

Two remarks on composition operators on the Dirichlet space

Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza

► **To cite this version:**

Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza. Two remarks on composition operators on the Dirichlet space. 15 pages. 2014. <hal-00982257v2>

HAL Id: hal-00982257

<https://hal-univ-artois.archives-ouvertes.fr/hal-00982257v2>

Submitted on 10 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Two remarks on composition operators on the Dirichlet space

*Daniel Li, Hervé Queffélec,
Luis Rodríguez-Piazza**

July 10, 2014

Abstract. *We show that the decay of approximation numbers of compact composition operators on the Dirichlet space \mathcal{D} can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on \mathcal{D} of self-maps of the disk all of whose powers are norm-bounded in \mathcal{D} .*

Mathematics Subject Classification. Primary: 47B33 – Secondary: 46E22; 47B06 ; 47B32

Key-words. approximation numbers – Carleson embedding – composition operator – cusp map – Dirichlet space

1 Introduction

Recall that if φ is an analytic self-map of \mathbb{D} , a so-called *Schur function*, the composition operator C_φ associated to φ is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$

The Littlewood subordination principle ([4], p. 30) tells us that C_φ maps the Hardy space H^2 to itself for every Schur function φ . Also recall that if H is a Hilbert space and $T: H \rightarrow H$ a bounded linear operator, the n -th approximation number $a_n(T)$ of T is defined as

$$(1.1) \quad a_n(T) = \inf\{\|T - R\|; \text{rank } R < n\}, \quad n = 1, 2, \dots$$

In [12], working on that Hardy space H^2 (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators C_φ , and proved among other facts the following:

*Supported by a Spanish research project MTM 2012-05622.

Theorem 1.1 *Let $(\varepsilon_n)_{n \geq 1}$ be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator C_φ on H^2 such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

As a consequence, there are composition operators on H^2 which are compact but in no Schatten class.

The last item had been previously proved by Carroll and Cowen ([3]), the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of \mathbb{D} generates a bounded composition operator on \mathcal{D} . When this is the case, we will say that φ is a *symbol* (understanding “of \mathcal{D} ”). Note that every symbol is necessarily in \mathcal{D} .

In [11], we have performed a similar study on that Dirichlet space \mathcal{D} , and established several results on approximation numbers in that new setting, in particular the existence of symbols φ for which C_φ is compact without being in any Schatten class S_p . But we have not been able in [11] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

Theorem 1.2 *For every sequence $(\varepsilon_n)_{n \geq 1}$ of positive numbers tending to 0, there exists a compact composition operator C_φ on the Dirichlet space \mathcal{D} such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Turning now to the question of necessary or sufficient conditions for a Schur function φ to be a symbol, we can observe that, since $(z^n/\sqrt{n})_{n \geq 1}$ is an orthonormal sequence in \mathcal{D} and since formally $C_\varphi(z^n) = \varphi^n$, a necessary condition is as follows:

$$(1.2) \quad \varphi \text{ is a symbol} \implies \|\varphi^n\|_{\mathcal{D}} = O(\sqrt{n}).$$

It is worth noting that, for any Schur function, one has:

$$\varphi \in \mathcal{D} \implies \|\varphi^n\|_{\mathcal{D}} = O(n)$$

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer $n \geq 1$:

$$\begin{aligned} \|\varphi^n\|_{\mathcal{D}}^2 &= |\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |\varphi(z)|^{2(n-1)} |\varphi'(z)|^2 dA(z) \\ &\leq |\varphi(0)|^2 + \int_{\mathbb{D}} n^2 |\varphi'(z)|^2 dA(z) \leq n^2 \|\varphi\|_{\mathcal{D}}^2, \end{aligned}$$

giving the result.

Now, the following sufficient condition was given in [5]:

$$(1.3) \quad \|\varphi^n\|_{\mathcal{D}} = O(1) \implies \varphi \text{ is a symbol.}$$

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

Theorem 1.3 *Let $(M_n)_{n \geq 1}$ be an arbitrary sequence of positive numbers tending to ∞ . Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

- 1) $\|\varphi^n\|_{\mathcal{D}} = O(M_n)$ as $n \rightarrow \infty$;
- 2) φ is not a symbol on \mathcal{D} .

The organization of that paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

2 Notation and background.

We denote by \mathbb{D} the open unit disk of the complex plane and by A the normalized area measure $dx dy/\pi$ of \mathbb{D} . The unit circle is denoted by $\mathbb{T} = \partial\mathbb{D}$. The notation $A \lesssim B$ indicates that $A \leq cB$ for some positive constant c .

A Schur function is an analytic self-map of \mathbb{D} and the associated composition operator is defined, formally, by $C_\varphi(f) = f \circ \varphi$. The operator C_φ maps the space $\mathcal{H}ol(\mathbb{D})$ of holomorphic functions on \mathbb{D} into itself.

The Dirichlet space \mathcal{D} is the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$(2.1) \quad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has:

$$(2.2) \quad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.$$

Then $\|\cdot\|_{\mathcal{D}}$ is a norm on \mathcal{D} , making \mathcal{D} a Hilbert space, and $\|\cdot\|_{H^2} \leq \|\cdot\|_{\mathcal{D}}$. For further information on the Dirichlet space, the reader may see [1] or [16].

The Bergman space \mathfrak{B} is the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that:

$$\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2 dA(z) < +\infty.$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}$. If $f \in \mathcal{D}$, one has by definition:

$$\|f\|_{\mathcal{D}}^2 = \|f'\|_{\mathfrak{B}}^2 + |f(0)|^2.$$

Recall that, whereas every Schur function φ generates a bounded composition operator C_φ on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [14], Proposition 3.12, for instance).

We denote by $b_n(T)$ the n -th *Bernstein number* of the operator $T: H \rightarrow H$, namely:

$$(2.3) \quad b_n(T) = \sup_{\dim E=n} \left(\inf_{f \in S_E} \|Tx\| \right)$$

where S_E denotes the unit sphere of E . It is easy to see ([11]) that

$$b_n(T) = a_n(T) \quad \text{for all } n \geq 1.$$

(recall that the approximation numbers are defined in (1.1)).

If φ is a Schur function, let

$$(2.4) \quad n_\varphi(w) = \#\{z \in \mathbb{D}; \varphi(z) = w\} \geq 0$$

be the associated *counting function*. If $f \in \mathcal{D}$ and $g = f \circ \varphi$, the change of variable formula provides us with the useful following equation ([17], [11]):

$$(2.5) \quad \int_{\mathbb{D}} |g'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w)$$

(the integrals might be infinite). In those terms, a necessary and sufficient condition for φ to be a symbol is as follows ([17], Theorem 1). Let:

$$(2.6) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} \int_{S(\xi, h)} n_\varphi dA$$

where $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$ is the Carleson window centered at ξ and of size h . Then φ is a symbol if and only if:

$$(2.7) \quad \sup_{0 < h < 1} \frac{1}{h^2} \rho_\varphi(h) < \infty.$$

This is not difficult to prove. In view of (2.5), the boundedness of C_φ amounts to the existence of a constant C such that:

$$\int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |f'(z)|^2 dA(z), \quad \forall f \in \mathcal{D}.$$

Since $f' = h$ runs over \mathfrak{B} as f runs over \mathcal{D} , and with equal norms, the above condition reads:

$$\int_{\mathbb{D}} |h(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |h(z)|^2 dA(z), \quad \forall h \in \mathfrak{B}.$$

This exactly means that the measure $n_\varphi dA$ is a Carleson measure for \mathfrak{B} . Such measures have been characterized in [7] and that characterization gives (2.7).

But this condition is very abstract and difficult to test, and sometimes more “concrete” sufficient conditions are desirable. In [11], we proved that, even if the Schur function extends continuously to $\overline{\mathbb{D}}$, no Lipschitz condition of order α , $0 < \alpha < 1$, on φ is sufficient for ensuring that φ is a symbol. It is worth noting that the limiting case $\alpha = 1$, so restrictive it is, guarantees the result.

Proposition 2.1 *Suppose that the Schur function φ is in the analytic Lipschitz class on the unit disk, i.e. satisfies:*

$$|\varphi(z) - \varphi(w)| \leq C |z - w|, \quad \forall z, w \in \mathbb{D}.$$

Then C_φ is bounded on \mathcal{D} .

Proof. Let $f \in \mathcal{D}$; one has:

$$\begin{aligned} \|C_\varphi(f)\|_{\mathcal{D}}^2 &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) \\ &\leq |f(\varphi(0))|^2 + \|\varphi'\|_{\infty}^2 \int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z). \end{aligned}$$

This integral is nothing but $\|C_\varphi(f')\|_{\mathfrak{B}}^2$ and hence, since C_φ is bounded on the Bergman space \mathfrak{B} , we have, for some constant K_1 :

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z) \leq K_1^2 \|f'\|_{\mathfrak{B}}^2 \leq K_1^2 \|f\|_{\mathcal{D}}^2.$$

On the other hand,

$$|f(\varphi(0))| \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{H^2} \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{\mathcal{D}},$$

and we get

$$\|C_\varphi(f)\|_{\mathcal{D}}^2 \leq K^2 \|f\|_{\mathcal{D}}^2,$$

with $K^2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1}$. □

3 Proof of Theorem 1.2

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

Theorem 3.1 *For every sequence (ε_n) of positive numbers with limit 0, there exists a compact composition operator C_φ on \mathcal{D} such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Before entering really in the proof, we may remark that, without loss of generality, by replacing ε_n with $\inf(2^{-8}, \sup_{k \geq n} \varepsilon_k)$, we can, and do, assume that $(\varepsilon_n)_n$ decreases and $\varepsilon_1 \leq 2^{-8}$.

Moreover, we can assume that $(\varepsilon_n)_n$ decreases “slowly”, as said in the following lemma.

Lemma 3.2 *Let (ε_i) be a decreasing sequence with limit zero and let $0 < \rho < 1$. Then, there exists another sequence $(\widehat{\varepsilon}_i)$, decreasing with limit zero, such that $\widehat{\varepsilon}_i \geq \varepsilon_i$ and $\widehat{\varepsilon}_{i+1} \geq \rho \widehat{\varepsilon}_i$, for every $i \geq 1$.*

Proof. We define inductively $\widehat{\varepsilon}_i$ by $\widehat{\varepsilon}_1 = \varepsilon_1$ and

$$\widehat{\varepsilon}_{i+1} = \max(\rho \widehat{\varepsilon}_i, \varepsilon_{i+1}).$$

It is seen by induction that $\widehat{\varepsilon}_i \geq \varepsilon_i$ and that $\widehat{\varepsilon}_i$ decreases to a limit $a \geq 0$. If $\widehat{\varepsilon}_i = \varepsilon_i$ for infinitely many indices i , we have $a = 0$. In the opposite case, $\widehat{\varepsilon}_{i+1} = \rho \widehat{\varepsilon}_i$ from some index i_0 onwards, and again $a = 0$ since $\rho < 1$. \square

We will take $\rho = 1/2$ and assume for the sequel that $\varepsilon_{i+1} \geq \varepsilon_i/2$.

Proof of Theorem 3.1. We first construct a subdomain $\Omega = \Omega_\theta$ of \mathbb{D} defined by a cuspidal inequality:

$$(3.1) \quad \Omega = \{z = x + iy \in \mathbb{D}; |y| < \theta(1-x), 0 < x < 1\},$$

where $\theta: [0, 1] \rightarrow [0, 1[$ is a continuous increasing function such that

$$(3.2) \quad \theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1-x.$$

Note that since $1-x \leq \sqrt{1-x^2}$, the condition $|y| < \theta(1-x)$ implies that $z = x + iy \in \mathbb{D}$. Note also that $1 \in \overline{\Omega}$ and that Ω is a Jordan domain.

We introduce a parameter δ with $\varepsilon_1 \leq \delta \leq 1 - \varepsilon_1$. We put:

$$(3.3) \quad \theta(\delta^j) = \varepsilon_j \delta^j$$

and we extend θ to an increasing continuous function from $(0, 1)$ into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

$$(3.4) \quad \theta(h) \leq h \quad \text{and} \quad \theta(h) = o(h) \text{ as } h \rightarrow 0.$$

Indeed, if $\delta^{j+1} \leq h < \delta^j$, we have $\theta(h)/h \leq \theta(\delta^j)/\delta^{j+1} = \varepsilon_j/\delta$, which is $\leq \varepsilon_1/\delta \leq 1$ and which tends to 0 with h .

We define now $\varphi = \varphi_\theta: \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ as a continuous map which is a Riemann map from \mathbb{D} onto Ω , and with $\varphi(1) = 1$ (a cusp-type map). Since φ is univalent, one has $n_\varphi = \mathbb{1}_\Omega$, and since Ω is bounded, φ defines a symbol on \mathcal{D} , by (2.7). Moreover, (3.4) implies that $A[S(\xi, h) \cap \Omega] \leq h\theta(h)$ for every $\xi \in \mathbb{T}$; hence, ρ_φ being defined in (2.6), one has $\rho_\varphi(h) = o(h^2)$ as $h \rightarrow 0^+$. In view of [17], this little-oh condition guarantees the compactness of $C_\varphi: \mathcal{D} \rightarrow \mathcal{D}$.

It remains to minorate its approximation numbers.

The measure $\mu = n_\varphi dA$ is a Carleson measure for the Bergman space \mathfrak{B} , and it was proved in [10] that $C_\varphi^* C_\varphi$ is unitarily equivalent to the Toeplitz operator $T_\mu = I_\mu^* I_\mu: \mathfrak{B} \rightarrow \mathfrak{B}$ defined by:

$$(3.5) \quad T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w) = \int_{\mathbb{D}} f(w) K_w(z) dA(w),$$

where $I_\mu: \mathfrak{B} \rightarrow L^2(\mu)$ is the canonical inclusion and K_w the reproducing kernel of \mathfrak{B} at w , i.e. $K_w(z) = \frac{1}{(1-\bar{w}z)^2}$.

Actually, we can get rid of the analyticity constraint in considering, instead of T_μ , the operator $S_\mu = I_\mu I_\mu^*: L^2(\mu) \rightarrow L^2(\mu)$, which corresponds to the arrows:

$$L^2(\mu) \xrightarrow{I_\mu^*} \mathfrak{B} \xrightarrow{I_\mu} L^2(\mu).$$

We use the relation (3.5) which implies:

$$(3.6) \quad a_n(C_\varphi) = a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)}.$$

We set:

$$(3.7) \quad c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \varepsilon_j \delta^j$$

One has $r_j = \varepsilon_j(1 - c_j)/2$.

Lemma 3.3 *The disks $\Delta_j = D(c_j, r_j)$, $j \geq 1$, are disjoint and contained in Ω .*

Proof. If $z = x + iy \in \Delta_j$, then $1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \varepsilon_j/2) = 2\delta^j(1 - \varepsilon_j/2) \geq \delta^j$ and $|y| < r_j = \theta(\delta^j)$; hence $|y| < \theta(\delta^j) \leq \theta(1 - x)$ and $z \in \Omega$. On the other hand, $c_{j+1} - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta)\delta^j \geq 2\varepsilon_1\delta^j \geq 2\varepsilon_j\delta^j = 2r_j > r_j + r_{j+1}$; hence $\Delta_j \cap \Delta_{j+1} = \emptyset$. \square

We will next need a description of S_μ .

Lemma 3.4 *For every $g \in L^2(\mu)$ and every $z \in \mathbb{D}$:*

$$(3.8) \quad I_\mu^*g(z) = \int_{\Omega} \frac{g(w)}{(1 - \bar{w}z)^2} dA(w)$$

$$(3.9) \quad S_\mu g(z) = \left(\int_{\Omega} \frac{g(w)}{(1 - \bar{w}z)^2} dA(w) \right) \mathbb{1}_{\Omega}(z).$$

Proof. K_w being the reproducing kernel of \mathfrak{B} , we have for any pair of functions $f \in \mathfrak{B}$ and $g \in L^2(\mu)$:

$$\begin{aligned} \langle I_\mu^*g, f \rangle_{\mathfrak{B}} &= \langle g, I_\mu f \rangle_{L^2(\mu)} = \int_{\Omega} g(w) \overline{f(w)} dA(w) = \int_{\Omega} g(w) \langle K_w, f \rangle_{\mathfrak{B}} dA(w) \\ &= \left\langle \int_{\Omega} g(w) K_w dA(w), f \right\rangle_{\mathfrak{B}}, \end{aligned}$$

so that $I_\mu^*g = \int_{\Omega} g(w) K_w dA(w)$, giving the result. \square

In the rest of the proof, we fix a positive integer n and put:

$$(3.10) \quad f_j = \frac{1}{r_j} \mathbb{1}_{\Delta_j}, \quad j = 1, \dots, n.$$

Let:

$$E = \text{span}(f_1, \dots, f_n).$$

This is an n -dimensional subspace of $L^2(\mu)$.

The Δ_j 's being disjoint, the sequence (f_1, \dots, f_n) is orthonormal in $L^2(\mu)$. Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 d\mu = \int f_j^2 n_\varphi dA = \int_{\Delta_j} \frac{1}{r_j^2} dA = 1.$$

We now estimate from below the Bernstein numbers of I_μ^* . To that effect, we compute the scalar products $m_{i,j} = \langle I_\mu^*(f_i), I_\mu^*(f_j) \rangle$. One has:

$$\begin{aligned} m_{i,j} &= \langle f_i, S_\mu(f_j) \rangle = \int_{\Omega} f_i(z) \overline{S_\mu f_j(z)} dA(z) \\ &= \iint_{\Omega \times \Omega} \frac{f_i(z) \overline{f_j(w)}}{(1 - w\bar{z})^2} dA(z) dA(w) \\ &= \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \frac{1}{(1 - w\bar{z})^2} dA(z) dA(w). \end{aligned}$$

Lemma 3.5 *We have*

$$(3.11) \quad m_{i,i} \geq \frac{\varepsilon_i^2}{32}, \quad \text{and} \quad |m_{i,j}| \leq \varepsilon_i \varepsilon_j \delta^{j-i} \quad \text{for } i < j.$$

Proof. Set $\varepsilon'_i = \frac{r_i}{1-c_i^2} = \frac{\varepsilon_i}{2(1+c_i)}$. One has $\frac{\varepsilon_i}{4} \leq \varepsilon'_i \leq \frac{\varepsilon_i}{2}$. We observe that (recall that $A(\Delta_i) = r_i^2$):

$$m_{i,i} - \varepsilon_i'^2 = \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left[\frac{1}{(1 - w\bar{z})^2} - \frac{1}{(1 - c_i^2)^2} \right] dA(z) dA(w).$$

Therefore, using the fact that, for $z \in \Delta_i$ and $w \in \mathbb{D}$:

$$|1 - w\bar{z}| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \varepsilon_i \left(\frac{1 - c_i}{2} \right) \geq (1 - c_i) \left(1 - \frac{\varepsilon_i}{2} \right) \geq \frac{1 - c_i}{2}$$

and then the mean-value theorem, we get:

$$\begin{aligned} |m_{i,i} - \varepsilon_i'^2| &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left| \frac{1}{(1 - w\bar{z})^2} - \frac{1}{(1 - c_i^2)^2} \right| dA(z) dA(w) \\ &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \frac{32 r_i}{(1 - c_i)^3} dA(z) dA(w) \\ &= \frac{32 r_i^3}{(1 - c_i)^3} \leq 32 \times 8 \varepsilon_i'^3 \leq \frac{\varepsilon_i'^2}{2}, \end{aligned}$$

since $\varepsilon_i \leq \varepsilon_1 \leq 2^{-8}$ implies that $\varepsilon'_i \leq 1/(32 \times 16)$. This gives us the lower bound $m_{i,i} \geq \varepsilon_i'^2/2 \geq \varepsilon_i^2/32$.

Next, for $i < j$:

$$\begin{aligned} |m_{i,j}| &\leq \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \left| \frac{1}{(1-w\bar{z})^2} \right| dA(z) dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1-c_i)^2} r_i^2 r_j^2 \\ &= \frac{4 \varepsilon_i \varepsilon_j \delta^{i+j}}{4 \delta^{2i}} = \varepsilon_i \varepsilon_j \delta^{j-i}, \end{aligned}$$

and that ends the proof of Lemma 3.5. \square

We further write the $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ as $M = D + R$ where D is the diagonal matrix $m_i = m_{i,i}$ with $m_i \geq \frac{\varepsilon_i^2}{32}$, $1 \leq i \leq n$. Observe that M is nothing but the matrix of S_μ on the orthonormal basis (f_1, \dots, f_n) of E , so that we can identify M and S_μ on E .

Now the following lemma will end the proof of Theorem 3.1.

Lemma 3.6 *If $\delta \leq 1/200$, we have:*

$$(3.12) \quad \|D^{-1}R\| \leq 1/2.$$

Indeed, by the ideal property of Bernstein numbers, Neumann's lemma and the relations:

$$M = D(I + D^{-1}R), \quad \text{and} \quad D = MQ \quad \text{with} \quad \|Q\| \leq 2,$$

we have $b_n(D) \leq b_n(M) \|Q\| \leq 2 b_n(M)$, that is:

$$a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\varepsilon_n^2}{64},$$

since the n first approximation numbers of the diagonal matrix D (the matrices being viewed as well as operators on the Hilbertian space \mathbb{C}^n with its canonical basis) are $m_{1,1}, \dots, m_{n,n}$. It follows that, using (3.6):

$$(3.13) \quad a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\varepsilon_n}{8}.$$

In view of (3.6), we have as well $a_n(C_\varphi) \geq \varepsilon_n/8$, and we are done. \square

Proof of Lemma 3.6. Write $M = (m_{i,j}) = D(I + N)$ with $N = D^{-1}R$. One has:

$$(3.14) \quad N = (\nu_{i,j}), \quad \text{with} \quad \nu_{i,i} = 0 \quad \text{and} \quad \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i.$$

We shall show that $\|N\| \leq 1/2$ by using the (unweighted) Schur test, which we recall ([6], Problem 45):

Proposition 3.7 *Let $(a_{i,j})_{1 \leq i,j \leq n}$ be a matrix of complex numbers. Suppose that there exist two positive numbers $\alpha, \beta > 0$ such that:*

1. $\sum_{j=1}^n |a_{i,j}| \leq \alpha$ for all i ;
2. $\sum_{i=1}^n |a_{i,j}| \leq \beta$ for all j .

Then, the (Hilbertian) norm of this matrix satisfies $\|A\| \leq \sqrt{\alpha\beta}$.

It is essential for our purpose to note that:

$$(3.15) \quad i < j \implies |\nu_{i,j}| \leq 32 \delta^{j-i},$$

$$(3.16) \quad i > j \implies |\nu_{i,j}| \leq 32 (2\delta)^{i-j}.$$

Indeed, we see from (3.11) and (3.14) that, for $i < j$:

$$|\nu_{i,j}| = \frac{|m_{i,j}|}{m_{i,i}} \leq 32 \varepsilon_i \varepsilon_j \varepsilon_i^{-2} \delta^{j-i} \leq 32 \delta^{j-i}$$

since $\varepsilon_j \leq \varepsilon_i$. Secondly, using $\varepsilon_j/\varepsilon_i \leq 2^{i-j}$ for $i > j$ (recall that we assumed that $\varepsilon_{k+1} \geq \varepsilon_k/2$), as well as $|m_{i,j}| = |m_{j,i}|$, we have, for $i > j$:

$$|\nu_{i,j}| = \frac{|m_{j,i}|}{m_{i,i}} \leq 32 \frac{\varepsilon_j}{\varepsilon_i} \delta^{i-j} \leq 32 (2\delta)^{i-j}.$$

Now, for fixed i , (3.15) gives:

$$\begin{aligned} \sum_{j=1}^n |\nu_{i,j}| &= \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32 \left(\sum_{j>i} \delta^{j-i} + \sum_{j<i} (2\delta)^{i-j} \right) \\ &\leq 32 \left(\frac{\delta}{1-\delta} + \frac{2\delta}{1-2\delta} \right) \leq 32 \frac{3\delta}{1-2\delta} \leq \frac{96}{198} \leq \frac{1}{2}, \end{aligned}$$

since $\delta \leq 1/200$. Hence:

$$(3.17) \quad \sup_i \left(\sum_j |\nu_{i,j}| \right) \leq 1/2.$$

In the same manner, but using (3.16) instead of (3.15), one has:

$$(3.18) \quad \sup_j \left(\sum_i |\nu_{i,j}| \right) \leq 1/2.$$

Now, (3.17), (3.18) and the Schur criterion recalled above give:

$$\|N\| \leq \sqrt{1/2 \times 1/2} = 1/2,$$

as claimed. \square

Remark. We could reverse the point of view in the preceding proof: start from θ and see what lower bound for $a_n(C_\varphi)$ emerges. For example, if $\theta(h) \approx h$ as is the case for lens maps (see [11]), we find again that $a_n(C_\varphi) \geq \delta_0 > 0$ and that C_φ is not compact. But if $\theta(h) \approx h^{1+\alpha}$ with $\alpha > 0$, the method only gives $a_n(C_\varphi) \gtrsim e^{-\alpha n}$ (which is always true: see [11], Theorem 2.1), whereas the methods of [11] easily give $a_n(C_\varphi) \gtrsim e^{-\alpha\sqrt{n}}$. Therefore, this μ -method seems to be sharp when we are close to non-compactness, and to be beaten by those of [11] for “strongly compact” composition operators.

3.1 Optimality of the EKSJ result

El Fallah, Kellay, Shabankhah and Youssfi proved in [5] the following: if φ is a Schur function such that $\varphi \in \mathcal{D}$ and $\|\varphi^p\|_{\mathcal{D}} = O(1)$ as $p \rightarrow \infty$, then φ is a symbol on \mathcal{D} . We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

Theorem 3.8 *Let $(M_p)_{p \geq 1}$ be an arbitrary sequence of positive numbers such that $\lim_{p \rightarrow \infty} M_p = \infty$. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

- 1) $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$ as $p \rightarrow \infty$;
- 2) φ is not a symbol on \mathcal{D} .

Remark. We first observe that we cannot replace \lim by \limsup in Theorem 3.8. Indeed, since $\varphi \in \mathcal{D}$, the measure $\mu = n_\varphi dA$ is finite, and

$$\|\varphi^p\|_{\mathcal{D}}^2 = p^2 \int_{\mathbb{D}} |w|^{2p-2} d\mu(w) \geq c p^2 \left(\int_{\mathbb{D}} |w|^2 d\mu(w) \right)^{p-1} \geq c \delta^p,$$

where c and δ are positive constants.

Proof of Theorem 3.8. We may, and do, assume that (M_p) is non-decreasing and integer-valued. Let $(l_n)_{n \geq 1}$ be an non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let Ω be the subdomain of the right half-plane \mathbb{C}_0 defined as follows. We set:

$$\varepsilon_n = -\log(1 - 2^{-n}) \sim 2^{-n},$$

and we consider the (essentially) disjoint boxes ($k = 0, 1, \dots$):

$$B_{k,n} = B_{0,n} + 2k\pi i,$$

with:

$$B_{0,n} = \{u \in \mathbb{C}; \varepsilon_{n+1} \leq \Re u \leq \varepsilon_n \text{ and } |\Im u| \leq 2^{-n}\pi\},$$

as well as the union

$$T_n = \bigcup_{0 < k < l_n} B_{k,2n},$$

which is a kind of broken tower above the "basis" $B_{0,2n}$ of even index.

We also consider, for $1 \leq k \leq l_n - 1$, very thin vertical pipes $P_{k,n}$ connecting $B_{k,2n}$ and $B_{k-1,2n}$, of side lengths 4^{-2n} and $2\pi(1 - 2^{-2n})$ respectively:

$$P_{k,n} = P_{0,n} + 2k\pi i,$$

and we set:

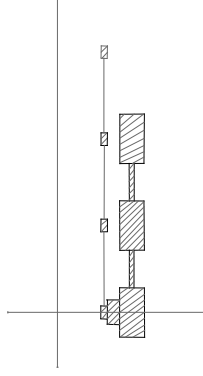
$$P_n = \bigcup_{1 \leq k < l_n} P_{k,n}$$

Finally, we set:

$$F = \left(\bigcup_{n=2}^{\infty} B_{0,n} \right) \cup \left(\bigcup_{n=1}^{\infty} T_n \right) \cup \left(\bigcup_{n=1}^{\infty} P_n \right)$$

and:

$$\Omega = \overset{\circ}{F}$$



Then Ω is a simply connected domain. Indeed, it is connected thanks to the $B_{0,n}$ and the P_n , since the $P_{k,n}$ were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of n out of two in the union of sets $B_{k,n}$ defining F .

Let now $f: \mathbb{D} \rightarrow \Omega$ be a Riemann map, and $\varphi = e^{-f}: \mathbb{D} \rightarrow \mathbb{D}$.

We introduce the Carleson window $W = W(1, h)$ defined as:

$$W(1, h) = \{z \in \mathbb{D}; 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi h\}.$$

This is a variant of the sets $S(1, h)$ of Section 2. We also introduce the Hastings-Luecking half-windows W'_n defined by:

$$W'_n = \{z \in \mathbb{D}; 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi 2^{-n}\}.$$

We will also need the sets:

$$E_n = e^{-(T_n \cup B_{0,2n+1} \cup P_n)} = e^{-(B_{0,2n} \cup B_{0,2n+1} \cup P_{0,n})},$$

for which one has:

$$\varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_n.$$

Next, we consider the measure $\mu = n_\varphi dA$, and a Carleson window $W = W(1, h)$ with $h = 2^{-2N}$. We observe that $W'_{2N} \subseteq W$ and claim that:

Lemma 3.9 *One has:*

- 1) $w \in W'_{2N} \implies n_\varphi(w) \geq l_N$;
- 2) $\|\varphi^p\|_{\mathcal{D}}^2 \lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p^4 n^{-n}}$.

Proof of Lemma 3.9. 1) Let $w = r e^{i\theta} \in W'_{2N}$ with $1 - 2^{-2N} < r < 1 - 2^{-2N-1}$ and $|\theta| < \pi 2^{-2N}$. As $-(\log r + i\theta) \in B_{0,2N}$, one has $-(\log r + i\theta) = f(z_0)$ for some $z_0 \in \mathbb{D}$. Similarly, $-(\log r + i\theta) + 2k\pi i$, for $1 \leq k < l_N$, belongs to $B_{k,2N}$ and can be written as $f(z_k)$, with $z_k \in \mathbb{D}$. The z_k 's, $0 \leq k < l_N$, are distinct and satisfy $\varphi(z_k) = e^{-f(z_k)} = e^{-f(z_0)} = w$ for $0 \leq k < l_N$, thanks to the $2\pi i$ -periodicity of the exponential function.

2) We have $A(E_n) \lesssim e^{-2\varepsilon_{2n+2}} 4^{-2n} \leq 4^{-2n}$ (the term $e^{-2\varepsilon_{2n+2}}$ coming from the Jacobian of e^{-z}) and we observe that

$$w \in E_n \implies |w|^{2p-2} \leq (1 - 2^{-2n-1})^{2p-2} \lesssim e^{-p4^{-n}}.$$

It is easy to see that $n_\varphi(w) \leq l_n$ for $w \in E_n$; thus we obtain, forgetting the constant term $|\varphi(0)|^{2p} \leq 1$, using (2.5) and keeping in mind the fact that $n_\varphi(w) = 0$ for $w \notin \varphi(\mathbb{D})$:

$$\begin{aligned} \|\varphi^p\|_{\mathcal{D}}^2 &= p^2 \int_{\varphi(\mathbb{D})} |w|^{2p-2} n_\varphi(w) dA(w) \\ &\leq p^2 \left(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} n_\varphi(w) dA(w) \right) \\ &\leq p^2 \left(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} l_n dA(w) \right) \\ &\lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p4^{-n}}, \end{aligned}$$

ending the proof of Lemma 3.9. □

End of the proof of Theorem 3.8. Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

$$\mu(W) \geq \mu(W'_{2N}) = \int_{W'_{2N}} n_\varphi dA \geq l_N A(W'_{2N}) \gtrsim l_N h^2,$$

which implies that $\sup_{0 < h < 1} h^{-2} \mu[W(1, h)] = +\infty$ and shows that C_φ is not bounded on \mathcal{D} by Zorboska's criterion ([17], Theorem 1), recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers (l_n) so as to have $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$. To this effect, we first observe that, if one sets $F(x) = x^2 e^{-x}$, we have:

$$p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} = \sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim 1.$$

Indeed, let s be the integer such that $4^s \leq p < 4^{s+1}$. We have:

$$\sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim \sum_{n=1}^s \frac{4^n}{p} + \sum_{n>s} F(4^{-(n-s-1)}) \lesssim 1 + \sum_{n=0}^{\infty} F(4^{-n}) < \infty,$$

where we used that F is increasing on $(0, 1)$ and satisfies $F(x) \lesssim \min(x^2, 1/x)$ for $x > 0$. We finally choose the non-decreasing sequence (l_n) of integers as:

$$l_n = \min(n, M_n^2).$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\begin{aligned} \|\varphi^p\|_{\mathcal{D}}^2 &\lesssim p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p 4^{-n}} l_n \\ &\leq p^2 \sum_{n=1}^p 16^{-n} e^{-p 4^{-n}} l_p + p^2 \sum_{n>p} 16^{-n} l_n \\ &\lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2 4^{-p} \lesssim M_p^2, \end{aligned}$$

as desired. This choice of (l_n) gives us an unbounded composition operator on \mathcal{D} such that $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$, which ends the proof of Theorem 3.8. \square

References

- [1] N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick, The Dirichlet space: a survey, *New York J. Math.* 17A (2011), 45–86.
- [2] B. Carl and I. Stephani, Entropy, Compactness and the Approximation of Operators, *Cambridge Tracts in Mathematics*, vol. 98 (1990).
- [3] T. Carroll and C. Cowen, Compact composition operators not in the Schatten classes, *J. Oper. Theory*, no. 26 (1991), 109–120.
- [4] C. Cowen and B. MacCluer, Composition operators on spaces of analytic functions, *CRC Press* (1994).
- [5] O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi, Level sets and composition operators on the Dirichlet space, *J. Funct. Anal.* 260, no. 6 (2011), 1721–1733.
- [6] P. Halmos, A Hilbert space problem book, Second Edition, *Graduate Texts in Mathematics* 19, Springer-Verlag (1982).
- [7] W. H. Hastings, A Carleson theorem for Bergman spaces, *Proc. Amer. Math. Soc.* 52 (1975), 237–241.
- [8] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on Bergman-Orlicz spaces, *Trans. Amer. Math. Soc.* 365, no. 8 (2013), 3943–3970.
- [9] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some new properties of composition operators associated to lens maps, *Israel J. Math.* 195, no. 2 (2013), 801–824.

- [10] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on the Dirichlet space and capacity of sets of contact points, *J. Funct. Anal.* 264 (2013), no. 4, 895–919.
- [11] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Approximation numbers of composition operators on the Dirichlet space, *Arkiv för Mat.* doi. 10.1007/s11512-013-0194-z
- [12] D. Li, H. Queffélec and L. Rodríguez-Piazza, On approximation numbers of composition operators, *J. Approx. Theory* 164, no. 4 (2012), 431–459.
- [13] D. Li, H. Queffélec and L. Rodríguez-Piazza, Estimates for approximation numbers of some classes of composition operators on the Hardy space, *Ann. Acad. Sci. Fenn. Math.*, Vol. 38 (2013), 1–18.
- [14] B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* 38, no. 4 (1986), 878–906.
- [15] A. Pietsch, s -numbers of operators in Banach spaces, *Studia Math.* LI (1974), 201–223.
- [16] W. T. Ross, The classical Dirichlet space, Recent advances in operator-related function theory, 171–197, *Contemp. Math.* 393, Amer. Math. Soc., Providence, RI (2006).
- [17] N. Zorboska, Composition operators on weighted Dirichlet spaces, *Proc. Amer. Math. Soc.* 126, no. 7 (1998), 2013–2023.

Daniel Li, Univ Lille Nord de France,
 U-Artois, Laboratoire de Mathématiques de Lens EA 2462
 & Fédération CNRS Nord-Pas-de-Calais FR 2956,
 Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18,
 F-62 300 LENS, FRANCE
 daniel.li@euler.univ-artois.fr

Hervé Queffélec, Univ Lille Nord de France,
 USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS
 Nord-Pas-de-Calais FR 2956,
 F-59 655 VILLENEUVE D’ASCQ Cedex, FRANCE
 Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza, Universidad de Sevilla,
 Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS,
 Apartado de Correos 1160,
 41 080 SEVILLA, SPAIN
 piazza@us.es