{T} \} \cap \Omega] \leq \theta(h) \) for every \( \xi \in \mathbb{T} \); hence, \( \rho_\varphi \) being defined in (2.6), one has \( \rho_\varphi(h) = o(h^2) \) as \( h \rightarrow 0^+ \). In view of [17], this little-oh condition guarantees the compactness of \( C_\varphi \mathcal{D} \rightarrow \mathcal{D} \).

It remains to minorate its approximation numbers.

The measure \( \mu = n_\varphi dA \) is a Carleson measure for the Bergman space \( \mathcal{B} \), and it was proved in [10] that \( C_\varphi \) is unitarily equivalent to the Toeplitz operator \( T_\mu = T_\mu^* T_\mu: \mathcal{B} \rightarrow \mathcal{B} \) defined by:

\[
T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w) = \int_{\mathbb{D}} f(w) K_w(z) dA(w), \tag{3.5}
\]
where $I_\mu : \mathcal{B} \rightarrow L^2(\mu)$ is the canonical inclusion and $K_w$ the reproducing kernel of $\mathcal{B}$ at $w$, i.e. $K_w(z) = \frac{1}{1 - wz}$.

Actually, we can get rid of the analyticity constraint in considering, instead of $T_\mu$, the operator $S_\mu = I_\mu I_\mu^* : L^2(\mu) \rightarrow L^2(\mu)$, which corresponds to the arrows:

$$L^2(\mu) \xrightarrow{I_\mu^*} \mathcal{B} \xrightarrow{I_\mu} L^2(\mu).$$

We use the relation (3.5) which implies:

$$a_n(C_\varphi) = a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)}.$$

We set:

$$c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \epsilon_j \delta^j$$

One has $r_j = \epsilon_j(1 - c_j)/2$.

**Lemma 3.3** The disks $\Delta_j = D(c_j, r_j)$, $j \geq 1$, are disjoint and contained in $\Omega$.

**Proof.** If $z = x + iy \in \Delta_j$, then $1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \epsilon_j/2) = 2\delta^j(1 - \epsilon_j/2) \geq \delta^j$ and $|y| < r_j = \theta(\delta^j)$; hence $|y| < \theta(\delta^j) \leq \theta(1 - x)$ and $z \in \Omega$. On the other hand, $c_j + 1 - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta)\delta^j \geq 2\epsilon_j\delta^j \geq 2\epsilon_j\delta^j = 2r_j > r_j + r_{j+1}$; hence $\Delta_j \cap \Delta_{j+1} = \emptyset$. □

We will next need a description of $S_\mu$.

**Lemma 3.4** For every $g \in L^2(\mu)$ and every $z \in \mathbb{D}$:

$$S_\mu g(z) = \left( \int_\Omega \frac{g(w)}{(1 - wz)^2} dA(w) \right) \mathbb{1}_\Omega(z).$$

**Proof.** $K_w$ being the reproducing kernel of $\mathcal{B}$, we have for any pair of functions $f \in \mathcal{B}$ and $g \in L^2(\mu)$:

$$\langle I_\mu^* g, f \rangle_{\mathcal{B}} = \langle g, I_\mu f \rangle_{L^2(\mu)} = \int_\Omega g(w)\overline{f(w)} dA(w) = \int_\Omega g(w) \langle K_w, f \rangle_{\mathcal{B}} dA(w) = \langle \int_\Omega g(w)K_w dA(w), f \rangle_{\mathcal{B}},$$

so that $I_\mu^* g = \int_\Omega g(w)K_w dA(w)$, giving the result. □

In the rest of the proof, we fix a positive integer $n$ and put:

$$f_j = \frac{1}{r_j} \mathbb{1}_{\Delta_j}, \quad j = 1, \ldots, n.$$

Let:

$$E = \text{span} \{ f_1, \ldots, f_n \}. $$
This is an $n$-dimensional subspace of $L^2(\mu)$.

The $\Delta_j$'s being disjoint, the sequence $(f_1, \ldots, f_n)$ is orthonormal in $L^2(\mu)$. Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 \, d\mu = \int f_j^2 \, n \, dA = \int_{\Delta_j} \frac{1}{r_j^2} \, dA = 1.$$  

We now estimate from below the Bernstein numbers of $I^*_\mu$. To that effect, we compute the scalar products $m_{i,j} = \langle I^*_\mu(f_i), I^*_\mu(f_j) \rangle$. One has:

$$m_{i,j} = \langle f_i, S_\mu(f_j) \rangle = \int_{\Omega} f_i(z) S_\mu(f_j(z)) \, dA(z)$$

$$= \int_{\Omega \times \Omega} f_i(z) f_j(w) \frac{1}{(1 - w \overline{z})^2} \, dA(z) \, dA(w)$$

$$= \frac{1}{r_i r_j} \int_{\Delta_i \times \Delta_j} \frac{1}{(1 - w \overline{z})^2} \, dA(z) \, dA(w).$$

**Lemma 3.5** We have

$$(3.11) \quad m_{i,i} \geq \frac{\epsilon_i^2}{32}, \quad \text{and} \quad |m_{i,j}| \leq \epsilon_i \epsilon_j \delta^{i-j} \quad \text{for } i < j.$$  

**Proof.** Set $\epsilon_i' = \frac{\epsilon_i}{\epsilon_{i-1}} = \frac{\bar{c}_i}{2(1+c_i)}$. One has $\frac{4}{e} \leq \epsilon_i' \leq \frac{4}{r_i}$. We observe that (recall that $A(\Delta_i) = r_i^2$):

$$m_{i,i} - \epsilon_i'^2 = \frac{1}{r_i^2} \int_{\Delta_i \times \Delta_i} \left[ \frac{1}{(1 - w \overline{z})^2} - \frac{1}{(1 - \epsilon_i'^2)^2} \right] \, dA(z) \, dA(w).$$

Therefore, using the fact that, for $z \in \Delta_i$ and $w \in \mathbb{D}$:

$$|1 - w \overline{z}| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \epsilon_i \left(1 - \frac{1-c_i}{2}\right) \geq (1-c_i) \left(1 - \frac{\epsilon_i}{2}\right) \geq 1 - c_i$$

and then the mean-value theorem, we get:

$$|m_{i,j} - \epsilon_i'^2| \leq \frac{1}{r_i^2} \int_{\Delta_i \times \Delta_i} \left| \frac{1}{(1 - w \overline{z})^2} - \frac{1}{(1 - \epsilon_i'^2)^2} \right| \, dA(z) \, dA(w)$$

$$\leq \frac{1}{r_i^2} \int_{\Delta_i \times \Delta_i} \frac{32 r_i}{(1-c_i)^3} \, dA(z) \, dA(w)$$

$$= \frac{32 r_i^3}{(1-c_i)^3} \leq 32 \times 8 \epsilon_i'^3 \leq \frac{\epsilon_i'^2}{2},$$

since $\epsilon_i \leq \epsilon_1 \leq 2^{-8}$ implies that $\epsilon_i' \leq 1/(32 \times 16)$. This gives us the lower bound $m_{i,i} \geq \epsilon_i'^2/2 \geq \epsilon_i'^2/32$. 

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Next, for $i < j$:

$$
|m_{i,j}| \leq \frac{1}{r_i r_j} \int_{\Delta_i \times \Delta_j} \left| \frac{1}{(1-wz)^2} \right| dA(z) dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1-c_i)^2 r_i^2 r_j^2}
$$

$$
= \frac{4 \epsilon_i \epsilon_j \delta^{i+j}}{4 \delta^{2i}} = \epsilon_i \epsilon_j \delta^{i-j},
$$

and that ends the proof of Lemma 3.5. □

We further write the $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ as $M = D + R$ where $D$ is the diagonal matrix $m_i = m_{i,i}$ with $m_i \geq \frac{\epsilon_i^2}{64}$, $1 \leq i \leq n$. Observe that $M$ is nothing but the matrix of $S_\mu$ on the orthonormal basis $(f_1, \ldots, f_n)$ of $E$, so that we can identify $M$ and $S_\mu$ on $E$.

Now the following lemma will end the proof of Theorem 3.1.

**Lemma 3.6** If $\delta \leq \frac{1}{200}$, we have:

$$
\|D^{-1}R\| \leq \frac{1}{2}.
$$

Indeed, by the ideal property of Bernstein numbers, Neumann’s lemma and the relations:

$$
M = D(I + D^{-1}R), \quad \text{and} \quad D = MQ \quad \text{with} \quad \|Q\| \leq 2,
$$

we have $b_n(D) \leq b_n(M) \|Q\| \leq 2 b_n(M)$, that is:

$$
a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\epsilon_n^2}{64},
$$

since the $n$ first approximation numbers of the diagonal matrix $D$ (the matrices being viewed as well as operators on the Hilbertian space $C^n$ with its canonical basis) are $m_{1,1}, \ldots, m_{n,n}$. It follows that, using (3.6):

$$
(3.13) \quad a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\epsilon_n}{8}.
$$

In view of (3.6), we have as well $a_n(C_\varphi) \geq \epsilon_n/8$, and we are done. □

**Proof of Lemma 3.6.** Write $M = (m_{i,j}) = D(I + N)$ with $N = D^{-1}R$. One has:

$$
(3.14) \quad N = (\nu_{i,j}), \quad \text{with} \quad \nu_{i,i} = 0 \quad \text{and} \quad \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i.
$$

We shall show that $\|N\| \leq 1/2$ by using the (unweighted) Schur test, which we recall ([6], Problem 45):

**Proposition 3.7** Let $(a_{i,j})_{1 \leq i,j \leq n}$ be a matrix of complex numbers. Suppose that there exist two positive numbers $\alpha, \beta > 0$ such that:

1. $\sum_{j=1}^{n} |a_{i,j}| \leq \alpha$ for all $i$;
2. $\sum_{i=1}^{n} |a_{i,j}| \leq \beta$ for all $j$.

Then, the (Hilbertian) norm of this matrix satisfies $\|A\| \leq \sqrt{\alpha \beta}$.

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It is essential for our purpose to note that:

\[(3.15) \quad i < j \implies |\nu_{i,j}| \leq 32 \delta^{i-j},\]

\[(3.16) \quad i > j \implies |\nu_{i,j}| \leq 32 (2\delta)^{i-j}.\]

Indeed, we see from (3.11) and (3.14) that, for \(i < j\):

\[|\nu_{i,j}| = \frac{|m_{i,j}|}{m_{i,i}} \leq 32 \varepsilon_j \varepsilon_i^{-2} \delta^{i-j} \leq 32 \delta^{i-j},\]

since \(\varepsilon_j \leq \varepsilon_i\). Secondly, using \(\varepsilon_j/\varepsilon_i \leq 2^{i-j}\) for \(i > j\) (recall that we assumed that \(\varepsilon_{k+1} \geq \varepsilon_k/2\)), as well as \(|m_{i,j}| = |m_{j,i}|\), we have, for \(i > j\):

\[|\nu_{i,j}| = \frac{|m_{j,i}|}{m_{i,i}} \leq 32 \frac{\varepsilon_i}{\varepsilon_j} \delta^{i-j} \leq 32 (2\delta)^{i-j}.\]

Now, for fixed \(i\), (3.15) gives:

\[
\sum_{j=1}^{n} |\nu_{i,j}| = \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32 \left( \sum_{j>i} \delta^{i-j} + \sum_{j<i} (2\delta)^{i-j} \right) \\
\leq 32 \left( \frac{\delta}{1-\delta} + \frac{2\delta}{1-2\delta} \right) \leq 32 \frac{3\delta}{1-2\delta} \leq \frac{96}{198} \leq \frac{1}{2},
\]

since \(\delta \leq 1/200\). Hence:

\[(3.17) \quad \sup_i \left( \sum_j |\nu_{i,j}| \right) \leq 1/2.\]

In the same manner, but using (3.16) instead of (3.15), one has:

\[(3.18) \quad \sup_j \left( \sum_i |\nu_{i,j}| \right) \leq 1/2.\]

Now, (3.17), (3.18) and the Schur criterion recalled above give:

\[\|N\| \leq \sqrt{1/2 \times 1/2} = 1/2,\]

as claimed. \(\square\)

**Remark.** We could reverse the point of view in the preceding proof: start from \(\theta\) and see what lower bound for \(a_n(C_\phi)\) emerges. For example, if \(\theta(h) \approx h\) as is the case for lens maps (see [11]), we find again that \(a_n(C_\phi) \geq \delta_0 > 0\) and that \(C_\phi\) is not compact. But if \(\theta(h) \approx h^{1+\alpha}\) with \(\alpha > 0\), the method only gives \(a_n(C_\phi) \gtrsim e^{-\alpha n}\) (which is always true: see [11], Theorem 2.1), whereas the methods of [11] easily give \(a_n(C_\phi) \gtrsim e^{-\alpha \sqrt{n}}\). Therefore, this \(\mu\)-method seems to be sharp when we are close to non-compactness, and to be beaten by those of [11] for “strongly compact” composition operators.
3.1 Optimality of the EKSY result

El Fallah, Kellay, Shabankhah and Youssfi proved in [5] the following: if \( \varphi \) is a Schur function such that \( \varphi \in D \) and \( \| \varphi^p \|_D = O(1) \) as \( p \to \infty \), then \( \varphi \) is a symbol on \( D \). We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

**Theorem 3.8** Let \( (M_p)_{p \geq 1} \) be an arbitrary sequence of positive numbers such that \( \lim_{p \to \infty} M_p = \infty \). Then, there exists a Schur function \( \varphi \in D \) such that:

1) \( \| \varphi^p \|_D = \Theta(M_p) \) as \( p \to \infty \);
2) \( \varphi \) is not a symbol on \( D \).

**Remark.** We first observe that we cannot replace \( \lim \) by \( \limsup \) in Theorem 3.8. Indeed, since \( \varphi \in D \), the measure \( \mu = n \varphi \, dA \) is finite, and

\[
\| \varphi^p \|_2^2 = \int_D |w|^{2p-2} d\mu(w) \geq c p^2 \left( \int_D |w|^{2} d\mu(w) \right)^{p-1} \geq c \delta^p,
\]

where \( c \) and \( \delta \) are positive constants.

**Proof of Theorem 3.8.** We may, and do, assume that \( (M_p) \) is non-decreasing and integer-valued. Let \( (l_n)_{n \geq 1} \) be an non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let \( \Omega \) be the subdomain of the right half-plane \( \mathbb{C}_0 \) defined as follows. We set:

\[
\varepsilon_n = -\log(1 - 2^{-n}) \sim 2^{-n},
\]

and we consider the (essentially) disjoint boxes \( (k = 0, 1, \ldots) \):

\[
B_{k,n} = B_{0,n} + 2k\pi i,
\]

with:

\[
B_{0,n} = \{ u \in \mathbb{C} : \varepsilon_{n+1} \leq \Re u \leq \varepsilon_n \text{ and } |\Im u| \leq 2^{-n} \pi \},
\]

as well as the union

\[
T_n = \bigcup_{0 < k < l_n} B_{k,2n},
\]

which is a kind of broken tower above the "basis" \( B_{0,2n} \) of even index.

We also consider, for \( 1 \leq k \leq l_n - 1 \), very thin vertical pipes \( P_{k,n} \) connecting \( B_{k,2n} \) and \( B_{k-1,2n} \), of side lengths \( 4^{-2n} \) and \( 2\pi(1 - 2^{-2n}) \) respectively:

\[
P_{k,n} = P_{0,n} + 2k\pi i,
\]

and we set:

\[
P_n = \bigcup_{1 \leq k < l_n} P_{k,n}
\]

Finally, we set:

\[
F = \left( \bigcup_{n=2}^\infty B_{0,n} \right) \cup \left( \bigcup_{n=1}^\infty T_n \right) \cup \left( \bigcup_{n=1}^\infty P_n \right)
\]
and:

$$\Omega = \hat{F}.$$

Then $\Omega$ is a simply connected domain. Indeed, it is connected thanks to the $B_{0,n}$ and the $P_n$, since the $P_{k,n}$ were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of $n$ out of two in the union of sets $B_{k,n}$ defining $F$.

Let now $f : \mathbb{D} \to \Omega$ be a Riemann map, and $\varphi = e^{-f} : \mathbb{D} \to \mathbb{D}$.

We introduce the Carleson window $W = W(1, h)$ defined as:

$$W(1, h) = \{ z \in \mathbb{D} : 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi h \}.$$ 

This is a variant of the sets $S(1, h)$ of Section 2. We also introduce the Hastings-Luecking half-windows $W'_n$ defined by:

$$W'_n = \{ z \in \mathbb{D} : 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi 2^{-n} \}.$$

We will also need the sets:

$$E_n = e^{-\left( T_n \cup B_{0,2n+1} \cup P_n \right)} = e^{-\left( B_{0,2n} \cup B_{0,2n+1} \cup P_{0,n} \right)},$$

for which one has:

$$\varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_n.$$ 

Next, we consider the measure $\mu = n_{\varphi} dA$, and a Carleson window $W = W(1, h)$ with $h = 2^{-2n}$. We observe that $W'_{2n} \subseteq W$ and claim that:

**Lemma 3.9** One has:

1) $w \in W'_{2n} \implies n_{\varphi}(w) \geq l_N$;

2) $\| \varphi^p \|^2_D \leq p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p 4^{-n}}$. 

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Proof of Lemma 3.9. 1) Let \( w = re^{i\theta} \in W'_{2N} \) with \( 1 - 2^{-2N} < r < 1 - 2^{-2N-1} \) and \( |\theta| < 2\pi - 2tN \). As \(-(\log r + i\theta) \in B_{0,2N}, \) one has \(-(\log r + i\theta) = f(z_0)\) for some \( z_0 \in \mathbb{D}. \) Similarly, \(-(\log r + i\theta) + 2k\pi i, \) for \( 1 \leq k < l_N, \) belongs to \( B_{k,2N} \) and can be written as \( f(z_k)\), with \( z_k \in \mathbb{D}. \) The \( z_k\)'s, \( 0 \leq k < l_N, \) are distinct and satisfy \( \varphi(z_k) = e^{-f(z_k)} = e^{-f(z_0)} = w \) for \( 0 \leq k < l_N, \) thanks to the \( 2\pi i\)-periodicity of the exponential function.

2) We have \( A(E_n) \lesssim e^{-2r_n+2} 4^{-2n} \leq 4^{-2n} \) (the term \( e^{-2r_n+2} \) coming from the Jacobian of \( e^{-z} \)) and we observe that

\[
w \in E_n \quad \Longrightarrow \quad |w|^{2p-2} \leq (1 - 2^{-2n-1})^{2p-2} \lesssim e^{-p4^{-n}}.
\]

It is easy to see that \( n_\varphi(w) \leq l_n \) for \( w \in E_n; \) thus we obtain, forgetting the constant term \( |\varphi(0)|^{2p} \leq 1, \) using (2.5) and keeping in mind the fact that \( n_\varphi(w) = 0 \) for \( w \notin \varphi(\mathbb{D}); \)

\[
\|\varphi^p\|^2 = p^2 \int_{\varphi(\mathbb{D})} |w|^{2p-2} n_\varphi(w) dA(w)
\leq p^2 \left( \sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} n_\varphi(w) dA(w) \right)
\leq p^2 \left( \sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} l_n dA(w) \right)
\lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p4^{-n}},
\]

ending the proof of Lemma 3.9. \( \square \)

End of the proof of Theorem 3.8. Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

\[
\mu(W) \geq \mu(W'_{2N}) = \int_{W'_{2N}} n_\varphi dA \geq l_N A(W'_{2N}) \gtrsim l_N h^2,
\]

which implies that \( \sup_{0<h<1} h^{-2}\mu[W(1,h)] = +\infty \) and shows that \( C_\varphi \) is not bounded on \( \mathcal{D} \) by Zorboska’s criterion ([17], Theorem 1), recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers \( (l_n) \) so as to have \( \|\varphi^p\|_{\mathcal{D}} = O(M_p). \) To this effect, we first observe that, if one sets \( F(x) = x^{2}e^{-x}, \) we have:

\[
p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} = \sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim 1.
\]

Indeed, let \( s \) be the integer such that \( 4^s \leq p < 4^{s+1} \). We have:

\[
\sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim \sum_{n=1}^{s} \frac{4^n}{p} + \sum_{n>s} F(4^{-(n-s-1)}) \lesssim 1 + \sum_{n=0}^{\infty} F(4^{-n}) < \infty,
\]

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where we used that $F$ is increasing on $(0, 1)$ and satisfies $F(x) \lesssim \min(x^2, 1/x)$ for $x > 0$. We finally choose the non-decreasing sequence $(l_n)$ of integers as:

$$l_n = \min(n, M_2^n).$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\|\varphi\|^2_D \lesssim p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} l_n \lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2 4^{-p} \lesssim M_p^2,$$

as desired. This choice of $(l_n)$ gives us an unbounded composition operator on $\mathcal{D}$ such that $\|\varphi\|_D = O(M_p)$, which ends the proof of Theorem 3.8. □

References


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