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Two remarks on composition operators on the Dirichlet space

*Daniel Li, Hervé Queffélec,
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Abstract. *We show that the decay of approximation numbers of compact composition operators on the Dirichlet space \mathcal{D} can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on \mathcal{D} of self-maps of the disk all of whose powers are norm-bounded in \mathcal{D} .*

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1 Introduction

Recall that if φ is an analytic self-map of \mathbb{D} , a so-called *Schur function*, the composition operator C_φ associated to φ is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$

The Littlewood subordination principle ([4], p. 30) tells us that C_φ maps the Hardy space H^2 to itself for every Schur function φ . Also recall that if H is a Hilbert space and $T: H \rightarrow H$ a bounded linear operator, the n -th approximation number $a_n(T)$ of T is defined as

$$(1.1) \quad a_n(T) = \inf\{\|T - R\|; \text{rank } R < n\}, \quad n = 1, 2, \dots$$

In [12], working on that Hardy space H^2 (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators C_φ , and proved among other facts the following:

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Theorem 1.1 *Let $(\varepsilon_n)_{n \geq 1}$ be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator C_φ on H^2 such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

As a consequence, there are composition operators on H^2 which are compact but in no Schatten class.

The last item had been previously proved by Carroll and Cowen ([3]), the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of \mathbb{D} generates a bounded composition operator on \mathcal{D} . When this is the case, we will say that φ is a *symbol* (understanding “of \mathcal{D} ”). Note that every symbol is necessarily in \mathcal{D} .

In [11], we have performed a similar study on that Dirichlet space \mathcal{D} , and established several results on approximation numbers in that new setting, in particular the existence of symbols φ for which C_φ is compact without being in any Schatten class S_p . But we have not been able in [11] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

Theorem 1.2 *For every sequence $(\varepsilon_n)_{n \geq 1}$ of positive numbers tending to 0, there exists a compact composition operator C_φ on the Dirichlet space \mathcal{D} such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Turning now to the question of necessary or sufficient conditions for a Schur function φ to be a symbol, we can observe that, since $(z^n/\sqrt{n})_{n \geq 1}$ is an orthonormal sequence in \mathcal{D} and since formally $C_\varphi(z^n) = \varphi^n$, a necessary condition is as follows:

$$(1.2) \quad \varphi \text{ is a symbol} \implies \|\varphi^n\|_{\mathcal{D}} = O(\sqrt{n}).$$

It is worth noting that, for any Schur function, one has:

$$\varphi \in \mathcal{D} \implies \|\varphi^n\|_{\mathcal{D}} = O(n)$$

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer $n \geq 1$:

$$\begin{aligned} \|\varphi^n\|_{\mathcal{D}}^2 &= |\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |\varphi(z)|^{2(n-1)} |\varphi'(z)|^2 dA(z) \\ &\leq |\varphi(0)|^2 + \int_{\mathbb{D}} n^2 |\varphi'(z)|^2 dA(z) \leq n^2 \|\varphi\|_{\mathcal{D}}^2, \end{aligned}$$

giving the result.

Now, the following sufficient condition was given in [5]:

$$(1.3) \quad \|\varphi^n\|_{\mathcal{D}} = O(1) \implies \varphi \text{ is a symbol.}$$

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

Theorem 1.3 *Let $(M_n)_{n \geq 1}$ be an arbitrary sequence of positive numbers tending to ∞ . Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

- 1) $\|\varphi^n\|_{\mathcal{D}} = O(M_n)$ as $n \rightarrow \infty$;
- 2) φ is not a symbol on \mathcal{D} .

The organization of that paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

2 Notation and background.

We denote by \mathbb{D} the open unit disk of the complex plane and by A the normalized area measure $dx dy/\pi$ of \mathbb{D} . The unit circle is denoted by $\mathbb{T} = \partial\mathbb{D}$. The notation $A \lesssim B$ indicates that $A \leq cB$ for some positive constant c .

A Schur function is an analytic self-map of \mathbb{D} and the associated composition operator is defined, formally, by $C_\varphi(f) = f \circ \varphi$. The operator C_φ maps the space $\mathcal{H}ol(\mathbb{D})$ of holomorphic functions on \mathbb{D} into itself.

The Dirichlet space \mathcal{D} is the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$(2.1) \quad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has:

$$(2.2) \quad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.$$

Then $\|\cdot\|_{\mathcal{D}}$ is a norm on \mathcal{D} , making \mathcal{D} a Hilbert space, and $\|\cdot\|_{H^2} \leq \|\cdot\|_{\mathcal{D}}$. For further information on the Dirichlet space, the reader may see [1] or [16].

The Bergman space \mathfrak{B} is the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that:

$$\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2 dA(z) < +\infty.$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}$. If $f \in \mathcal{D}$, one has by definition:

$$\|f\|_{\mathcal{D}}^2 = \|f'\|_{\mathfrak{B}}^2 + |f(0)|^2.$$

Recall that, whereas every Schur function φ generates a bounded composition operator C_φ on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [14], Proposition 3.12, for instance).

We denote by $b_n(T)$ the n -th *Bernstein number* of the operator $T: H \rightarrow H$, namely:

$$(2.3) \quad b_n(T) = \sup_{\dim E=n} \left(\inf_{f \in S_E} \|Tx\| \right)$$

where S_E denotes the unit sphere of E . It is easy to see ([11]) that

$$b_n(T) = a_n(T) \quad \text{for all } n \geq 1.$$

(recall that the approximation numbers are defined in (1.1)).

If φ is a Schur function, let

$$(2.4) \quad n_\varphi(w) = \#\{z \in \mathbb{D}; \varphi(z) = w\} \geq 0$$

be the associated *counting function*. If $f \in \mathcal{D}$ and $g = f \circ \varphi$, the change of variable formula provides us with the useful following equation ([17], [11]):

$$(2.5) \quad \int_{\mathbb{D}} |g'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w)$$

(the integrals might be infinite). In those terms, a necessary and sufficient condition for φ to be a symbol is as follows ([17], Theorem 1). Let:

$$(2.6) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} \int_{S(\xi, h)} n_\varphi dA$$

where $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$ is the Carleson window centered at ξ and of size h . Then φ is a symbol if and only if:

$$(2.7) \quad \sup_{0 < h < 1} \frac{1}{h^2} \rho_\varphi(h) < \infty.$$

This is not difficult to prove. In view of (2.5), the boundedness of C_φ amounts to the existence of a constant C such that:

$$\int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |f'(z)|^2 dA(z), \quad \forall f \in \mathcal{D}.$$

Since $f' = h$ runs over \mathfrak{B} as f runs over \mathcal{D} , and with equal norms, the above condition reads:

$$\int_{\mathbb{D}} |h(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |h(z)|^2 dA(z), \quad \forall h \in \mathfrak{B}.$$

This exactly means that the measure $n_\varphi dA$ is a Carleson measure for \mathfrak{B} . Such measures have been characterized in [7] and that characterization gives (2.7).

But this condition is very abstract and difficult to test, and sometimes more “concrete” sufficient conditions are desirable. In [11], we proved that, even if the Schur function extends continuously to $\overline{\mathbb{D}}$, no Lipschitz condition of order α , $0 < \alpha < 1$, on φ is sufficient for ensuring that φ is a symbol. It is worth noting that the limiting case $\alpha = 1$, so restrictive it is, guarantees the result.

Proposition 2.1 *Suppose that the Schur function φ is in the analytic Lipschitz class on the unit disk, i.e. satisfies:*

$$|\varphi(z) - \varphi(w)| \leq C |z - w|, \quad \forall z, w \in \mathbb{D}.$$

Then C_φ is bounded on \mathcal{D} .

Proof. Let $f \in \mathcal{D}$; one has:

$$\begin{aligned} \|C_\varphi(f)\|_{\mathcal{D}}^2 &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) \\ &\leq |f(\varphi(0))|^2 + \|\varphi'\|_\infty^2 \int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z). \end{aligned}$$

This integral is nothing but $\|C_\varphi(f')\|_{\mathfrak{B}}^2$ and hence, since C_φ is bounded on the Bergman space \mathfrak{B} , we have, for some constant K_1 :

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z) \leq K_1^2 \|f'\|_{\mathfrak{B}}^2 \leq K_1^2 \|f\|_{\mathcal{D}}^2.$$

On the other hand,

$$|f(\varphi(0))| \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{H^2} \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{\mathcal{D}},$$

and we get

$$\|C_\varphi(f)\|_{\mathcal{D}}^2 \leq K^2 \|f\|_{\mathcal{D}}^2,$$

with $K^2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1}$. □

3 Proof of Theorem 1.2

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

Theorem 3.1 *For every sequence (ε_n) of positive numbers with limit 0, there exists a compact composition operator C_φ on \mathcal{D} such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Before entering really in the proof, we may remark that, without loss of generality, by replacing ε_n with $\inf(2^{-8}, \sup_{k \geq n} \varepsilon_k)$, we can, and do, assume that $(\varepsilon_n)_n$ decreases and $\varepsilon_1 \leq 2^{-8}$.

Moreover, we can assume that $(\varepsilon_n)_n$ decreases “slowly”, as said in the following lemma.

Lemma 3.2 *Let (ε_i) be a decreasing sequence with limit zero and let $0 < \rho < 1$. Then, there exists another sequence $(\widehat{\varepsilon}_i)$, decreasing with limit zero, such that $\widehat{\varepsilon}_i \geq \varepsilon_i$ and $\widehat{\varepsilon}_{i+1} \geq \rho \widehat{\varepsilon}_i$, for every $i \geq 1$.*

Proof. We define inductively $\widehat{\varepsilon}_i$ by $\widehat{\varepsilon}_1 = \varepsilon_1$ and

$$\widehat{\varepsilon}_{i+1} = \max(\rho \widehat{\varepsilon}_i, \varepsilon_{i+1}).$$

It is seen by induction that $\widehat{\varepsilon}_i \geq \varepsilon_i$ and that $\widehat{\varepsilon}_i$ decreases to a limit $a \geq 0$. If $\widehat{\varepsilon}_i = \varepsilon_i$ for infinitely many indices i , we have $a = 0$. In the opposite case, $\widehat{\varepsilon}_{i+1} = \rho \widehat{\varepsilon}_i$ from some index i_0 onwards, and again $a = 0$ since $\rho < 1$. \square

We will take $\rho = 1/2$ and assume for the sequel that $\varepsilon_{i+1} \geq \varepsilon_i/2$.

Proof of Theorem 3.1. We first construct a subdomain $\Omega = \Omega_\theta$ of \mathbb{D} defined by a cuspidal inequality:

$$(3.1) \quad \Omega = \{z = x + iy \in \mathbb{D}; |y| < \theta(1-x), 0 < x < 1\},$$

where $\theta: [0, 1] \rightarrow [0, 1[$ is a continuous increasing function such that

$$(3.2) \quad \theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1-x.$$

Note that since $1-x \leq \sqrt{1-x^2}$, the condition $|y| < \theta(1-x)$ implies that $z = x + iy \in \mathbb{D}$. Note also that $1 \in \overline{\Omega}$ and that Ω is a Jordan domain.

We introduce a parameter δ with $\varepsilon_1 \leq \delta \leq 1 - \varepsilon_1$. We put:

$$(3.3) \quad \theta(\delta^j) = \varepsilon_j \delta^j$$

and we extend θ to an increasing continuous function from $(0, 1)$ into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

$$(3.4) \quad \theta(h) \leq h \quad \text{and} \quad \theta(h) = o(h) \text{ as } h \rightarrow 0.$$

Indeed, if $\delta^{j+1} \leq h < \delta^j$, we have $\theta(h)/h \leq \theta(\delta^j)/\delta^{j+1} = \varepsilon_j/\delta$, which is $\leq \varepsilon_1/\delta \leq 1$ and which tends to 0 with h .

We define now $\varphi = \varphi_\theta: \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ as a continuous map which is a Riemann map from \mathbb{D} onto Ω , and with $\varphi(1) = 1$ (a cusp-type map). Since φ is univalent, one has $n_\varphi = \mathbb{1}_\Omega$, and since Ω is bounded, φ defines a symbol on \mathcal{D} , by (2.7). Moreover, (3.4) implies that $A[S(\xi, h) \cap \Omega] \leq h\theta(h)$ for every $\xi \in \mathbb{T}$; hence, ρ_φ being defined in (2.6), one has $\rho_\varphi(h) = o(h^2)$ as $h \rightarrow 0^+$. In view of [17], this little-oh condition guarantees the compactness of $C_\varphi: \mathcal{D} \rightarrow \mathcal{D}$.

It remains to minorate its approximation numbers.

The measure $\mu = n_\varphi dA$ is a Carleson measure for the Bergman space \mathfrak{B} , and it was proved in [10] that $C_\varphi^* C_\varphi$ is unitarily equivalent to the Toeplitz operator $T_\mu = I_\mu^* I_\mu: \mathfrak{B} \rightarrow \mathfrak{B}$ defined by:

$$(3.5) \quad T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w) = \int_{\mathbb{D}} f(w) K_w(z) dA(w),$$

where $I_\mu: \mathfrak{B} \rightarrow L^2(\mu)$ is the canonical inclusion and K_w the reproducing kernel of \mathfrak{B} at w , i.e. $K_w(z) = \frac{1}{(1-\bar{w}z)^2}$.

Actually, we can get rid of the analyticity constraint in considering, instead of T_μ , the operator $S_\mu = I_\mu I_\mu^*: L^2(\mu) \rightarrow L^2(\mu)$, which corresponds to the arrows:

$$L^2(\mu) \xrightarrow{I_\mu^*} \mathfrak{B} \xrightarrow{I_\mu} L^2(\mu).$$

We use the relation (3.5) which implies:

$$(3.6) \quad a_n(C_\varphi) = a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)}.$$

We set:

$$(3.7) \quad c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \varepsilon_j \delta^j$$

One has $r_j = \varepsilon_j(1 - c_j)/2$.

Lemma 3.3 *The disks $\Delta_j = D(c_j, r_j)$, $j \geq 1$, are disjoint and contained in Ω .*

Proof. If $z = x + iy \in \Delta_j$, then $1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \varepsilon_j/2) = 2\delta^j(1 - \varepsilon_j/2) \geq \delta^j$ and $|y| < r_j = \theta(\delta^j)$; hence $|y| < \theta(\delta^j) \leq \theta(1 - x)$ and $z \in \Omega$. On the other hand, $c_{j+1} - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta)\delta^j \geq 2\varepsilon_1\delta^j \geq 2\varepsilon_j\delta^j = 2r_j > r_j + r_{j+1}$; hence $\Delta_j \cap \Delta_{j+1} = \emptyset$. \square

We will next need a description of S_μ .

Lemma 3.4 *For every $g \in L^2(\mu)$ and every $z \in \mathbb{D}$:*

$$(3.8) \quad I_\mu^*g(z) = \int_{\Omega} \frac{g(w)}{(1 - \bar{w}z)^2} dA(w)$$

$$(3.9) \quad S_\mu g(z) = \left(\int_{\Omega} \frac{g(w)}{(1 - \bar{w}z)^2} dA(w) \right) \mathbb{1}_{\Omega}(z).$$

Proof. K_w being the reproducing kernel of \mathfrak{B} , we have for any pair of functions $f \in \mathfrak{B}$ and $g \in L^2(\mu)$:

$$\begin{aligned} \langle I_\mu^*g, f \rangle_{\mathfrak{B}} &= \langle g, I_\mu f \rangle_{L^2(\mu)} = \int_{\Omega} g(w) \overline{f(w)} dA(w) = \int_{\Omega} g(w) \langle K_w, f \rangle_{\mathfrak{B}} dA(w) \\ &= \left\langle \int_{\Omega} g(w) K_w dA(w), f \right\rangle_{\mathfrak{B}}, \end{aligned}$$

so that $I_\mu^*g = \int_{\Omega} g(w) K_w dA(w)$, giving the result. \square

In the rest of the proof, we fix a positive integer n and put:

$$(3.10) \quad f_j = \frac{1}{r_j} \mathbb{1}_{\Delta_j}, \quad j = 1, \dots, n.$$

Let:

$$E = \text{span}(f_1, \dots, f_n).$$

This is an n -dimensional subspace of $L^2(\mu)$.

The Δ_j 's being disjoint, the sequence (f_1, \dots, f_n) is orthonormal in $L^2(\mu)$. Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 d\mu = \int f_j^2 n_\varphi dA = \int_{\Delta_j} \frac{1}{r_j^2} dA = 1.$$

We now estimate from below the Bernstein numbers of I_μ^* . To that effect, we compute the scalar products $m_{i,j} = \langle I_\mu^*(f_i), I_\mu^*(f_j) \rangle$. One has:

$$\begin{aligned} m_{i,j} &= \langle f_i, S_\mu(f_j) \rangle = \int_{\Omega} f_i(z) \overline{S_\mu f_j(z)} dA(z) \\ &= \iint_{\Omega \times \Omega} \frac{f_i(z) \overline{f_j(w)}}{(1 - w\bar{z})^2} dA(z) dA(w) \\ &= \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \frac{1}{(1 - w\bar{z})^2} dA(z) dA(w). \end{aligned}$$

Lemma 3.5 *We have*

$$(3.11) \quad m_{i,i} \geq \frac{\varepsilon_i^2}{32}, \quad \text{and} \quad |m_{i,j}| \leq \varepsilon_i \varepsilon_j \delta^{j-i} \quad \text{for } i < j.$$

Proof. Set $\varepsilon'_i = \frac{r_i}{1-c_i^2} = \frac{\varepsilon_i}{2(1+c_i)}$. One has $\frac{\varepsilon_i}{4} \leq \varepsilon'_i \leq \frac{\varepsilon_i}{2}$. We observe that (recall that $A(\Delta_i) = r_i^2$):

$$m_{i,i} - \varepsilon_i'^2 = \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left[\frac{1}{(1 - w\bar{z})^2} - \frac{1}{(1 - c_i^2)^2} \right] dA(z) dA(w).$$

Therefore, using the fact that, for $z \in \Delta_i$ and $w \in \mathbb{D}$:

$$|1 - w\bar{z}| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \varepsilon_i \left(\frac{1 - c_i}{2} \right) \geq (1 - c_i) \left(1 - \frac{\varepsilon_i}{2} \right) \geq \frac{1 - c_i}{2}$$

and then the mean-value theorem, we get:

$$\begin{aligned} |m_{i,i} - \varepsilon_i'^2| &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left| \frac{1}{(1 - w\bar{z})^2} - \frac{1}{(1 - c_i^2)^2} \right| dA(z) dA(w) \\ &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \frac{32 r_i}{(1 - c_i)^3} dA(z) dA(w) \\ &= \frac{32 r_i^3}{(1 - c_i)^3} \leq 32 \times 8 \varepsilon_i'^3 \leq \frac{\varepsilon_i'^2}{2}, \end{aligned}$$

since $\varepsilon_i \leq \varepsilon_1 \leq 2^{-8}$ implies that $\varepsilon'_i \leq 1/(32 \times 16)$. This gives us the lower bound $m_{i,i} \geq \varepsilon_i'^2/2 \geq \varepsilon_i^2/32$.

Next, for $i < j$:

$$\begin{aligned} |m_{i,j}| &\leq \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \left| \frac{1}{(1 - w\bar{z})^2} \right| dA(z) dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1 - c_i)^2} r_i^2 r_j^2 \\ &= \frac{4 \varepsilon_i \varepsilon_j \delta^{i+j}}{4 \delta^{2i}} = \varepsilon_i \varepsilon_j \delta^{j-i}, \end{aligned}$$

and that ends the proof of Lemma 3.5. \square

We further write the $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ as $M = D + R$ where D is the diagonal matrix $m_i = m_{i,i}$ with $m_i \geq \frac{\varepsilon_i^2}{32}$, $1 \leq i \leq n$. Observe that M is nothing but the matrix of S_μ on the orthonormal basis (f_1, \dots, f_n) of E , so that we can identify M and S_μ on E .

Now the following lemma will end the proof of Theorem 3.1.

Lemma 3.6 *If $\delta \leq 1/200$, we have:*

$$(3.12) \quad \|D^{-1}R\| \leq 1/2.$$

Indeed, by the ideal property of Bernstein numbers, Neumann's lemma and the relations:

$$M = D(I + D^{-1}R), \quad \text{and} \quad D = MQ \quad \text{with} \quad \|Q\| \leq 2,$$

we have $b_n(D) \leq b_n(M) \|Q\| \leq 2 b_n(M)$, that is:

$$a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\varepsilon_n^2}{64},$$

since the n first approximation numbers of the diagonal matrix D (the matrices being viewed as well as operators on the Hilbertian space \mathbb{C}^n with its canonical basis) are $m_{1,1}, \dots, m_{n,n}$. It follows that, using (3.6):

$$(3.13) \quad a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\varepsilon_n}{8}.$$

In view of (3.6), we have as well $a_n(C_\varphi) \geq \varepsilon_n/8$, and we are done. \square

Proof of Lemma 3.6. Write $M = (m_{i,j}) = D(I + N)$ with $N = D^{-1}R$. One has:

$$(3.14) \quad N = (\nu_{i,j}), \quad \text{with} \quad \nu_{i,i} = 0 \quad \text{and} \quad \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i.$$

We shall show that $\|N\| \leq 1/2$ by using the (unweighted) Schur test, which we recall ([6], Problem 45):

Proposition 3.7 *Let $(a_{i,j})_{1 \leq i,j \leq n}$ be a matrix of complex numbers. Suppose that there exist two positive numbers $\alpha, \beta > 0$ such that:*

1. $\sum_{j=1}^n |a_{i,j}| \leq \alpha$ for all i ;
2. $\sum_{i=1}^n |a_{i,j}| \leq \beta$ for all j .

Then, the (Hilbertian) norm of this matrix satisfies $\|A\| \leq \sqrt{\alpha\beta}$.

It is essential for our purpose to note that:

$$(3.15) \quad i < j \implies |\nu_{i,j}| \leq 32 \delta^{j-i},$$

$$(3.16) \quad i > j \implies |\nu_{i,j}| \leq 32 (2\delta)^{i-j}.$$

Indeed, we see from (3.11) and (3.14) that, for $i < j$:

$$|\nu_{i,j}| = \frac{|m_{i,j}|}{m_{i,i}} \leq 32 \varepsilon_i \varepsilon_j \varepsilon_i^{-2} \delta^{j-i} \leq 32 \delta^{j-i}$$

since $\varepsilon_j \leq \varepsilon_i$. Secondly, using $\varepsilon_j/\varepsilon_i \leq 2^{i-j}$ for $i > j$ (recall that we assumed that $\varepsilon_{k+1} \geq \varepsilon_k/2$), as well as $|m_{i,j}| = |m_{j,i}|$, we have, for $i > j$:

$$|\nu_{i,j}| = \frac{|m_{j,i}|}{m_{i,i}} \leq 32 \frac{\varepsilon_j}{\varepsilon_i} \delta^{i-j} \leq 32 (2\delta)^{i-j}.$$

Now, for fixed i , (3.15) gives:

$$\begin{aligned} \sum_{j=1}^n |\nu_{i,j}| &= \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32 \left(\sum_{j>i} \delta^{j-i} + \sum_{j<i} (2\delta)^{i-j} \right) \\ &\leq 32 \left(\frac{\delta}{1-\delta} + \frac{2\delta}{1-2\delta} \right) \leq 32 \frac{3\delta}{1-2\delta} \leq \frac{96}{198} \leq \frac{1}{2}, \end{aligned}$$

since $\delta \leq 1/200$. Hence:

$$(3.17) \quad \sup_i \left(\sum_j |\nu_{i,j}| \right) \leq 1/2.$$

In the same manner, but using (3.16) instead of (3.15), one has:

$$(3.18) \quad \sup_j \left(\sum_i |\nu_{i,j}| \right) \leq 1/2.$$

Now, (3.17), (3.18) and the Schur criterion recalled above give:

$$\|N\| \leq \sqrt{1/2 \times 1/2} = 1/2,$$

as claimed. \square

Remark. We could reverse the point of view in the preceding proof: start from θ and see what lower bound for $a_n(C_\varphi)$ emerges. For example, if $\theta(h) \approx h$ as is the case for lens maps (see [11]), we find again that $a_n(C_\varphi) \geq \delta_0 > 0$ and that C_φ is not compact. But if $\theta(h) \approx h^{1+\alpha}$ with $\alpha > 0$, the method only gives $a_n(C_\varphi) \gtrsim e^{-\alpha n}$ (which is always true: see [11], Theorem 2.1), whereas the methods of [11] easily give $a_n(C_\varphi) \gtrsim e^{-\alpha\sqrt{n}}$. Therefore, this μ -method seems to be sharp when we are close to non-compactness, and to be beaten by those of [11] for “strongly compact” composition operators.

3.1 Optimality of the EKSJ result

El Fallah, Kellay, Shabankhah and Youssfi proved in [5] the following: if φ is a Schur function such that $\varphi \in \mathcal{D}$ and $\|\varphi^p\|_{\mathcal{D}} = O(1)$ as $p \rightarrow \infty$, then φ is a symbol on \mathcal{D} . We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

Theorem 3.8 *Let $(M_p)_{p \geq 1}$ be an arbitrary sequence of positive numbers such that $\lim_{p \rightarrow \infty} M_p = \infty$. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

- 1) $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$ as $p \rightarrow \infty$;
- 2) φ is not a symbol on \mathcal{D} .

Remark. We first observe that we cannot replace \lim by \limsup in Theorem 3.8. Indeed, since $\varphi \in \mathcal{D}$, the measure $\mu = n_\varphi dA$ is finite, and

$$\|\varphi^p\|_{\mathcal{D}}^2 = p^2 \int_{\mathbb{D}} |w|^{2p-2} d\mu(w) \geq c p^2 \left(\int_{\mathbb{D}} |w|^2 d\mu(w) \right)^{p-1} \geq c \delta^p,$$

where c and δ are positive constants.

Proof of Theorem 3.8. We may, and do, assume that (M_p) is non-decreasing and integer-valued. Let $(l_n)_{n \geq 1}$ be an non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let Ω be the subdomain of the right half-plane \mathbb{C}_0 defined as follows. We set:

$$\varepsilon_n = -\log(1 - 2^{-n}) \sim 2^{-n},$$

and we consider the (essentially) disjoint boxes ($k = 0, 1, \dots$):

$$B_{k,n} = B_{0,n} + 2k\pi i,$$

with:

$$B_{0,n} = \{u \in \mathbb{C}; \varepsilon_{n+1} \leq \Re u \leq \varepsilon_n \text{ and } |\Im u| \leq 2^{-n}\pi\},$$

as well as the union

$$T_n = \bigcup_{0 < k < l_n} B_{k,2n},$$

which is a kind of broken tower above the "basis" $B_{0,2n}$ of even index.

We also consider, for $1 \leq k \leq l_n - 1$, very thin vertical pipes $P_{k,n}$ connecting $B_{k,2n}$ and $B_{k-1,2n}$, of side lengths 4^{-2n} and $2\pi(1 - 2^{-2n})$ respectively:

$$P_{k,n} = P_{0,n} + 2k\pi i,$$

and we set:

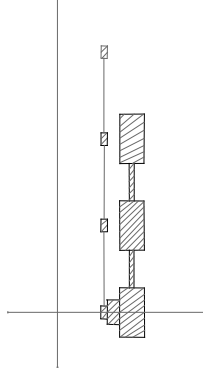
$$P_n = \bigcup_{1 \leq k < l_n} P_{k,n}$$

Finally, we set:

$$F = \left(\bigcup_{n=2}^{\infty} B_{0,n} \right) \cup \left(\bigcup_{n=1}^{\infty} T_n \right) \cup \left(\bigcup_{n=1}^{\infty} P_n \right)$$

and:

$$\Omega = \overset{\circ}{F}$$



Then Ω is a simply connected domain. Indeed, it is connected thanks to the $B_{0,n}$ and the P_n , since the $P_{k,n}$ were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of n out of two in the union of sets $B_{k,n}$ defining F .

Let now $f: \mathbb{D} \rightarrow \Omega$ be a Riemann map, and $\varphi = e^{-f}: \mathbb{D} \rightarrow \mathbb{D}$.

We introduce the Carleson window $W = W(1, h)$ defined as:

$$W(1, h) = \{z \in \mathbb{D}; 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi h\}.$$

This is a variant of the sets $S(1, h)$ of Section 2. We also introduce the Hastings-Luecking half-windows W'_n defined by:

$$W'_n = \{z \in \mathbb{D}; 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi 2^{-n}\}.$$

We will also need the sets:

$$E_n = e^{-(T_n \cup B_{0,2n+1} \cup P_n)} = e^{-(B_{0,2n} \cup B_{0,2n+1} \cup P_{0,n})},$$

for which one has:

$$\varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_n.$$

Next, we consider the measure $\mu = n_\varphi dA$, and a Carleson window $W = W(1, h)$ with $h = 2^{-2N}$. We observe that $W'_{2N} \subseteq W$ and claim that:

Lemma 3.9 *One has:*

- 1) $w \in W'_{2N} \implies n_\varphi(w) \geq l_N$;
- 2) $\|\varphi^p\|_{\mathcal{D}}^2 \lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p^4 n}$.

Proof of Lemma 3.9. 1) Let $w = r e^{i\theta} \in W'_{2N}$ with $1 - 2^{-2N} < r < 1 - 2^{-2N-1}$ and $|\theta| < \pi 2^{-2N}$. As $-(\log r + i\theta) \in B_{0,2N}$, one has $-(\log r + i\theta) = f(z_0)$ for some $z_0 \in \mathbb{D}$. Similarly, $-(\log r + i\theta) + 2k\pi i$, for $1 \leq k < l_N$, belongs to $B_{k,2N}$ and can be written as $f(z_k)$, with $z_k \in \mathbb{D}$. The z_k 's, $0 \leq k < l_N$, are distinct and satisfy $\varphi(z_k) = e^{-f(z_k)} = e^{-f(z_0)} = w$ for $0 \leq k < l_N$, thanks to the $2\pi i$ -periodicity of the exponential function.

2) We have $A(E_n) \lesssim e^{-2\varepsilon_{2n+2}} 4^{-2n} \leq 4^{-2n}$ (the term $e^{-2\varepsilon_{2n+2}}$ coming from the Jacobian of e^{-z}) and we observe that

$$w \in E_n \implies |w|^{2p-2} \leq (1 - 2^{-2n-1})^{2p-2} \lesssim e^{-p4^{-n}}.$$

It is easy to see that $n_\varphi(w) \leq l_n$ for $w \in E_n$; thus we obtain, forgetting the constant term $|\varphi(0)|^{2p} \leq 1$, using (2.5) and keeping in mind the fact that $n_\varphi(w) = 0$ for $w \notin \varphi(\mathbb{D})$:

$$\begin{aligned} \|\varphi^p\|_{\mathcal{D}}^2 &= p^2 \int_{\varphi(\mathbb{D})} |w|^{2p-2} n_\varphi(w) dA(w) \\ &\leq p^2 \left(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} n_\varphi(w) dA(w) \right) \\ &\leq p^2 \left(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} l_n dA(w) \right) \\ &\lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p4^{-n}}, \end{aligned}$$

ending the proof of Lemma 3.9. □

End of the proof of Theorem 3.8. Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

$$\mu(W) \geq \mu(W'_{2N}) = \int_{W'_{2N}} n_\varphi dA \geq l_N A(W'_{2N}) \gtrsim l_N h^2,$$

which implies that $\sup_{0 < h < 1} h^{-2} \mu[W(1, h)] = +\infty$ and shows that C_φ is not bounded on \mathcal{D} by Zorboska's criterion ([17], Theorem 1), recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers (l_n) so as to have $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$. To this effect, we first observe that, if one sets $F(x) = x^2 e^{-x}$, we have:

$$p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} = \sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim 1.$$

Indeed, let s be the integer such that $4^s \leq p < 4^{s+1}$. We have:

$$\sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim \sum_{n=1}^s \frac{4^n}{p} + \sum_{n>s} F(4^{-(n-s-1)}) \lesssim 1 + \sum_{n=0}^{\infty} F(4^{-n}) < \infty,$$

where we used that F is increasing on $(0, 1)$ and satisfies $F(x) \lesssim \min(x^2, 1/x)$ for $x > 0$. We finally choose the non-decreasing sequence (l_n) of integers as:

$$l_n = \min(n, M_n^2).$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\begin{aligned} \|\varphi^p\|_{\mathcal{D}}^2 &\lesssim p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p 4^{-n}} l_n \\ &\leq p^2 \sum_{n=1}^p 16^{-n} e^{-p 4^{-n}} l_p + p^2 \sum_{n>p} 16^{-n} l_n \\ &\lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2 4^{-p} \lesssim M_p^2, \end{aligned}$$

as desired. This choice of (l_n) gives us an unbounded composition operator on \mathcal{D} such that $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$, which ends the proof of Theorem 3.8. \square

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