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Two remarks on composition operators on the Dirichlet space

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We show that the decay of approximation numbers of compact composition operators on the Dirichlet space $D$ can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on $D$ of self-maps of the disk all of whose powers are norm-bounded in $D$.

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1 Introduction

Recall that if $\varphi$ is an analytic self-map of $\mathbb{D}$, a so-called Schur function, the composition operator $C_\varphi$ associated to $\varphi$ is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$ 

The Littlewood subordination principle ([4], p. 30) tells us that $C_\varphi$ maps the Hardy space $H^2$ to itself for every Schur function $\varphi$. Also recall that if $H$ is a Hilbert space and $T: H \to H$ a bounded linear operator, the $n$-th approximation number $a_n(T)$ of $T$ is defined as

$$a_n(T) = \inf \{ \| T - R \| : \text{rank} \; R < n \}, \quad n = 1, 2, \ldots .$$

In [12], working on that Hardy space $H^2$ (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators $C_\varphi$, and proved among other facts the following:

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Theorem 1.1 Let \((\varepsilon_n)_{n \geq 1}\) be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator \(C_\varphi\) on \(H^2\) such that 
\[
\liminf_{n \to \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.
\]
As a consequence, there are composition operators on \(H^2\) which are compact but in no Schatten class.

The last item had been previously proved by Carroll and Cowen ([3]), the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of \(D\) generates a bounded composition operator on \(D\). When this is the case, we will say that \(\varphi\) is a symbol (understanding “of \(D\)”). Note that every symbol is necessarily in \(D\).

In [11], we have performed a similar study on that Dirichlet space \(D\), and established several results on approximation numbers in that new setting, in particular the existence of symbols \(\varphi\) for which \(C_\varphi\) is compact without being in any Schatten class \(S_p\). But we have not been able to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

Theorem 1.2 For every sequence \((\varepsilon_n)_{n \geq 1}\) of positive numbers tending to 0, there exists a compact composition operator \(C_\varphi\) on the Dirichlet space \(D\) such that 
\[
\liminf_{n \to \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.
\]

Turning now to the question of necessary or sufficient conditions for a Schur function \(\varphi\) to be a symbol, we can observe that, since \((z^n/\sqrt{n})_{n \geq 1}\) is an orthonormal sequence in \(D\) and since formally \(C_\varphi(z^n) = \varphi^n\), a necessary condition is as follows:

\[
(1.2) \quad \varphi \text{ is a symbol } \implies \|\varphi^n\|_D = O(\sqrt{n}).
\]

It is worth noting that, for any Schur function, one has:

\[
\varphi \in D \implies \|\varphi^n\|_D = O(n)
\]

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer \(n \geq 1\):

\[
\|\varphi^n\|^2_D = |\varphi(0)|^{2n} + \int_D n^2 |\varphi(z)|^{2(n-1)}|\varphi'(z)|^2 dA(z)
\leq |\varphi(0)|^2 + \int_D n^2 |\varphi'(z)|^2 dA(z) \leq n^2 \|\varphi\|^2_D,
\]
giving the result.
Now, the following sufficient condition was given in [5]:

\[ \| \varphi^n \|_D = O(1) \implies \varphi \text{ is a symbol}. \]  

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

**Theorem 1.3** Let \((M_n)_{n \geq 1}\) be an arbitrary sequence of positive numbers tending to \(\infty\). Then, there exists a Schur function \(\varphi \in \mathcal{D}\) such that:

1) \(\| \varphi^n \|_D = O(M_n)\) as \(n \to \infty\);  
2) \(\varphi\) is not a symbol on \(\mathcal{D}\).

The organization of that paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

### 2 Notation and background.

We denote by \(\mathbb{D}\) the open unit disk of the complex plane and by \(A\) the normalized area measure \(dx\,dy/\pi\) of \(\mathbb{D}\). The unit circle is denoted by \(\partial \mathbb{D}\).

A Schur function is an analytic self-map of \(\mathbb{D}\) and the associated composition operator is defined, formally, by \(C_{\varphi}(f) = f \circ \varphi\). The operator \(C_{\varphi}\) maps the space \(\mathcal{H}ol(\mathbb{D})\) of holomorphic functions on \(\mathbb{D}\) into itself.

The Dirichlet space \(\mathcal{D}\) is the space of analytic functions \(f: \mathbb{D} \to \mathbb{C}\) such that:

\[ \| f \|_D^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty. \]

If \(f(z) = \sum_{n=0}^{\infty} c_n z^n\), one has:

\[ \| f \|_D^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2. \]

Then \(\| \cdot \|_D\) is a norm on \(\mathcal{D}\), making \(\mathcal{D}\) a Hilbert space, and \(\| \cdot \|_{H^2} \leq \| \cdot \|_D\). For further information on the Dirichlet space, the reader may see [1] or [16].

The Bergman space \(\mathcal{B}\) is the space of analytic functions \(f: \mathbb{D} \to \mathbb{C}\) such that:

\[ \| f \|_B^2 := \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < +\infty. \]

If \(f(z) = \sum_{n=0}^{\infty} c_n z^n\), one has \(\| f \|_B^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}\). If \(f \in \mathcal{D}\), one has by definition:

\[ \| f \|_D^2 = \| f' \|_B^2 + |f(0)|^2. \]
Recall that, whereas every Schur function \( \varphi \) generates a bounded composition operator \( C_\varphi \) on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [14], Proposition 3.12, for instance).

We denote by \( b_n(T) \) the \( n \)-th Bernstein number of the operator \( T : H \to H \), namely:

\[
(2.3) \quad b_n(T) = \sup_{\dim E=n} \left( \inf_{f \in S_E} \|Tx\| \right)
\]

where \( S_E \) denotes the unit sphere of \( E \). It is easy to see ([11]) that

\[
b_n(T) = a_n(T) \quad \text{for all } n \geq 1.
\]

(recall that the approximation numbers are defined in (1.1)).

If \( \varphi \) is a Schur function, let

\[
(2.4) \quad n_\varphi(w) = \# \{ z \in \mathbb{D} ; \varphi(z) = w \} \geq 0
\]

be the associated counting function. If \( f \in \mathcal{D} \) and \( g = f \circ \varphi \), the change of variable formula provides us with the useful following equation ([17], [11]):

\[
(2.5) \quad \int_{\mathbb{D}} |g'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w)
\]

(the integrals might be infinite). In those terms, a necessary and sufficient condition for \( \varphi \) to be a symbol is as follows ([17], Theorem 1). Let:

\[
(2.6) \quad \rho_\varphi(h) = \sup_{\xi \in T} \int_{S(\xi,h)} n_\varphi dA,
\]

where \( S(\xi,h) = \mathbb{D} \cap D(\xi,h) \) is the Carleson window centered at \( \xi \) and of size \( h \).

Then \( \varphi \) is a symbol if and only if:

\[
(2.7) \quad \sup_{0<h<1} \frac{1}{h^2} \rho_\varphi(h) < \infty.
\]

This is not difficult to prove. In view of (2.5), the boundedness of \( C_\varphi \) amounts to the existence of a constant \( C \) such that:

\[
\int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |f'(z)|^2 dA(z), \quad \forall f \in \mathcal{D}.
\]

Since \( f' = h \) runs over \( \mathfrak{B} \) as \( f \) runs over \( \mathcal{D} \), and with equal norms, the above condition reads:

\[
\int_{\mathbb{D}} |h(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |h(z)|^2 dA(z), \quad \forall h \in \mathfrak{B}.
\]

This exactly means that the measure \( n_\varphi dA \) is a Carleson measure for \( \mathfrak{B} \). Such measures have been characterized in [7] and that characterization gives (2.7).
But this condition is very abstract and difficult to test, and sometimes more “concrete” sufficient conditions are desirable. In [11], we proved that, even if the Schur function extends continuously to $\mathbb{D}$, no Lipschitz condition of order $\alpha$, $0 < \alpha < 1$, on $\varphi$ is sufficient for ensuring that $\varphi$ is a symbol. It is worth noting that the limiting case $\alpha = 1$, so restrictive it is, guarantees the result.

**Proposition 2.1** Suppose that the Schur function $\varphi$ is in the analytic Lipschitz class on the unit disk, i.e. satisfies:

$$|\varphi(z) - \varphi(w)| \leq C|z - w|, \quad \forall z, w \in \mathbb{D}.$$ 

Then $C_\varphi$ is bounded on $\mathbb{D}$.

**Proof.** Let $f \in \mathbb{D}$; one has:

$$\|C_\varphi(f)\|_B^2 = |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z)$$

$$\leq |f(\varphi(0))|^2 + \|\varphi'\|_\infty^2 \int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z).$$

This integral is nothing but $\|C_\varphi(f')\|_B^2$ and hence, since $C_\varphi$ is bounded on the Bergman space $\mathcal{B}$, we have, for some constant $K_1$:

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z) \leq K_1^2 \|f'\|_B^2 \leq K_1^2 \|f\|_D^2.$$ 

On the other hand,

$$|f(\varphi(0))| \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{H^2} \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{\mathbb{D}},$$

and we get

$$\|C_\varphi(f)\|_D^2 \leq K_2^2 \|f\|_D^2,$$

with $K_2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1}$. \hfill $\Box$

## 3 Proof of Theorem 1.2

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

**Theorem 3.1** For every sequence $(\epsilon_n)$ of positive numbers with limit 0, there exists a compact composition operator $C_\varphi$ on $\mathbb{D}$ such that

$$\liminf_{n \to \infty} \frac{a_n(C_\varphi)}{\epsilon_n} > 0.$$ 

Before entering really in the proof, we may remark that, without loss of generality, by replacing $\epsilon_n$ with $\inf\{2^{-n}, \sup_{k \geq n} \epsilon_k\}$, we can, and do, assume that $(\epsilon_n)_n$ decreases and $\epsilon_1 \leq 2^{-8}$.

Moreover, we can assume that $(\epsilon_n)_n$ decreases “slowly”, as said in the following lemma.
Lemma 3.2 Let \((\varepsilon_i)\) be a decreasing sequence with limit zero and let \(0 < \rho < 1\). Then, there exists another sequence \((\hat{\varepsilon}_i)\), decreasing with limit zero, such that \(\hat{\varepsilon}_i \geq \varepsilon_i\) and \(\hat{\varepsilon}_{i+1} \geq \rho \hat{\varepsilon}_i\), for every \(i \geq 1\).

Proof. We define inductively \(\hat{\varepsilon}_i\) by \(\hat{\varepsilon}_1 = \varepsilon_1\) and

\[
\hat{\varepsilon}_{i+1} = \max(\rho \hat{\varepsilon}_i, \varepsilon_{i+1}).
\]

It is seen by induction that \(\hat{\varepsilon}_i \geq \varepsilon_i\) and that \(\hat{\varepsilon}_i\) decreases to a limit \(a \geq 0\). If \(\hat{\varepsilon}_i = \varepsilon_i\) for infinitely many indices \(i\), we have \(a = 0\). In the opposite case, \(\hat{\varepsilon}_{i+1} = \rho \hat{\varepsilon}_i\) from some index \(i_0\) onwards, and again \(a = 0\) since \(\rho < 1\). \(\square\)

We will take \(\rho = 1/2\) and assume for the sequel that \(\varepsilon_{i+1} \geq \varepsilon_i/2\).

Proof of Theorem 3.1. We first construct a subdomain \(\Omega = \Omega_\theta\) of \(\mathbb{D}\) defined by a cuspidal inequality:

\[
\Omega = \{z = x + iy \in \mathbb{D}; \ |y| < \theta(1-x), \ 0 < x < 1\},
\]

where \(\theta: [0,1] \to [0,1]\) is a continuous increasing function such that

\[
\theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1 - x.
\]

Note that since \(1 - x \leq \sqrt{1 - x^2}\), the condition \(|y| < \theta(1-x)\) implies that \(z = x + iy \in \mathbb{D}\). Note also that \(1 \in \overline{\Omega}\) and that \(\Omega\) is a Jordan domain.

We introduce a parameter \(\delta\) with \(\varepsilon_1 \leq \delta \leq 1 - \varepsilon_1\). We put:

\[
\theta(\delta^j) = \varepsilon_j \delta^j
\]

and we extend \(\theta\) to an increasing continuous function from \((0,1)\) into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

\[
\theta(h) \leq h \quad \text{and} \quad \theta(h) = o(h) \text{ as } h \to 0.
\]

Indeed, if \(\delta^j+1 \leq h < \delta^j\), we have \(\theta(h)/h \leq \theta(\delta^j)/\delta^j+1 = \varepsilon_j/\delta^j\), which is \(\leq \varepsilon_1/\delta \leq 1\) and which tends to 0 with \(\delta\).

We define now \(\varphi = \varphi_\theta\): \(\mathbb{D} \to \overline{\Omega}\) as a continuous map which is a Riemann map from \(\mathbb{D}\) onto \(\Omega\), and with \(\varphi(1) = 1\) (a cusp-type map). Since \(\varphi\) is univalent, one has \(n_\varphi = 1\), and since \(\Omega\) is bounded, \(\varphi\) defines a symbol on \(\mathcal{D}\), by (2.7). Moreover, (3.4) implies that \(A[S(\xi,h)] \leq \theta(h)\) for every \(\xi \in \mathbb{T}\); hence, \(\rho_\varphi\) being defined in (2.6), one has \(\rho_\varphi(h) = o(h^2)\) as \(h \to 0^+\). In view of [17], this little-oh condition guarantees the compactness of \(C_\varphi: \mathcal{D} \to \mathcal{D}\).

It remains to miniorate its approximation numbers.

The measure \(\mu = n_\varphi \, dA\) is a Carleson measure for the Bergman space \(\mathcal{B}\), and it was proved in [10] that \(C_\varphi^* C_\varphi\) is unitarily equivalent to the Toeplitz operator \(T_\mu = T_\mu^* T_\mu: \mathcal{B} \to \mathcal{B}\) defined by:

\[
T_\mu f(z) = \int_\mathbb{D} \frac{f(w)}{(1 - \overline{w}z)^2} \, dA(w) = \int_\mathbb{D} f(w) K_w(z) \, dA(w),
\]

(3.5)
where \( I_\mu : \mathcal{B} \to L^2(\mu) \) is the canonical inclusion and \( K_w \) the reproducing kernel of \( \mathcal{B} \) at \( w \), i.e. \( K_w(z) = \frac{1}{(1-wz)^2} \).

Actually, we can get rid of the analyticity constraint in considering, instead of \( T_\mu \), the operator \( S_\mu = I_\mu I_\mu^* : L^2(\mu) \to L^2(\mu) \), which corresponds to the arrows:

\[
L^2(\mu) \xrightarrow{I_\mu^*} \mathcal{B} \xrightarrow{I_\mu} L^2(\mu).
\]

We use the relation (3.5) which implies:

\[
(3.6) \quad a_n(C_\varphi) = a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)}.
\]

We set:

\[
(3.7) \quad c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \varepsilon_j \delta^j.
\]

One has \( r_j = \varepsilon_j(1 - c_j)/2 \).

**Lemma 3.3** The disks \( \Delta_j = D(c_j, r_j), \ j \geq 1, \) are disjoint and contained in \( \Omega \).

**Proof.** If \( z = x + iy \in \Delta_j \), then \( 1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \varepsilon_j/2) = 2\delta^j(1 - \varepsilon_j/2) \geq \delta^j \) and \( |y| < r_j = \theta(\delta^j) \); hence \( |y| < \theta(\delta^j) \leq \theta(1 - x) \) and \( z \in \Omega \). On the other hand, \( c_{j+1} - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta^j) \delta^j \geq 2\varepsilon_j \delta^j \geq 2\varepsilon_j \delta^j = 2r_j > r_j + r_{j+1} \); hence \( \Delta_j \cap \Delta_{j+1} = \emptyset \). \( \square \)

We will next need a description of \( S_\mu \).

**Lemma 3.4** For every \( g \in L^2(\mu) \) and every \( z \in \mathbb{D} \):

\[
(3.8) \quad I_\mu^* g(z) = \int_{\Omega} \frac{g(w)}{(1 - wz)^2} dA(w)
\]

\[
(3.9) \quad S_\mu g(z) = \left( \int_{\Omega} \frac{g(w)}{(1 - wz)^2} dA(w) \right) \mathbb{I}_{\Omega}(z).
\]

**Proof.** \( K_w \) being the reproducing kernel of \( \mathcal{B} \), we have for any pair of functions \( f \in \mathcal{B} \) and \( g \in L^2(\mu) \):

\[
\langle I_\mu^* g, f \rangle_\mathcal{B} = \langle g, I_\mu f \rangle_{L^2(\mu)} = \int_{\Omega} g(w)f(w) dA(w) = \int_{\Omega} g(w) \langle K_w, f \rangle_\mathcal{B} dA(w)
\]

so that \( I_\mu^* g = \int_{\Omega} g(w)K_w dA(w), \) giving the result. \( \square \)

In the rest of the proof, we fix a positive integer \( n \) and put:

\[
(3.10) \quad f_j = \frac{1}{r_j} \mathbb{1}_{\Delta_j}, \quad j = 1, \ldots, n.
\]

Let:

\[
E = \text{span} \{f_1, \ldots, f_n\}.
\]
This is an $n$-dimensional subspace of $L^2(\mu)$.

The $\Delta_j$’s being disjoint, the sequence $(f_1, \ldots, f_n)$ is orthonormal in $L^2(\mu)$. Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 \, d\mu = \int f_j^2 \, n_\varphi \, dA = \int_{\Delta_j} \frac{1}{r_j^2} \, dA = 1.$$ 

We now estimate from below the Bernstein numbers of $I_\mu^\ast$. To that effect, we compute the scalar products $m_{i,j} = \langle I_\mu^\ast(f_i), I_\mu^\ast(f_j) \rangle$. One has:

$$m_{i,j} = \langle f_i, S_\mu(f_j) \rangle = \frac{1}{r_ir_j} \int_{\Delta_i \times \Delta_j} 1 \left( \frac{1}{1 - wz} \right)^2 dA(z) dA(w).$$

**Lemma 3.5** We have

(3.11) $m_{i,i} \geq \frac{\varepsilon_i^2}{32}$, and $|m_{i,j}| \leq \varepsilon_i \varepsilon_j \delta^{j-i}$ for $i < j$.

**Proof.** Set $\varepsilon_i' = \frac{\varepsilon_i}{1+\varepsilon_i}$. One has $\frac{4}{1+\varepsilon_i} \leq \varepsilon_i' \leq \frac{8}{1+\varepsilon_i}$. We observe that (recall that $A(\Delta_i) = r_i^2$):

$$m_{i,i} - \varepsilon_i'^2 = \frac{1}{r_i^2} \int_{\Delta_i \times \Delta_i} \left[ \frac{1}{(1 - wz)^2} - \frac{1}{(1 - \varepsilon_i'^2)^2} \right] dA(z) dA(w).$$

Therefore, using the fact that, for $z \in \Delta_i$ and $w \in D$:

$$|1 - wz| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \varepsilon_i \left( \frac{1 - c_i}{2} \right) \geq (1 - c_i) \left( 1 - \frac{\varepsilon_i}{2} \right) \geq \frac{1 - c_i}{2}$$

and then the mean-value theorem, we get:

$$|m_{i,i} - \varepsilon_i'^2| \leq \frac{1}{r_i^2} \int_{\Delta_i \times \Delta_i} \left| \frac{1}{(1 - wz)^2} - \frac{1}{(1 - \varepsilon_i'^2)^2} \right| dA(z) dA(w)$$

$$\leq \frac{1}{r_i^2} \int_{\Delta_i \times \Delta_i} \frac{32 r_i}{(1 - c_i)^3} dA(z) dA(w)$$

$$= \frac{32 r_i^3}{(1 - c_i)^3} \leq 32 \times 8 \varepsilon_i'^3 \leq \frac{\varepsilon_i'^2}{2},$$

since $\varepsilon_i \leq \varepsilon_1 \leq 2^{-8}$ implies that $\varepsilon_i' \leq 1/(32 \times 16)$. This gives us the lower bound $m_{i,i} \geq \varepsilon_i'^2/2 \geq \varepsilon_i^2/32$. 

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Next, for \( i < j \):
\[
|m_{i,j}| \leq \frac{1}{r_i r_j} \int_{\Delta_i \times \Delta_j} \frac{1}{(1 - w z)^2} dA(z) dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1 - c_i)^2 r_i r_j} = \frac{4 \varepsilon_i \varepsilon_j \delta^{i+j}}{4 \delta^{4n}} = \varepsilon_i \varepsilon_j \delta^{j-i},
\]
and that ends the proof of Lemma 3.5.

We further write the \( n \times n \) matrix \( M = (m_{i,j})_{1 \leq i,j \leq n} \) as \( M = D + R \) where \( D \) is the diagonal matrix \( m_i = m_{i,i} \) with \( m_i \geq \frac{\varepsilon^2}{64} \), \( 1 \leq i \leq n \). Observe that \( M \) is nothing but the matrix of \( S_\mu \) on the orthonormal basis \((f_1, \ldots, f_n)\) of \( E \), so that we can identify \( M \) and \( S_\mu \) on \( E \).

Now the following lemma will end the proof of Theorem 3.1.

**Lemma 3.6** If \( \delta \leq 1/200 \), we have:
\begin{equation}
\| D^{-1} R \| \leq 1/2. \tag{3.12}
\end{equation}

Indeed, by the ideal property of Bernstein numbers, Neumann’s lemma and the relations:
\[
M = D(I + D^{-1} R), \quad \text{and} \quad D = MQ \quad \text{with} \quad \| Q \| \leq 2,
\]
we have \( b_n(D) \leq b_n(M) \| Q \| \leq 2 b_n(M) \), that is:
\[
a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\varepsilon^2}{64},
\]
since the \( n \) first approximation numbers of the diagonal matrix \( D \) (the matrices being viewed as well as operators on the Hilbertian space \( C^n \) with its canonical basis) are \( m_{1,1}, \ldots, m_{n,n} \). It follows that, using (3.6):
\begin{equation}
a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\varepsilon_n}{8}. \tag{3.13}
\end{equation}

In view of (3.6), we have as well \( a_n(C_\mu) \geq \varepsilon_n/8 \), and we are done. \( \square \)

**Proof of Lemma 3.6.** Write \( M = (m_{i,j}) = D(I + N) \) with \( N = D^{-1} R \). One has:
\begin{equation}
N = (\nu_{i,j}), \quad \text{with} \quad \nu_{i,i} = 0 \quad \text{and} \quad \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i. \tag{3.14}
\end{equation}

We shall show that \( \| N \| \leq 1/2 \) by using the (unweighted) Schur test, which we recall ([6], Problem 45):

**Proposition 3.7** Let \((a_{i,j})_{1 \leq i,j \leq n}\) be a matrix of complex numbers. Suppose that there exist two positive numbers \( \alpha, \beta > 0 \) such that:
1. \( \sum_{j=1}^n |a_{i,j}| \leq \alpha \) for all \( i \);
2. \( \sum_{i=1}^n |a_{i,j}| \leq \beta \) for all \( j \).

Then, the (Hilbertian) norm of this matrix satisfies \( \| A \| \leq \sqrt{\alpha \beta} \).
It is essential for our purpose to note that:

\[(3.15)\quad i < j \iff |\nu_{i,j}| \leq 32 \delta^{j-i},\]

\[(3.16)\quad i > j \iff |\nu_{i,j}| \leq 32 (2\delta)^{i-j} .\]

Indeed, we see from (3.11) and (3.14) that, for \(i < j \):

\[|\nu_{i,j}| = \left| \frac{m_{i,j}}{m_{i,i}} \right| \leq 32 \varepsilon_i \varepsilon_j \varepsilon_i^{-2} \delta^{j-i} \leq 32 \delta^{j-i},\]

since \(\varepsilon_j \leq \varepsilon_i\). Secondly, using \(\varepsilon_j/\varepsilon_i \leq 2^{i-j}\) for \(i > j\) (recall that we assumed that \(\varepsilon_{k+1} \geq \varepsilon_k/2\), as well as \(|m_{i,j}| = |m_{j,i}|\), we have, for \(i > j\):

\[|\nu_{i,j}| = \left| \frac{m_{j,i}}{m_{i,i}} \right| \leq 32 \frac{\varepsilon_i}{\varepsilon_i} \delta^{j-i} \leq 32 (2\delta)^{i-j}.\]

Now, for fixed \(i\), (3.15) gives:

\[\sum_{j=1}^{n} |\nu_{i,j}| = \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32 \left( \sum_{j>i} \delta^{j-i} + \sum_{j<i} (2\delta)^{i-j} \right)
\leq 32 \left( \frac{\delta}{1-\delta} + \frac{2 \delta}{1-2\delta} \right) \leq 32 \frac{3 \delta}{1-2\delta} \leq \frac{96}{198} \leq \frac{1}{2},\]

since \(\delta \leq 1/200\). Hence:

\[(3.17)\quad \sup_{i} \left( \sum_{j} |\nu_{i,j}| \right) \leq 1/2.\]

In the same manner, but using (3.16) instead of (3.15), one has:

\[(3.18)\quad \sup_{j} \left( \sum_{i} |\nu_{i,j}| \right) \leq 1/2.\]

Now, (3.17), (3.18) and the Schur criterion recalled above give:

\[\|N\| \leq \sqrt{1/2 \times 1/2} = 1/2,\]

as claimed. \(\square\)

**Remark.** We could reverse the point of view in the preceding proof: start from \(\theta\) and see what lower bound for \(a_n(C_{\phi})\) emerges. For example, if \(\theta(h) \approx h\) as is the case for lens maps (see [11]), we find again that \(a_n(C_{\phi}) \geq \delta_0 > 0\) and that \(C_{\phi}\) is not compact. But if \(\theta(h) \approx h^{1+\alpha}\) with \(\alpha > 0\), the method only gives \(a_n(C_{\phi}) \gtrsim e^{-\alpha n}\) (which is always true: see [11], Theorem 2.1), whereas the methods of [11] easily give \(a_n(C_{\phi}) \gtrsim e^{-\alpha \sqrt{n}}\). Therefore, this \(\mu\)-method seems to be sharp when we are close to non-compactness, and to be beaten by those of [11] for “strongly compact” composition operators.
3.1 Optimality of the EKSY result

El Fallah, Kellay, Shabankhah and Youssfi proved in [5] the following: if $\varphi$ is a Schur function such that $\varphi \in \mathcal{D}$ and $\|\varphi^p\|_{\mathcal{D}} = O(1)$ as $p \to \infty$, then $\varphi$ is a symbol on $\mathcal{D}$. We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

**Theorem 3.8** Let $(M_p)_{p \geq 1}$ be an arbitrary sequence of positive numbers such that $\lim_{p \to \infty} M_p = \infty$. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:

1) $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$ as $p \to \infty$;
2) $\varphi$ is not a symbol on $\mathcal{D}$.

**Remark.** We first observe that we cannot replace $\lim$ by $\lim \sup$ in Theorem 3.8. Indeed, since $\varphi \in \mathcal{D}$, the measure $\mu = n\varphi \, dA$ is finite, and

$$\|\varphi^p\|_{\mathcal{D}}^2 = p^2 \int_{\mathbb{B}} |w|^{2p-2} \, d\mu(w) \geq c p^2 \left( \int_{\mathbb{B}} |w|^2 \, d\mu(w) \right)^{p-1} \geq c \delta^p,$$

where $c$ and $\delta$ are positive constants.

**Proof of Theorem 3.8.** We may, and do, assume that $(M_p)$ is non-decreasing and integer-valued. Let $(l_n)_{n \geq 1}$ be a non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let $\Omega$ be the subdomain of the right half-plane $\mathbb{C}_0$ defined as follows. We set:

$$\varepsilon_n = - \log(1 - 2^{-n}) \sim 2^{-n},$$

and we consider the (essentially) disjoint boxes $(k = 0, 1, \ldots)$:

$$B_{k,n} = B_{0,n} + 2k\pi i,$$

with:

$$B_{0,n} = \{ u \in \mathbb{C} : \varepsilon_{n+1} \leq \Re u \leq \varepsilon_n \text{ and } |\Im u| \leq 2^{-n}\pi \},$$

as well as the union

$$T_n = \bigcup_{0 < k < l_n} B_{k,2n},$$

which is a kind of broken tower above the "basis" $B_{0,2n}$ of even index.

We also consider, for $1 \leq k \leq l_n - 1$, very thin vertical pipes $P_{k,n}$ connecting $B_{k,2n}$ and $B_{k-1,2n}$, of side lengths $4^{-2n}$ and $2\pi(1 - 2^{-2n})$ respectively:

$$P_{k,n} = P_{0,n} + 2k\pi i,$$

and we set:

$$P_n = \bigcup_{1 \leq k < l_n} P_{k,n}.$$

Finally, we set:

$$F = \left( \bigcup_{n=2}^\infty B_{0,n} \right) \cup \left( \bigcup_{n=1}^\infty T_n \right) \cup \left( \bigcup_{n=1}^\infty P_n \right)$$
and:
\[ \Omega = \mathring{\mathcal{G}}. \]

Then \( \Omega \) is a simply connected domain. Indeed, it is connected thanks to the \( B_{0,n} \) and the \( P_{n} \), since the \( P_{k,n} \) were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of \( n \) out of two in the union of sets \( B_{k,n} \) defining \( \mathcal{G} \).

Let now \( f : \mathbb{D} \to \Omega \) be a Riemann map, and \( \varphi = e^{-f} : \mathbb{D} \to \mathbb{D} \).

We introduce the Carleson window \( W = W(1, h) \) defined as:
\[ W(1, h) = \{ z \in \mathbb{D} ; 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi h \}. \]
This is a variant of the sets \( S(1, h) \) of Section 2. We also introduce the Hastings-Luecking half-windows \( W'_{n} \) defined by:
\[ W'_{n} = \{ z \in \mathbb{D} ; 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi 2^{-n} \}. \]
We will also need the sets:
\[ E_{n} = e^{- \left( T_{n} \cup \mathring{B}_{0,2n+1} \cup P_{n} \right)} = e^{- \left( \mathring{B}_{0,2n} \cup \mathring{B}_{0,2n+1} \cup P_{0,n} \right)}, \]
for which one has:
\[ \varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_{n}. \]

Next, we consider the measure \( \mu = n_{\varphi} \, dA \), and a Carleson window \( W = W(1, h) \) with \( h = 2^{-2N} \). We observe that \( W'_{2N} \subseteq W \) and claim that:

**Lemma 3.9** One has:
1) \( w \in W'_{2N} \implies n_{\varphi}(w) \geq l_{N} \);
2) \( \| \varphi^p \|_{D}^{2} \lesssim p^2 \sum_{n=1}^{\infty} l_{n} 16^{-n} e^{-p 4^{-n}}. \)
Proof of Lemma 3.9. 1) Let \( w = re^{i\theta} \in W'_{2N} \) with \( 1 - 2^{-2N} < r < 1 - 2^{-2N-1} \) and \( |\theta| < \pi 2^{-2N} \). As \( -(\log r + i\theta) \in B_{0,2N} \), one has \( -(\log r + i\theta) = f(z_0) \) for some \( z_0 \in \mathbb{D} \). Similarly, \( -(\log r + i\theta) + 2k\pi i \), for \( 1 \leq k < l_N \), belongs to \( B_{k,2N} \) and can be written as \( f(z_k) \), with \( z_k \in \mathbb{D} \). The \( z_k \)'s, \( 0 \leq k < l_N \), are distinct and satisfy \( \varphi(z_k) = e^{-f(z_k)} = e^{-f(z_0)} = w \) for \( 0 \leq k < l_N \), thanks to the \( 2\pi i \)-periodicity of the exponential function.

2) We have \( A(E_n) \lesssim e^{-2\pi n+2} e^{-2n} \lesssim 4^{-2n} \) (the term \( e^{-2\pi n+2} \) coming from the Jacobian of \( e^{-\varphi} \)) and we observe that

\[
|w|^{2p-2} \leq (1 - 2^{-2n-1})^{2p-2} \lesssim e^{-p4^{-n}}.
\]

It is easy to see that \( n_\varphi(w) \leq l_n \) for \( w \in E_n \); thus we obtain, forgetting the constant term \( |\varphi(0)|^{2p} \leq 1 \), using (2.5) and keeping in mind the fact that \( n_\varphi(w) = 0 \) for \( w \notin \varphi(\mathcal{D}) \):

\[
\|\varphi^p\|_D^2 = p^2 \int_{\varphi(\mathcal{D})} |w|^{2p-2} n_\varphi(w) \, dA(w)
\]

\[
\leq p^2 \left( \sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} n_\varphi(w) \, dA(w) \right)
\]

\[
\leq p^2 \left( \sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} l_n \, dA(w) \right)
\]

\[
\lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p4^{-n}},
\]

ending the proof of Lemma 3.9. \( \square \)

End of the proof of Theorem 3.8. Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

\[
\mu(W) \geq \mu(W_{2N}) = \int_{W_{2N}} n_\varphi \, dA \geq l_N A(W_{2N}') \gtrsim l_N h^2,
\]

which implies that \( \sup_{0<h<1} h^{-2}\mu(W(1,h)) = +\infty \) and shows that \( C_\varphi \) is not bounded on \( \mathcal{D} \) by Zorboska’s criterion ([17], Theorem 1), recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers \( (l_n) \) so as to have \( \|\varphi^p\|_D = O(M_p) \). To this effect, we first observe that, if one sets \( F(x) = x^2 e^{-x} \), we have:

\[
p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} = \sum_{n=1}^{\infty} F\left( \frac{p}{4^n} \right) \lesssim 1.
\]

Indeed, let \( s \) be the integer such that \( 4^s \leq p < 4^{s+1} \). We have:

\[
\sum_{n=1}^{\infty} F\left( \frac{p}{4^n} \right) \lesssim \sum_{n=1}^{s} \frac{4^n}{p} + \sum_{n>s} F(4^{-n-s-1}) \lesssim 1 + \sum_{n=0}^{\infty} F(4^{-n}) < \infty,
\]

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where we used that $F$ is increasing on $(0, 1)$ and satisfies $F(x) \lesssim \min(x^2, 1/x)$ for $x > 0$. We finally choose the non-decreasing sequence $(l_n)$ of integers as:

$$l_n = \min(n, M^n).$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\|\varphi^p\|_D^2 \lesssim p^2 \sum_{n=1}^\infty 16^{-n} e^{-p^{4-n}l_n} \lesssim l_p + p^2 \sum_{n>p} 16^{-n} l_n \lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2 4^{-p} \lesssim M^2_p,$$

as desired. This choice of $(l_n)$ gives us an unbounded composition operator on $D$ such that $\|\varphi^p\|_D = O(M^p)$, which ends the proof of Theorem 3.8. □

References


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