# Poincaré duality of the basic intersection cohomology of a Killing foliation

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**Abstract**

We prove that the basic intersection cohomology $\mathbb{H}^{*}_{\overline{p}}(M/\mathcal{F})$, where $\mathcal{F}$ is the singular foliation determined by an isometric action of a Lie group $G$ on a compact manifold $M$, verifies the Poincaré duality property.


Basic cohomology theories are cohomology theories taking into account the particular structure of a foliated manifold and defined using differential forms. The notion was introduced by B. Reinhardt in 1959, cf. [23], in his fundamental study of Riemannian foliations. He proved that the basic cohomology of a compact manifold with a Riemannian foliation is finite dimensional and that the Poincaré duality works in this cohomology.

However, the proofs had some gaps, and the theorems were considered as conjectures. In the paper [8], based his PhD thesis, Carrière published an example of a 1-dimensional Riemannian foliation (a Riemannian flow) on a compact 3-manifold for which the top basic cohomology, of degree 2, is trivial. Therefore the basic cohomology of this Riemannian foliation cannot satisfy the Poincaré duality property. The paper also presented a classification of Riemannian flows on compact 3-manifolds.

Using this classification Y. Carrière noticed that the basic cohomology of foliations given by isometric flows has the Poincaré duality property, and that the only class of flows whose basic cohomology does not satisfy the Poincaré duality property consists of foliations which cannot be defined by isometric flows.

That remark led him to formulate a conjecture that the non-triviality of the top basic cohomology space, which is necessary for the Poincaré duality to hold, is equivalent to the tautness of the Riemannian foliation, i.e. the existence of a bundle-like metric for which all the leaves of the foliation are minimal sub manifolds. For dimension one foliations it is equivalent to the fact that the foliation is defined by an isometric action, a Killing vector field.

For over a decade, the conjecture was the subject of intensive study by a group of *foliators* and was finally solved by Masa, cf. [14], and refined Álvarez, cf. [1]. The best account of the development of the theory up to 1995 can be found in Tondeur's book [33]. For more recent developments see [27].


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The case of singular Riemannian foliations (SRF for short) is much more subtle. The suspension of a linear flow on a sphere provides an example of a singular isometric flow for which the top-dimensional basic cohomology group is isomorphic to $\mathbb{R}$, but this cohomology fails to satisfy the Poincaré duality property (see for example [27]). Therefore in the case of SRFs there is no direct connection between an SRF being defined by an isometric action of a Lie group, the non-triviality of the top dimensional basic cohomology group and the Poincaré duality property of the basic cohomology. It seems that the main reason is the fact that the basic cohomology does not take into account the structure of the set of singular orbits. Moreover, V. Miquel Molina and the second author proved that an SRF cannot be taut in the classical sense, cf. [15]. All these facts stress that in the research into the geometrical properties of SRFs and their relation to their topological ones we have to use more subtle instruments which take into account the structure of the set of singularities. The well-developed theory of singularities provide such tools: perverse forms and the intersection cohomology.

The intersection homology was introduced and studied by M. Goresky and R. MacPherson in 80's in the setting of pseudo manifolds. They established a generalized version of the Poincaré duality property. First versions of the theory used PL machinery [10] and sheaf theory [11]. The first version of the intersection cohomology by means of (perverse) differential forms was proposed by J.-L. Brylinski [7] (see also [28]). This point of view requires some extra data to introduce the notion of perverse form (Thom-Mather system, blow-up, Riemannian metric, . . . ).

In [29] we adapted these notions to the foliated case and we defined basic perverse forms and basic intersection cohomology (BIC for short). We used the fact that on a manifold foliated by an SRF there is a natural stratification defined using the dimension of leaves, the dimension of leaf closures, and holonomy. A remarkable fact is that these perverse forms do not involve any extra data in order to be defined.

In his thesis, [24], J.I. Royo Prieto demonstrated the Poincaré duality property of the BIC for singular Riemannian flows and the singular version of the Molino-Sergiescu theorem, cf. [20]. Inspired by these results, we have started to investigate possible generalizations and to study these cohomologies and their relation to tautness; to this study we dedicated a series of papers, (cf. [25], [26], [27]).

In the paper [29] we studied singular Riemannian foliations with compact leaves on compact manifolds. The leaf space of such a foliation is a pseudomanifold. The basic intersection cohomology is isomorphic the the corresponding intersection cohomology of the leaf space, and the Poincaré duality property in the basic intersection cohomology is equivalent to the Poincaré duality property in the intersection cohomology of the leaf space. The relation between the Poincaré duality property, the non-triviality of the top dimensional BIC and the tautness in the case of SRFs is more complex.

Summing up our main results, let us say that in [30] we demonstrated that the BIC of SRFs defined by an isometric action of an abelian Lie group is finite dimensional and satisfies the Poincaré duality. In the most recent paper [31] we showed that the BIC for SRFs defined by an isometric action of a Lie group is finite dimensional.

In the paper we prove the Poincaré duality property for the BIC of SRF defined by an isometric action of a Lie group $G$ without any restriction imposed on the Lie group. We hope to extend our results to the case of OLF of [19] and general SRFs. However, there is an important technical difference. We have managed to demonstrate our result due to a nice description and properties of the stratification defined by the action of the closure of $G$: tubular neighborhoods of the strata can be assumed to be twisted products. So we can work with trivial tubular neighborhoods, which makes the calculations much easier. In the general case, tubular neighborhoods of the strata of the induced stratification are much more complicated.

Singular Riemannian foliations were introduced and studied by P. Molino. His book [18] contains the best introduction to the subject. The associated stratification was presented and studied in [17]. As a useful reading we also suggest [21, Section 9.3] of the book which contain a lot of information on isometric actions of Lie groups.

The classical reference for a compact group action is [5].

Lie group actions appear in the study of physical models as groups of symmetries of such models. Reduction theory is of great interest and the object of study of numerous papers and books, e.g.[21]

In the present article we use the notation introduced and developed in our previous papers on BIC [29] and isometric actions [30] and [31].

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In the sequel $M$ is a connected, second countable, Haussdorff, without boundary and smooth (of class $C^\infty$) manifold.

## 1. Killing Foliations.

A smooth action $\Phi\colon G\times M\to M$ of the Lie group $G$ is an *isometric action* when there exists a Riemannian metric $\mu$ on $M$ preserved by $G$. Moreover, the isometric action $\Phi$ is *tame* when the closure of $G$ in $\mathrm{Iso}\,(M,\mu)$ is compact. This is always the case when the manifold $M$ is compact (cf. [13, Section II, Theorem 1.2]).

The connected components of the orbits of a tame action determine a partition $\mathcal{F}$ on $M$. In fact, this partition is a singular Riemannian foliation that we shall call *Killing foliation* (cf. [18]).

Notice that $\mathcal{F}$ is also a conical foliation in the sense of [29]. So, its basic intersection cohomology can be used for the study of $\mathcal{F}$. In this work, we prove the Poincaré duality property of this cohomology.

If the action $\Phi$ is tame, then it is a restriction a smooth action $\Phi\colon K\times M\to M$ where $K$ is a compact Lie group containing $G$. The group $K$ is not unique. We always can choose $K$ in such a way that $G$ is dense in $K$. We shall say that $K$ is a *tamer group*.

Notice that the tamer group is not unique. Consider the two actions $\Phi\colon \mathbb{T}^2\times M\to M$ and $\Phi'\colon \mathbb{T}^3\times M\to M$ on $M=\mathbb{S}^1$ defined by $\Phi((u,v),z)=u\cdot v\cdot z$ and $\Phi'((u,v,w),z)=u\cdot v\cdot w\cdot z$. Take the subgroups $G=\{(e^{2\pi\alpha ti},e^{2\pi\beta ti})/t\in\mathbb{R}\}$ and $G'=\{(e^{2\pi\alpha ti},e^{2\pi\beta ti},e^{2\pi\gamma ti})/t\in\mathbb{R}\}$ where $\alpha,\beta,\gamma$ are three given reals which are $\mathbb{Q}$-independents. The two restrictions $\Phi\colon G\times M\to M$ and $\Phi'\colon G'\times M\to M$ define the one-leaf Killing foliation of $M$ but they have different tamer groups: $\mathbb{T}^2$ and $\mathbb{T}^3$.

For the rest of the work, let $\mathcal{F}$ be a Killing foliation. We fix an effective tame action $\Phi\colon G\times M\to M$, with $G$ connected, defining $\mathcal{F}$. We also fix a tamer group $K$. In this case, $G$ is normal in $K$ and the quotient group $K/G$ is commutative (see [22, 1.2 and Theorem 1.3]).

There are three key facts for this work: compactness of $K$ (which ensures the existence of invariant metrics, tubular neighborhoods, Molino's blow-up, isotropy type stratification), density of $G$ in $K$ (facilitates the computation of the intersection cohomology of a twisted product) and the normality of the subgroup $G$ in the group $K$ (which is the key factor in the compatibility between stratifications defined by $G$ and $K$).

We denote by $b=\dim G$ and $m=\dim M$. The induced foliation on the regular stratum $R_{\mathcal{F}}$ is regular, and its dimension will be denoted by $w=\dim\mathcal{F}$.

## 2. Stratification.

Classifying the points of $M$ following the dimension of the leaves of $\mathcal{F}$ one gets the *stratification* $\mathsf{S}_{\mathcal{F}}$. It is determined by the equivalence relation $x \sim y \Leftrightarrow \dim G_x = \dim G_y$. The elements of $\mathsf{S}_{\mathcal{F}}$ are called *strata*. The open stratum $R_{\mathcal{F}}$ is the *regular stratum*, and the other strata are the *singular strata*.

We fix a base point $p \in R_{\mathcal{F}}$ and we put $(G_p)_0 = L$, where $E_0$ stands for the identity component of the Lie group $E$ (the one containing the identity). When $G$ is abelian then $(G_p)_0 = L$ for each $p \in R_{\mathcal{F}}$. This implies that $L$ acts non-effectively on $R_{\mathcal{F}}$ and therefore on $M$. So, $L = \{e\}$.

This group $L$ is generic in the following sense.

**Proposition 2.1** *For each $x \in R_{\mathcal{F}}$ there exists $k \in K$ with $(G_x)_0 = kLk^{-1}$. Moreover, the choice $x \mapsto k$ can be done locally in a smooth way.*

*Proof.* We consider a point $x \in R_{\mathcal{F}}$ and we will find an open neighborhood $V \subset R_{\mathcal{F}}$ of $x$ and a smooth map $f : V \to K$ with $(G_y)_0 = f(y)(G_x)_0 f(y)^{-1}$, for each $y \in V$.

Since the Lie group $K$ is compact there exists a tube $K \times_H T$ around the point $x$ ([5, Chapter II, Theorem 5.4]) where $H = K_x$ and $T \subset R_{\mathcal{F}}$ is a transversal to the orbit $K(x)$ containing $x$. In particular, $\dim G_{<e,z>} = \dim G_x$ for each $z \in T$. Here $< k, z >$ is a generic point of the twisted product $K \times_H T$, and $e$ is the identity of $G$.

The group $G$ being normal in $K$, we get $G_{<k,z>} \overset{[5, pag.82]}{=} G \cap kH_z k^{-1} = k(G \cap H)_z k^{-1} = kG_{<e,z>}k^{-1}$ and therefore $K \times_H T \subset R_{\mathcal{F}}$ as $\dim G_{<k,z>} = \dim G_x$.

We consider a neighborhood $W \subset K$ of $e$. This neighborhood can be chosen small enough to ensure the existence of a smooth section $\sigma : \gamma(W) \to W$ of the canonical projection $\gamma : K \to K/H$. Let $\Gamma : K \times_H T \to K/H$ be the projection. So $V = \Gamma^{-1}(\gamma(W))$ is a neighborhood of $x$ in $K \times_H T \subset R_{\mathcal{F}}$. Denote by $f : V \to K$ the smooth map defined by $f(y) = \sigma(\Gamma(y))$. Let us consider a point $y =< k, z >\in V$. Since $kH = f(y)H$, then $y =< f(y), z' >$, and therefore $G_y = f(y)(G \cap H)_{z'} f(y)^{-1}$. On the other hand, $G_x = G \cap K_x = G \cap H$ and $\dim G_x = \dim G_y$ give $\dim(G \cap H) = \dim(G \cap H)_{z'}$ and therefore:

$$(G_y)_0 = f(y)\,((G \cap H)_{z'})_0\, f(y)^{-1} = f(y)\,(G \cap H)_0\, f(y)^{-1} = f(y)(G_x)_0 f(y)^{-1}.$$

This ends the proof. ♣

## 3. Presentation of the Poincaré duality property .

The basic intersection cohomology $\mathbb{H}^{*}_{\overline{p}}(M/\mathcal{F})$, relatively to the perversity $\overline{p}$, was introduced in [29, Section 3] for the study of conical foliations[1]. It coincides with the usual intersection cohomology when the leaves are compact [29, Theorem 1].

We define the *support* of a perverse form $\omega \in \Pi^{*}_{\mathcal{F}}(M)$ as supp $\omega = \overline{\{x \in M \backslash \Sigma_{\mathcal{F}} \ / \ \omega(x) \neq 0\}}$, where the closure is taken in $M$. We denote by $\Omega^{*}_{\overline{q},c}(M/\mathcal{F}) = \left\{\omega \in \Omega^{*}_{\overline{q}}(M/\mathcal{F}) \middle/ \text{ supp } \omega \text{ is compact}\right\}$ the complex of intersection basic differential forms with compact support relatively to the perversity $\overline{q}$. The cohomology $\mathbb{H}^{*}_{\overline{q},c}(M/\mathcal{F})$ of this complex is the *basic intersection cohomology with compact support* of $(M, \mathcal{F})$, relatively to the perversity[2] $\overline{q}$.

[1]We refer the reader to [31] for notation and main properties of this notion.
[2]We refer the reader to [30] for notation and main properties of this notion.

The goal of this work is to construct a non-degenerate pairing $P_M \colon \mathbb{H}^{*}_{\overline{p}}(M/\mathcal{F}) \times \mathbb{H}^{m-w-*}_{\overline{q},c}(M/\mathcal{F}) \longrightarrow \mathbb{R}$, inducing the Poincaré duality

$$P_M \colon \mathbb{H}^{*}_{\overline{p}}(M/\mathcal{F}) \longrightarrow \mathrm{Hom}\left(\mathbb{H}^{m-w-*}_{\overline{q},c}(M/\mathcal{F}), \mathbb{R}\right).$$

Here, $\overline{p}$ and $\overline{q}$ are complementary perversities, that is, $\overline{p} + \overline{q} = \overline{t}$, with

$$\overline{t}(S) = \operatorname{codim}_M \mathcal{F} - \operatorname{codim}_S \mathcal{F}_S - 2, \tag{1}$$

where $S$ is a singular stratum and $\mathcal{F}_S$ the restriction of $\mathcal{F}$ to $S$.

We have seen in [31, Theorem 4.4] that the discussed intersection cohomologies are finite dimensional when $M$ is compact. So the above Poincaré duality becomes the isomorphism

$$\mathbb{H}^{*}_{\overline{p}}(M/\mathcal{F}) \cong \mathbb{H}^{m-w-*}_{\overline{q}}(M/\mathcal{F}).$$

## 4. Twisted product.

Twisted products are local building blocks of SRFs. Their basic intersection cohomology have been calculated in [31, Proposition 5]. Below we present the compact support version of this result.

Let $K$ be a compact Lie group, $G$ a normal subgroup of $K$ and $H$ a closed subgroup of $K$. We consider a twisted product $K \times_H N$, where $N$ is a manifold endowed with an effective smooth action $\Theta \colon H \times N \to N$. The restriction $\Theta_0 \colon (G \cap H)_0 \times N \to N$ is a tame action, where the tamer group $H'$ is the closure of $(G \cap H)_0$ in $H$. The associated Killing foliation is denoted by $\mathcal{N}$. We denote by $\Phi \colon G \times (K \times_H N) \to (K \times_H N)$ the associated tame action, and the induced foliation by $\mathcal{W}$. The foliation defined by the tame (left or right!) action of $G$ on $K$ is denoted by $\mathcal{K}$. The foliation defined by the tame right action of $GH$[3] on $K$ is denoted by $\mathcal{E}$.

**Proposition 4.1** $\mathbb{H}^{*}_{\overline{q},c}(K \times_H N/\mathcal{W}) = \left(H^{*}(K/\mathcal{E}) \otimes \mathbb{H}^{*}_{\overline{q},c}(N/\mathcal{N})\right)^{H/H_0}$.

*Proof.* It suffices to follow [31, Proposition 5] taking into the account the fact that, given a differential form $\omega$ on $K \times_H R_{\mathcal{W}}$, we have:

$$\prod^{*} \omega \in \Omega^{*}_{\overline{q},c}(K \times N/\mathcal{E} \times \mathcal{N}) \Longleftrightarrow \omega \in \Omega^{*}_{\overline{q},c}(K \times_H N/\mathcal{W}).$$

It is so as the canonical projection $\prod \colon K \times N \to K \times_H N$ is an onto map and that the Lie groups $K$ and $H$ are compact. ♣

## 5. Tangent volume form.

In order to construct the pairing giving the Poincaré duality we need to introduce a particular tangent volume form of the orbits of $\Phi$.

We fix a bi-invariant metric $\nu$ on $\mathfrak{k}$, the Lie algebra of $K$ which exists since $K$ is compact (see for example [3, pag. 247]). Consider $\{u_1, \ldots, u_f\}$ an orthonormal basis of $\mathfrak{k}$ where $\{u_1, \cdots, u_b\}$ is a basis of $\mathfrak{g}$, $\{u_1, \cdots, u_w\}$ is a basis of $\mathfrak{l}^{\perp}$, the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$, and $\{u_{w+1}, \cdots, u_b\}$ is a basis of $\mathfrak{l}$. Here, $\mathfrak{g}$ (resp. $\mathfrak{l}$) denotes the Lie algebra of $G$ (resp. $L$). We take $\tau = u_1^* \wedge \cdots \wedge u_w^*$, the associated volume form of $\mathfrak{l}^{\perp}$.

[3] This is the minimal subgroup of $K$ containing $G$ and $H$. Normality of $G$ implies that $GH = \{g \cdot h / g \in G, h \in H\}$. The Lie algebra of $GH$ is the sum of the Lie algebras of $G$ and $H$.

We denote by $V_u$ the fundamental vector field on $M$ associated to $u \in \mathfrak{g}$. A *tangent volume form* of $\Phi$ is a $G$-invariant differential form $\eta \in \Pi^{w}_{\mathcal{F}}(M)$ verifying:

$$\eta(V_{v_1}(x), \ldots, V_{v_w}(x)) = \tau\left(\mathrm{Ad}\,(\ell^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell^{-1}) \cdot v_w\right), \tag{2}$$

where $\{v_1, \ldots, v_w\} \subset \mathfrak{g}$, $x \in R_{\mathcal{F}}$ and $(G_x)_0 = \ell L \ell^{-1}$. The existence of a tangent volume form implies that the foliation $\mathcal{F}$ is tangentially orientable on the regular stratum.

In the next Proposition we prove the existence of a tangent volume form under a suitable orientation conditions on the manifold and on the foliation. The tame action $\Phi$ is said to be *orientable* if

(i) the manifold $M$ is orientable, and

(ii) the adjoint action $\mathrm{Ad} : N_K(L) \times \mathfrak{l} \to \mathfrak{l}$ is orientation preserving.

We say that a Killing foliation is *orientable* if it is induced by an orientable action.

### 5.1 Remarks.

(a) Condition (ii) does not depend on the choice of the point $p$ defining $L$. Let us verify that fact. Choose another point $p' \in R_{\mathcal{F}}$, defining $L' = (G_{p'})_0$. Without loss of generality we can suppose that $p'$ is near enough to $p$ (the connectedness of $R_{\mathcal{F}}$) in order to apply [5, Chapter II, Corollary 5.4] and find $k \in K$ with $k^{-1} K_{p'} k \subset K_p$. Since $G$ is normal in $K$, we get $k^{-1} G_{p'} k \subset G_p$, and therefore $k^{-1}(G_{p'})_0 k \subset (G_p)_0$. Since $p, p' \in R_{\mathcal{F}}$, we conclude that $k^{-1}(G_{p'})_0 k = (G_p)_0$, that is $k^{-1} L' k = L$. This gives the claim since $\mathrm{Ad} : N_K(L') \times \mathfrak{l}' \to \mathfrak{l}'$ becomes $\mathrm{Ad} : k N_K(L) k^{-1} \times \mathrm{Ad}\,(k)(\mathfrak{l}) \to \mathrm{Ad}\,(k)(\mathfrak{l})$.

(b) By connectedness, the group $K$ preserves the orientation of $\mathfrak{g}$. So the condition (ii) is equivalent to

(ii') the adjoint action $\mathrm{Ad} : N_K(L) \times \mathfrak{l}^{\perp} \to \mathfrak{l}^{\perp}$ is orientation preserving.

(c) Condition (ii) is verified in the case where $N_K(L)$ is connected. In particular, when $G$ is abelian or when $\mathfrak{l} = 0$, that is, if $\dim \mathcal{F} = \dim G$.

(d) There are non-orientable actions on orientable manifolds. For example, consider the action of $G = \mathbb{S}^3$, on the twisted product $M = \mathbb{S}^3 \times_{N_{\mathbb{S}^3}(\mathbb{S}^1)} \mathbb{S}^3 = \mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{S}^2$ (cf. [5, pag. 80]). Here, $L = \mathbb{S}^1$ and the element $j \in N_{\mathbb{S}^3}(\mathbb{S}^1)$ acts on $\mathfrak{l} = \mathbb{R}$ by multiplying by $-1$, which does not preserve the orientation. Since $M/\mathcal{F} = \mathbb{RP}^2$ then the Poincaré duality property does not hold.

**Proposition 5.2** *For any orientable action there exists a $K$-invariant tangent volume form.*

*Proof.* Let $\Phi : G \times M \to M$ be an orientable action.

In order to decrease the depth of the stratification, we are going to use the Molino's blow up $\widehat{M}$ and the stratification $\mathsf{S}_{K,M}$ defined by the action of $K$ (see Appendix). We prove the following statement by induction on depth $\mathsf{S}_{K,M}$:

"There exists a $K$-invariant differential form $\overline{\eta} \in \Pi^{w}_{\mathcal{F}\times I}(M \times [0,1[^p)$ verifying

$$\overline{\eta}((V_{v_1}(x), 0), \ldots, (V_{v_w}(x), 0)) = \tau(\mathrm{Ad}\,(\ell^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell^{-1}) \cdot v_w), \tag{3}$$

where $\{v_1, \ldots, v_w\} \subset \mathfrak{g}$, $x \in R_{\mathcal{F}}$ and $(G_x)_0 = \ell L \ell^{-1}$ with $\ell \in K$."

The existence of $\eta$ is proven by taking $p = 0$.

*First case:* depth $\mathsf{S}_{K,M} = 0$.

Since $M = R_{\mathcal{F}}$ then we have $\Pi^{*}_{\mathcal{F}}(M \times [0,1[^p) = \Omega^{*}((M \times [0,1[^p)/(\mathcal{F} \times \mathcal{I}))$. Take $T(\mathcal{F} \times \mathcal{I})$ the sub-bundle of the tangent bundle $T(M \times [0,1[^p)$ formed by vectors tangents to the leaves of the foliation $\mathcal{F} \times \mathcal{I}$. Here $\mathcal{I}$ denotes the point wise foliation of $[0,1[^p$. Since the foliation $\mathcal{F}$ is $K$-invariant, then it suffices to define $\overline{\eta}$ on $T(\mathcal{F} \times \mathcal{I})$ and to extend it by 0 to a $K$-invariant complement of $T(\mathcal{F} \times \mathcal{I})$ in $T(M \times [0,1[^p)$. In fact, this restriction is given by (3). It remains to prove that $\overline{\eta}$ is well-defined, smooth on the tangent bundle $T(\mathcal{F} \times \mathcal{I})$ and $K$-invariant. Let us check that.

– *The definition (3) does not depend on $\ell$.* Let us consider $\ell' \in K$ with $(G_x)_0 = \ell' L \ell'^{-1}$. Then $\ell'^{-1}\ell \in N_K(L)$. This gives

$$\tau(\mathrm{Ad}\,(\ell'^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell'^{-1}) \cdot v_w) = \tau(\mathrm{Ad}\,(\ell'^{-1}\ell)\,\mathrm{Ad}\,(\ell^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell'^{-1}\ell)\,\mathrm{Ad}\,(\ell^{-1}) \cdot v_w).$$

Since the metric $\nu$ has been chosen to be bi-invariant, then the element $\mathrm{Ad}\,(\ell'^{-1}\ell)$ preserves the metric $\nu$. It also preserves the orientation of $\mathfrak{l}^{\perp}$ (see (ii')). So, we get

$$\tau(\mathrm{Ad}\,(\ell'^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell'^{-1}) \cdot v_w) = \tau(\mathrm{Ad}\,(\ell^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell^{-1}) \cdot v_w).$$

– *The definition (3) is smooth.* Consider $x \in M$. From Proposition 2.1 we know that there exist a neighborhood $V \subset M$ and a smooth map $f\colon V \to K$ such that $(G_y)_0 = f(y) L f(y)^{-1}$ for each $y \in V$. The previous point says that we can choose $\ell = f(y)$ in definition (3). So in this neighborhood we have $\overline{\eta}(V_{v_1}(y), \ldots, V_{v_w}(y)) = \tau(\mathrm{Ad}\,(f(y)^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(f(y)^{-1}) \cdot v_w)$, which is smooth.

– *The form $\overline{\eta}$ is $K$-invariant.* If $k \in K$ we get $(G_{k\cdot x})_0 = k\ell L \ell^{-1} k^{-1}$ and

$$\begin{aligned}
(k^*\overline{\eta})(V_{v_1}(x), \ldots, V_{v_w}(x)) &= \overline{\eta}(k_* V_{v_1}(x), \ldots, k_* V_{v_w}(x)) \\
&= \overline{\eta}(V_{\mathrm{Ad}\,(k)\cdot v_1}(k \cdot x) \ldots, V_{\mathrm{Ad}\,(k)\cdot v_w}(k \cdot x)) \\
&= \tau(\mathrm{Ad}\,(\ell^{-1}k^{-1})\,\mathrm{Ad}\,(k) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell^{-1}k^{-1})\,\mathrm{Ad}\,(k) \cdot v_w) \\
&= \tau(Ad(\ell^{-1}) \cdot v_1, \ldots, \mathrm{Ad}\,(\ell^{-1}) \cdot v_w) = \overline{\eta}(V_{v_1}(x) \ldots, V_{v_w}(x)).
\end{aligned}$$

*Second case:* depth $\mathsf{S}_{\mathcal{F}} > 0$.

By induction hypothesis there exists a $K$-invariant differential form $\overline{\eta}_0 \in \Pi^{w}_{\widehat{\mathcal{F}}}\big(\widehat{M} \times [0,1[^p\big)$ verifying (3). Associated to $\widehat{M}$, we have the $K$-equivariant imbedding $\sigma\colon M \backslash S_{min} \to \mathcal{L}^{-1}(M \backslash S_{min})$, defined by $\sigma(z) = (z,1)$. The differential form $\overline{\eta} = (\sigma \times \mathrm{identity}_{[0,1[^p})^* \eta_0$ belongs to $\Omega^{w}(R_{\mathcal{F}} \times [0,1[^p)$, it is $K$-invariant and verifies (3). It remains to prove that $\overline{\eta} \in \Pi^{w}_{\mathcal{F}\times\mathcal{I}}(M \times [0,1[^p)$, which is a local property.

So we can assume that $M = T_{min}$ and prove $(\nabla_{min} \times \mathrm{identity}_{[0,1[^p})^* \overline{\eta} \in \Pi^{w}_{\mathcal{F}\times\mathcal{I}}\big(D_{min} \times [0,1[^{p+1}\big)$ (cf. [30, 3.1.1 (e)]). This is the case since the map $\sigma \circ \nabla_{min} \colon D_{min} \times ]0,1[ \to D_{min} \times ]-1,1[$ is just the inclusion and we have $\overline{\eta}_0 \in \Pi^{w}_{\mathcal{F}\times\mathcal{I}}(D_{min} \times ]-1,1[ \times [0,1[^p)$. ♣

In this Proposition the density of $G$ in $K$ is used to ensure that $G$ is a normal subgroup of $K$.

**Proposition 5.3** *An invariant tangent volume form $\eta$ verifies:*

*(a) For each $\omega \in \Omega^{m-w-1}_{\bar{t}}(M/\mathcal{F})$ the product $\omega \wedge d\eta$ is 0.*

*(b) For each $\omega \in \Omega^{m-w}_{\bar{t},c}(M/\mathcal{F})$ the integral $\displaystyle\int_{R_{\mathcal{F}}} \omega \wedge \eta$ is finite and it does not depend on the choice of $\eta$.*

*(c) For each $\omega \in \Omega^{m-w-1}_{\bar{t},c}(M/\mathcal{F})$ the integral $\displaystyle\int_{R_{\mathcal{F}}} d(\omega \wedge \eta)$ is 0.*

*Proof.*

(a) Since the question is a local one, it is enough to prove $\omega \wedge d\eta = 0$ on an open subset $V \subset R_{\mathcal{F}}$ (cf. proof of Proposition 2.1).

For each $y \in V$ we have $(G_y)_0 = f(y)Lf(y)^{-1}$. Then $\{V_{\mathrm{Ad}(f(y))\cdot v_1}(x), \ldots, V_{\mathrm{Ad}(f(y))\cdot v_w}(x)\}$ is a basis of $T_yG(y)$. For degree reasons it suffices to prove that we have $i_{V_{\mathrm{Ad}(f(y))\cdot v_1}(x)} \cdots i_{V_{\mathrm{Ad}(f(y))\cdot v_w}(x)}(\omega \wedge d\eta) = 0$. Since $\omega$ is a basic form and $\eta$ is a $K$-invariant form, we can write

$$\begin{aligned}
i_{V_{\mathrm{Ad}(f(y))\cdot v_1}(y)} \cdots i_{V_{\mathrm{Ad}(f(y))\cdot v_w}(y)}(\omega \wedge d\eta) &= (-1)^w \omega \wedge i_{V_{\mathrm{Ad}(f(y))\cdot v_1}(y)} \cdots i_{V_{\mathrm{Ad}(f(y))\cdot v_w}(y)} d\eta = \omega \wedge d\left(i_{V_{\mathrm{Ad}(f(y))\cdot v_1}(y)} \cdots i_{V_{\mathrm{Ad}(f(y))\cdot v_w}(y)}\eta\right) \\
&= \omega \wedge d\left(\eta\left(V_{\mathrm{Ad}(f(y))\cdot v_1}(y), \ldots, V_{\mathrm{Ad}(f(y))\cdot v_w}(y)\right)\right) = \omega \wedge d\left(\tau(v_1, \ldots, v_w)\right) \\
&= \omega \wedge d1 = 0.
\end{aligned}$$

Notice that this property holds for any $\omega \in \Pi^{m-w-1}_{\mathcal{F}}(M)$.

(b) To demonstrate the finiteness, it suffices to prove that $\displaystyle\int_{R_{\mathcal{F}} \times [0,1[^p} \gamma < \infty$ where $\gamma \in \Pi^{w}_{\mathcal{F}\times\mathcal{I}}(M \times [0,1[^p)$ is of compact support. We proceed by induction on the depth of $\mathsf{S}_{K,M}$. When depth $\mathsf{S}_{K,M} = 0$ the result is clear since $M = R_{\mathcal{F}}$ (see Appendix).

In the general case we know that the result is true for $M \backslash S_{min} \times [0,1[^p$ and $(T_{min} \backslash S_{min}) \times [0,1[^p$ by induction. It remains to consider $T_{min} \times [0,1[^p$. We have seen that we can identify the perverse forms of $T_{min} \times [0,1[^p$ with the perverse forms of $D_{min} \times [0,1[^{p+1}$ through the map

$$\nabla_{min} \times Identity_{[0,1[^p} \colon D_{min} \times [0,1[ \times [0,1[^p \equiv D_{min} \times [0,1[^{p+1} \longrightarrow T_{min} \times [0,1[^p$$

(this is a general result for basic intersection cohomology proved in [30, 3.4.1 (d)]). Since this map is a diffeomorphism between $D_{min} \times ]0,1[ \times [0,1[^p$ and $(T_{min} \backslash S_{min}) \times [0,1[^p$, then we have

$$\int_{R_{\mathcal{F}_{T_{min}}} \times [0,1[^p} \gamma = \int_{R_{\mathcal{F}_{D_{min}}} \times ]0,1[ \times [0,1[^p} \gamma = \int_{R_{\mathcal{F}_{D_{min}}} \times [0,1[ \times [0,1[^p} \gamma = \int_{R_{\mathcal{F}_{D_{min}}} \times [0,1[^{p+1}} \gamma.$$

The induction hypothesis gives that this integral is finite.

Let $\mathcal{NF} \oplus \mathcal{TF}$ be a sub-bundle decomposition of $TR_{\mathcal{F}}$. Since $\omega$ is a basic form on $\mathbb{R}_{\mathcal{F}}$ then it vanishes on $\mathcal{TF}$. It is a top degree basic form, so $\omega \wedge \eta\,(\underbrace{a_1, \ldots a_{m-w}}_{\mathcal{NF}}, \underbrace{b_1, \ldots b_w}_{\mathcal{TF}}) = \omega(a_1, \ldots a_{m-w}) \cdot \eta(b_1, \ldots b_w)$ and $\eta(b_1, \ldots b_w)$ does not depend on $\eta$ (see (2)).

(c) Since supp $\omega$ is compact then it suffices to prove $\displaystyle\int_{U \cap R_{\mathcal{F}}} d(\omega \wedge \eta) = 0$ where $(U, \varphi)$ is a conical chart of $\mathcal{F}$ and $\omega \in \Omega^{\ell-1}_{\bar{t},c}(U/\mathcal{F})$ with supp $\omega \subset U$. Recall that the restriction $P_\varphi \colon \mathbb{R}^{m-n} \times \mathbb{S}^{n-1} \times ]0,1[ \longrightarrow U \cap R_{\mathcal{F}}$ is a diffeomorphism. Also, the pull-back $P_\varphi^* \colon \Pi^*_{\mathcal{F}}(U) \longrightarrow \Pi^*_{\mathcal{H}\times\mathcal{G}\times\mathcal{I}}\left(\mathbb{R}^{m-n}\times\mathbb{S}^{n-1}\times[0,1[\right)$ is a dgca isomorphism (this is a general result for basic intersection cohomology, see for example [30, Section 3.1]). So, we have:

$$\int_{U\cap R_{\mathcal{F}}} d(\omega\wedge\eta) \;=\; \int_{\mathbb{R}^{m-n}\times R_{\mathcal{G}}\times]0,1[} d(P_\varphi^*\omega\wedge P_\varphi^*\eta) = \int_{\mathbb{R}^{m-n}\times R_{\mathcal{G}}\times[0,1[} d(\omega\varphi\wedge\eta\varphi) \overset{Stokes}{=\!=} \int_{\mathbb{R}^{m-n}\times R_{\mathcal{G}}\times\{0\}} \omega\varphi\wedge\eta\varphi = 0$$

since, for any $\omega\in\Omega^{m-w-1}_{\overline{t}}(M/\mathcal{F})$, the restriction of $\omega\varphi$ to $\mathbb{R}^{m-n}\times R_{\mathcal{G}}\times\{0\}$ vanishes (this is a general result for basic intersection cohomology, see for example [30, 3.4.1 (c)]). ♣

## 6. The pairing.

In Section 9 we will prove the Poincaré duality property : $\mathbb{H}^*_{\overline{p}}(M/\mathcal{F}) \cong \mathrm{Hom}\,(\mathbb{H}^{m-w-*}_{\overline{q},c}(M/\mathcal{F});\mathbb{R})$, when $\mathcal{F}$ is orientable and the two perversities $\overline{p}$ and $\overline{q}$ are complementary. This isomorphism comes from the pairing $P_M$ constructed from the a tangent volume form $\eta$ (cf. Proposition 5.2) in the following way:

$$P_M\colon \Omega^*_{\overline{p}}(M/\mathcal{F})\times\Omega^{m-w-*}_{\overline{q},c}(M/\mathcal{F}) \longrightarrow \mathbb{R} \qquad \therefore \qquad (\alpha,\beta)\rightsquigarrow \int_{R_{\mathcal{F}}} \alpha\wedge\beta\wedge\eta. \tag{4}$$

Proposition 5.3 implies that this operator is well defined and that it induces the *pairing*

$$P_M\colon \mathbb{H}^*_{\overline{p}}(M/\mathcal{F})\times\mathbb{H}^{m-w-*}_{\overline{q},c}(M/\mathcal{F}) \longrightarrow \mathbb{R},$$

defined by $P_M([\alpha],[\beta]) = P_M(\alpha,\beta)$. Moreover, it does not depend on the choice of the tangent volume form $\eta$. The Poincaré duality property asserts that $P_M$ is a non degenerate pairing, that is, the operator

$$P_M\colon \mathbb{H}^*_{\overline{p}}(M/\mathcal{F}) \longrightarrow \mathrm{Hom}\left(\mathbb{H}^{m-w-*}_{\overline{q},c}(M/\mathcal{F}),\mathbb{R}\right)$$

defined by $P_M([\alpha])([\beta]) = \displaystyle\int_{R_{\mathcal{F}}} \alpha\wedge\beta\wedge\eta$ is an isomorphism.

## 7. Twisted product and Poincaré duality .

We first get the Poincaré duality property in the framework of a twisted product $K\times_H N$ (see Section 4). We fix a bi- invariant metric on the Lie algebra $\mathfrak{k}$ and we choose

$$B = \left\{u_1,\dots u_a,u_{a+1},\dots,u_w,u_{w+1},\dots,u_b,u_{b+1},\dots,u_c,u_{c+1},\dots,u_f\right\},\ \text{with } 0\le a\le w\le b\le c\le f$$

an orthonormal basis of $\mathfrak{k}$ with $\{u_1,\dots u_b\}$ basis of $\mathfrak{g}$, $\{u_{a+1},\dots u_c\}$ basis of the Lie algebra $\mathfrak{h}$ of $H$ and $\{u_{w+1},\dots u_b\}$ basis of $\mathfrak{l}$. Notice that $b = w$ when $G$ is abelian.

Let $\{\gamma_1,\dots,\gamma_f\}$ (reps. $\{\zeta_{a+1},\dots,\zeta_c\}$) be the dual forms associated to a basis $\{u_1,\dots,u_f\}$ (resp. $\{u_{a+1},\dots,u_c\}$ ) relatively to a bi -invariant metric $\nu$ on $K$ (resp. $H$-invariant Riemannian metric on $N$).

We shall use the following notation.

- $X_\bullet$ the fundamental vector fields of the right action of $G$ on $K$,
- $X^\bullet$ the fundamental vector fields of the left action of $G$ on $K$.

They are related $X_u(k) + X^{\mathrm{Ad}\,(k)\cdot u}(k) = 0$ if $k \in K$ and $u \in \mathfrak{g}$.

- $W_\bullet$ are the fundamental vector fields of the action $\Theta$.
- $V_\bullet = \prod_*(X^\bullet, 0)$ are the fundamental vector fields of the action $\Phi$.

First of all, we establish a relationship between the tangent volume forms of $\Phi$ and $\Theta_0$.

**Lemma 7.1** *If the action $\Phi\colon G \times (K \times_H N) \to (K \times_H N)$ is orientable then the action $\Theta_0\colon (G \cap H)_0 \times N \to N$ is also orientable. We denote by $\eta$ a tangent volume form of $\Phi$ (resp. $\eta_0$ of $\Theta_0$) associated to the metric $\nu$ (resp. of $\nu_{|\mathfrak{g}\cap\mathfrak{h}}$). Then*

$$(-1)^{ab}\gamma_{a+1} \wedge \cdots \wedge \gamma_b \wedge \prod^* \eta = \gamma_1 \wedge \cdots \wedge \gamma_b \wedge \eta_0 \text{ on the tangent bundle of } \mathcal{K} \times \mathcal{N}. \tag{5}$$

*Proof.* We fix a base point $v_0 \in R_{\mathcal{N}}$. Since $G_{<e,v_0>} = (G \cap H)_{v_0}$ then $< e, v_0 > \in R_{\mathcal{W}}$. We fix these two base points. So, we have the same $L$ for $\mathcal{N}$ and $\mathcal{W}$. The orientability of $\Theta_0$ comes now from the inclusion $N_{H'}(L) \subset N_K(L)$.

For the second part we use the foliated blow up $\prod\colon (K \times N, \mathcal{K} \times \mathcal{N}) \to (K \times_H N, \mathcal{W})$. We also recall that $\tau = u_1^* \wedge \cdots \wedge u_w^*$ and $\tau_0 = u_{a+1}^* \wedge \cdots \wedge u_w^*$ are the the associated volume forms of $\mathfrak{l}^\perp$ on $\mathfrak{g}$ and $\mathfrak{g} \cap \mathfrak{h}$, respectively.

The leaf of $\mathcal{K} \times \mathcal{N}$ at the point $(k, z) \in K \times N$ is generated by

$$\mathfrak{B} = \{X_{u_1}(k), \ldots, X_{u_b}(k), W_{\mathrm{Ad}\,(\ell)\cdot u_{a+1}}(z), \ldots, W_{\mathrm{Ad}\,(\ell)\cdot u_w}(z)\},$$

where $\ell \in K$ with $((H \cap G)_v)_0 = \ell L \ell^{-1}$. Notice that $(G_{<k,v>})_0 = k((G \cap H)_v)_0 k^{-1} = k\ell L(k\ell)^{-1}$.

The RHS of (5) applied to $\mathfrak{B}$ gives,

$$\eta_0(W_{\mathrm{Ad}\,(\ell)\cdot u_{a+1}}(z)), \ldots, W_{\mathrm{Ad}\,(\ell)\cdot u_w}(z) = \tau_0(u_{a+1}, \ldots, u_w) = (u_{a+1}^* \wedge \cdots \wedge u_w^*)(u_{a+1}, \ldots, u_w) = 1.$$

Using the fact that $\prod_* X_u(k) = -\prod_* W_u(z)$ if $u \in \mathfrak{g} \cap \mathfrak{h}$ (cf. [29, Proposition 5 ⟨5⟩] ) the LHS of (5) applied to $\mathfrak{B}$ gives:

$$\begin{aligned}
\prod^* \eta\, (X_{u_1}(k), \ldots, X_{u_a}(k), W_{\mathrm{Ad}\,(\ell)\cdot u_{a+1}}(z), \ldots, W_{\mathrm{Ad}\,(\ell)\cdot u_w}(z)) &= \\
(-1)^{w-a}\eta\Big(\prod\nolimits_* X_{u_1}(k), \ldots, \prod\nolimits_* X_{u_a}(k), \prod\nolimits_* X_{\mathrm{Ad}\,(\ell)\cdot u_{a+1}}(k), \ldots, \prod\nolimits_* X_{\mathrm{Ad}\,(\ell)\cdot u_w}(k)\Big) &= \\
(-1)^a\eta\Big(\prod\nolimits_* X^{\mathrm{Ad}\,(k)\cdot u_1}(k), \ldots, \prod\nolimits_* X^{\mathrm{Ad}\,(k)\cdot u_a}(k), \prod\nolimits_* X^{\mathrm{Ad}\,(k\ell)\cdot u_{a+1}}(k), \ldots, \prod\nolimits_* X^{\mathrm{Ad}\,(k\ell)\cdot u_w}(k)\Big) &= \\
(-1)^a\eta\,(V_{\mathrm{Ad}\,(k)\cdot u_1}(< k, v >) \ldots, V_{\mathrm{Ad}\,(k)\cdot u_a}(< k, v >), V_{\mathrm{Ad}\,(k\ell)\cdot u_{a+1}}(< k, v >), \ldots, V_{\mathrm{Ad}\,(k\ell)\cdot u_w}(< k, v >)) &= \\
(-1)^a\tau\Big(\mathrm{Ad}\,(\ell^{-1}) \cdot u_1, \ldots, \mathrm{Ad}\,(\ell^{-1}) \cdot u_a, u_{a+1}, \ldots, u_w\Big) = (-1)^a(u_1^* \wedge \cdots \wedge u_a^*)\Big(\mathrm{Ad}\,(\ell^{-1}) \cdot u_1 \ldots, \mathrm{Ad}\,(\ell^{-1}) \cdot u_a\Big)&,
\end{aligned}$$

up to the sign $(-1)^{a(b-a)}$, coming form the reordering: $(X_{u_1}, \ldots, X_{u_a}, X_{u_{a+1}}, \ldots, X_{u_b}) \mapsto (X_{u_{a+1}}, \ldots, X_{u_b}, X_{u_1}, \ldots, X_{u_a})$. Since $\ell \in H'$, the closure of $(G \cap H)_0$ on $H$, then $\mathrm{Ad}\,(\ell)$ preserves $\mathfrak{g}$ and $\mathfrak{h}$. The connectedness of $H'$ gives that the operator $\mathrm{Ad}\,(\ell)\colon (\mathfrak{g} \cap \mathfrak{h})^{\perp_\mathfrak{g}} \to (\mathfrak{g} \cap \mathfrak{h})^{\perp_\mathfrak{g}}$ is an orthogonal map preserving orientation. So

$$(u_1^* \wedge \cdots \wedge u_a^*)\Big(\mathrm{Ad}\,(\ell^{-1})(u_1) \ldots, \mathrm{Ad}\,(\ell^{-1})(u_a)\Big) = \det(\mathrm{Ad}\,(\ell^{-1}))(u_1^* \wedge \cdots \wedge u_a^*)\,(u_1, \ldots, u_a) \;=\; 1,$$

We obtain (5). ♣

**Proposition 7.2** *If $\mathcal{N}$ verifies the Poincaré duality property then $\mathcal{W}$ also verifies the Poincaré duality property .*

*Proof.* Let $\overline{p}$ and $\overline{q}$ two complementary perversities on $N$. We have $\dim N = m + c - a - f$ and $\dim \mathcal{N} = w - a$, where $w = \dim \mathcal{W}$ and $m = \dim K \times_H N$. By hypothesis, the pairing $P_N \colon \mathbb{H}^{*}_{\overline{p}}(N/\mathcal{N}) \times \mathbb{H}^{m+c-w-f-*}_{\overline{q},c}(N/\mathcal{N}) \to \mathbb{R}$ is non degenerate.

The foliation $\mathcal{E}$ of $K$ is generated by the vector fields $\{X_1, \ldots, X_c\}$. This gives $H^{*}(K/\mathcal{E}) = \bigwedge^{*}(\gamma_{c+1}, \ldots, \gamma_f)$. It is clear that the pairing $P \colon H^{*}(K/\mathcal{E}) \times H^{f-c-*}(K/\mathcal{E}) \longrightarrow \mathbb{R}$, defined by $P([\xi],[\chi]) = \displaystyle\int_K \xi \wedge \chi \wedge \gamma_1 \wedge \cdots \wedge \gamma_c$ is non degenerate (cf. [31, 4.1].

Since the first cohomology is finite dimensional then the pairing

$$P \otimes P_N \colon H^{*}(K/\mathcal{E}) \otimes \mathbb{H}^{*}_{\overline{p}}(N/\mathcal{N}) \times H^{f-c-*}(K/\mathcal{E}) \otimes \mathbb{H}^{m+c-w-f-*}_{\overline{q},c}(N/\mathcal{N}) \longrightarrow \mathbb{R}$$

is non degenerate. Notice that the group $H/H_0$ is finite. The group $H \cap G$ is normal on $H$, so the subgroup $(H \cap G)_0$ is also normal on $H$. The remark following Proposition 5.2 implies that we can suppose that the tangent volume form defining $P_N$ can be chosen $H$-invariant. On the other hand, the right-action of $H$ on $K$ preserves $\mathcal{E}$ and therefore $h^*\gamma_1 \wedge \cdots \wedge \gamma_c = \pm\gamma_1 \wedge \cdots \wedge \gamma_c$ for each $h \in H$. We conclude that the induced pairing

$$P \otimes P_N \colon \left(H^{*}(K/\mathcal{E}) \otimes \mathbb{H}^{*}_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0} \times \left(H^{f-c-*}(K/\mathcal{E}) \otimes \mathbb{H}^{m+c-w-f-*}_{\overline{q},c}(N/\mathcal{N})\right)^{H/H_0} \longrightarrow \mathbb{R}$$

is also non degenerate.

We also denote by $\overline{p}$ and $\overline{q}$ the associated perversities on $K \times_H N$, which also are two complementary perversities. Recall that the isomorphisms

$$\nabla_{min} \colon \left(H^{*}(K/\mathcal{E}) \otimes \mathbb{H}^{*}_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0} \longrightarrow \mathbb{H}^{*}_{\overline{p}}(K \times_H N),$$
$$\nabla_{min} \colon \left(H^{*}(K/\mathcal{E}) \otimes \mathbb{H}^{*}_{\overline{q},c}(N/\mathcal{N})\right)^{H/H_0} \longrightarrow \mathbb{H}^{*}_{\overline{q},c}(K \times_H N)$$

are characterized by

$$\prod{}^{*}\nabla_{min}([\xi] \otimes [\alpha]) = \left[\xi \wedge \left(\alpha + \underbrace{\sum_{b<i_1<\cdots<i_l\leq c} (-1)^l \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_l} \wedge (i_{W_{i_l}} \cdots i_{W_{i_1}} \alpha)}_{\overline{\alpha}}\right)\right] \tag{6}$$

(cf. [31, Proposition 5] and Proposition 4.1).

Let us consider the following diagram

$$\begin{tikzcd}
\left(H^{*}(K/\mathcal{E}) \otimes \mathbb{H}^{*}_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0} \times \left(H^{f-c-*}(K/\mathcal{E}) \otimes \mathbb{H}^{m+c-w-f-*}_{\overline{q},c}(N/\mathcal{N})\right)^{H/H_0} \arrow[r, "P\otimes P_N"] \arrow[d, "\nabla_{min}\times\nabla_{min}"'] & \mathbb{R} \arrow[d, "Identity"] \\
\mathbb{H}^{*}_{\overline{p}}(K \times_H N) \times \mathbb{H}^{m-w-*}_{\overline{q},c}(K \times_H N) \arrow[r, "P_{K\times_H N}"] & \mathbb{R},
\end{tikzcd}$$

where a suitable rearrangement in the top left term is necessary in order to apply $P \otimes P_N$. We will complete the proof if we show that this diagram commutes up to a non-zero constant depending on $\Phi$. In effect, vertical arrows are isomorphisms and top arrow is a non degenerate pairing.

Write $\nabla_{min}([\xi] \otimes [\alpha]) = [\xi \bullet \alpha]$. We have,

$$P_{K\times_H N}(\nabla_{min} \times \nabla_{min})([\xi] \otimes [\alpha], [\chi] \otimes [\beta]) = \int_{K\times_H R_N} \xi \bullet \alpha \wedge \chi \bullet \beta \wedge \eta.$$

Recall that we have denoted by $\{W_{u_{a+1}} = W_{a+1}, \dots, W_{u_c} = W_c\}$ the fundamental vector fields of the action $\Theta\colon H \times N \to N$ associated to the basis $\{u_{a+1}, \dots, u_c\}$. Let $\{\zeta_{a+1}, \dots, \zeta_c\}$ be the associated dual forms relatively to an $H$-invariant Riemannian metric on $R_{\mathcal{N}}$ . So $\frac{1}{2^{c-a}}(\gamma_{a+1} + \zeta_{a+1}) \wedge \cdots \wedge (\gamma_c + \zeta_c)$ is a differential form of $K \times R_{\mathcal{N}}$ giving a volume form on each fiber of $\prod$. Thus

$$P_{K\times_H N}(\nabla_{_{min}} \times \nabla_{_{min}})([\xi] \otimes [\alpha], [\chi] \otimes [\beta]) = \frac{1}{2^{c-a}} \int_{K\times R_{\mathcal{N}}} \prod^{*}(\xi \bullet \alpha \wedge \chi \bullet \beta \wedge \eta) \wedge (\gamma_{a+1} + \zeta_{a+1}) \wedge \cdots \wedge (\gamma_c + \zeta_c).$$

We claim that the integrand is the differential form $\xi \wedge \alpha \wedge \chi \wedge \beta \wedge \eta_0 \wedge \gamma_1 \wedge \cdots \wedge \gamma_c$, up to a non-zero constant $C$ depending on $\Phi$. We get

$$\begin{aligned} P_{K\times_H N}(\nabla_{_{min}} \times \nabla_{_{min}})([\xi] \otimes [\alpha], [\chi] \otimes [\beta]) &= \frac{(-1)^{|\alpha|\cdot|\chi| + c\cdot \dim N} C}{2^{c-a}} \int_K \xi \wedge \chi \wedge \gamma_1 \wedge \cdots \wedge \gamma_c \cdot \int_{R_{\mathcal{N}}} \alpha \wedge \beta \wedge \eta_0 \\ &= \frac{(-1)^{|\alpha|\cdot|\chi| + c\cdot \dim N} C}{2^{c-a}} P([\xi], [\chi]) \cdot P_{\mathcal{N}}([\alpha], [\beta]) \\ &= \frac{(-1)^{|\alpha|\cdot|\chi| + c\cdot \dim N} C}{2^{c-a}} (P \otimes P_{\mathcal{N}})([\xi] \otimes [\alpha], [\chi] \otimes [\beta]). \end{aligned}$$

This would end the proof after the verification of the claim, which is a local question.

We denote by $\mathcal{H}$ the foliation defined by the action of $H$ on $R_{\mathcal{N}}$. It is a regular foliation. Since $R_{\mathcal{H}}$ is dense in $R_{\mathcal{N}}$, it suffices to prove the claim on $K \times R_{\mathcal{H}}$. Let us consider a point $(k, z) \in K \times R_{\mathcal{H}}$. We need to find a non-zero constant $C$ depending on $\Phi$ such that

$$\xi \wedge (\alpha + \overline{\alpha}) \wedge \chi \wedge (\beta + \overline{\beta}) \wedge \prod^{*} \eta \wedge (\gamma_{a+1} + \zeta_{a+1}) \wedge \cdots \wedge (\gamma_c + \zeta_c) = C\, \xi \wedge \alpha \wedge \chi \wedge \beta \wedge \eta_0 \wedge \gamma_1 \wedge \cdots \wedge \gamma_c,$$

on the tangent space $T_k K \times T_z N$. Without loss of generality, we can suppose that this vector espace is generated by the family

$$\{X_1(k), \dots, X_f(k), W_{a+1}(z), \dots, W_w(z), W_{b+1}(z), \dots, W_v(z), V_1, \dots, V_{v'}\},$$

where $\{W_{a+1}(z), \dots, W_w(z)\}$ is a basis of $T_z\mathcal{N}$, $\{W_{a+1}(z), \dots, W_w(z), W_{b+1}(z), \dots, W_v(z)\}$ is a basis of $T_z\mathcal{H}$, $W_{w+1}(z) = \cdots = W_b(z) = W_{v+1}(z) = \cdots = W_c(z) = 0$ and $\{V_1, \dots, V_{v'}\}$ is a basis of the orthogonal of $T_z\mathcal{H}$ in $T_z N$.

The following differential forms vanish on the basis $\{X_1(k), \dots, X_b(k), W_{a+1}(z), \dots, W_w(z)\}$ of $T_k\mathcal{K} \times T_z\mathcal{N}$:

$$\xi \wedge \chi \in \bigwedge^{*}(\gamma_{c+1}, \dots, \gamma_f), \{\alpha, \beta, \overline{\alpha}, \overline{\beta}\} \subset \bigwedge^{*}(\gamma_{b+1}, \dots, \gamma_c) \otimes \Omega^{*}(R_{\mathcal{H}}/\mathcal{N}) \text{ and } \{\gamma_{b+1}, \dots, \gamma_c\}.$$

Moreover, the forms $\{\gamma_{w+1} + \zeta_{w+1}, \dots, \gamma_c + \zeta_c\}$ vanish on $\{X_{a+1}(k) + W_{a+1}(z), \dots, X_w(k) + W_w(z)\}$.

Applying Lemma 7.1 we conclude that it suffices to find a non-zero constant $C$ depending on $\Phi$ such that

$$\xi \wedge \chi \wedge (\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \wedge (\gamma_{b+1} + \zeta_{b+1}) \wedge \cdots \wedge (\gamma_c + \zeta_c) = C\frac{(-1)^{wb}}{2^{w-a}}\, \xi \wedge \chi \wedge \alpha \wedge \beta \wedge \gamma_{b+1} \wedge \cdots \wedge \gamma_c,$$

on the family $\{X_{b+1}(k), \dots, X_f(k), W_{b+1}(z), \dots, W_v(z), V_1, \dots, V_{v'}\}$,

The following differential forms vanish on $\{X_{c+1}(k), \dots, X_f(k)\}$:

$$\{\alpha, \beta, \overline{\alpha}, \overline{\beta}\} \subset \bigwedge^{*}(\gamma_{b+1}, \dots, \gamma_c) \otimes \Omega^{*}(R_{\mathcal{H}}/\mathcal{N}), \{\gamma_{b+1}, \dots, \gamma_c\} \text{ and } \{\zeta_{b+1}, \dots, \zeta_c\}.$$

We conclude that it suffices to find a non-zero constant $C$ depending on $\Phi$ such that

$$(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \wedge (\gamma_{b+1} + \zeta_{b+1}) \wedge \cdots \wedge (\gamma_c + \zeta_c) = C\frac{(-1)^{wb}}{2^{w-a}}\, \alpha \wedge \beta \wedge \gamma_{b+1} \wedge \cdots \wedge \gamma_c,$$

on the family $\{X_{b+1}(k), \dots, X_c(k), W_{b+1}(z), \dots, W_v(z), V_1, \dots, V_{v'}\}$,

Since $W_{v+1}(z) = \cdots = W_c(z) = 0$, the following differential forms vanish on $\{X_{v+1}(k), \dots, X_c(k)\}$:

$$\{\alpha, \beta, \overline{\alpha}, \overline{\beta}\} \subset \bigwedge^{*}(\gamma_{b+1}, \dots, \gamma_v) \otimes \Omega^{*}(R_{\mathcal{H}}/\mathcal{N}), \{\zeta_{b+1}, \dots, \zeta_c\} \text{ and } \{\gamma_{b+1}, \dots, \gamma_v\}.$$

(cf. (6)). We conclude that it suffices to find a non-zero constant $C$ depending on $\Phi$ such that

$$(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \wedge (\gamma_{b+1} + \zeta_{b+1}) \wedge \cdots \wedge (\gamma_v + \zeta_v) = C \frac{(-1)^{wb}}{2^{w-a}} \alpha \wedge \beta \wedge \gamma_{b+1} \wedge \cdots \wedge \gamma_v,$$

on the family $\{X_{b+1}(k), \dots, X_v(k), W_{b+1}(z), \dots, W_v(z), V_1, \dots, V_{v'}\}$,

We consider the canonical representation of $\alpha$:

$$\alpha = \alpha_0 + \sum_{b<i_1<\cdots<i_l \le v} \zeta_{i_1} \wedge \cdots \wedge \zeta_{i_l} \wedge \alpha_{i_1,\dots,i_l},$$

where $\alpha_{i_1,\dots,i_l} \in \Omega^{*}(R_{\mathcal{H}}/\mathcal{N})$ and $i_{W_j}\alpha_{i_1,\dots,i_l} = 0$ for any $j \in \{b+1, \dots, v\}$. A straightforward calculation gives

$$\alpha + \overline{\alpha} = \alpha_0 + \sum_{b<i_1<\cdots<i_l \le v} (\zeta_{i_1} - \gamma_{i_1}) \wedge \cdots \wedge (\zeta_{i_l} - \gamma_{i_l}) \wedge \alpha_{i_1,\dots,i_l},$$

Analogously for $\beta$.

Differential forms $\alpha \wedge \beta$ and $(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta})$ vanish respectively on $\{X_{b+1}(k), \dots, X_v(k)\}$ and $\{X_{b+1}(k) + W_{b+1}(z), \dots, X_v(k) + W_v(z)\}$. On the other hand, we have $(\gamma_j + \zeta_j)(X_j(k) + W_j(z)) = 2$ for $j \in \{b+1, \dots, v\}$. Then it suffices to find a non-zero constant $C$ depending on $\Phi$ such that

$$(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) = \frac{C(-1)^{wb}}{2^{w+v-a-b}} \alpha \wedge \beta$$

on the family $\{W_{b+1}(z), \dots, W_v(z), V_1, \dots, V_{v'}\}$,

Without loss of generality we can suppose $\alpha = \zeta_{b+1} \wedge \cdots \wedge \zeta_\ell \wedge \alpha_{b+1,\dots,\ell}$ and $\beta = \zeta_{\ell+1} \wedge \cdots \wedge \zeta_v \wedge \beta_{\ell+1,\dots,v}$ where $b < \ell \le v$. By orthogonality, differential forms $\{\zeta_{b+1}, \dots, \zeta_v, \gamma_{b+1}, \dots, \gamma_v\}$ vanish on $\{V_1, \dots, V_{v'}\}$. Then it suffices to find a non-zero constant $C$ depending on $\Phi$ such that

$$(\zeta_{b+1} - \gamma_{b+1}) \wedge \cdots \wedge (\zeta_v - \gamma_v) = C \frac{(-1)^{wb}}{2^{w+v-a-b}} \zeta_{b+1} \wedge \cdots \wedge \zeta_v$$

on the family $\{W_{b+1}(z), \dots, W_v(z)\}$. As $\gamma_j(W_{j'}(x)) = 0$ for any $j, j' \in \{b+1, \dots, v\}$, just take $C = (-1)^{wb} 2^{w+v-a-b}$. Notice that $b = \dim G$, $a = b - \dim G \cap H$, $w = \dim \mathcal{W}$ and $v = b + \dim \mathcal{H} - \dim \mathcal{N}$ do not depend on the choice of the point $(k, v)$ but on $\Phi$. ♣

## 8. Tubular neighborhoods and Poincaré duality .

In this section we consider a manifold $M$ endowed with a Killing foliation $\mathcal{F}$ induced by an orientable action $\Phi \colon G \times M \to M$, $G$ connected. We fix a tamer group $K$ and a tangent volume form $\eta$. We also fix a couple of complementary perversities $\overline{p}$ and $\overline{q}$.

We prove the Poincaré property of $\mathcal{F}$ by cutting the manifold $M$ into nicer pieces where this property is checked and by gluing these facts using Mayer-Vietoris technique and Bredon's Trick. These pieces are essentially tubular neighborhoods of strata. But we do not use the stratification $\mathsf{S}_{\mathcal{F}}$ (induced by $G$) since its properties are not strong enough. We work with the stratification $\mathsf{S}_{K,M}$ (induced by $K$). It possesses very useful properties because of the compactness of $K$.

**8.1 Mayer-Vietoris.** For any saturated open subset $U \subset M$ we introduce the following statement:

(7) $$\mathfrak{P}(U) = \text{“The pairing } P_U \colon \mathbb{H}^*_{\overline{p}}(U/\mathcal{F}) \longrightarrow \operatorname{Hom}\left(\mathbb{H}^{m-w-*}_{\overline{q},c}(U/\mathcal{F}), \mathbb{R}\right) \text{ is an isomorphism.}$$

Any covering $\{U, V\}$ of an open subset of $M$, made up of saturated ( union of leaves ) open subsets of $M$, possesses a subordinated partition of the unity made up of basic functions (see for example [26, Lemma 2.1.1]), which also are controlled functions [30, Remark 3.2.1 (b)]. This is the main ingredient permitting us to say that the two rows of the following commutative diagram are exact (see for example [4, Propositions 2.3 and 2.7])

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*_{\overline{p}}((U\cup V)/\mathcal{F}) \arrow[r] \arrow[d, "P_{U\cup V}"] & \Omega^*_{\overline{p}}(U/\mathcal{F}) \oplus \Omega^*_{\overline{p}}(V/\mathcal{F}) \arrow[r] \arrow[d, "P_U \oplus P_V"] & \Omega^*_{\overline{p}}((U\cap V)/\mathcal{F}) \arrow[r] \arrow[d, "P_{U\cap V}"] & 0 \\
0 \arrow[r] & \operatorname{Hom}(\Omega^*_{\overline{q},c}((U\cup V)/\mathcal{F}); \mathbb{R}) \arrow[r] & \left\{\begin{matrix} \operatorname{Hom}(\Omega^*_{\overline{q},c}(U/\mathcal{F}); \mathbb{R}) \\ \oplus \\ \operatorname{Hom}(\Omega^*_{\overline{q},c}(V/\mathcal{F}); \mathbb{R}) \end{matrix}\right\} \arrow[r] & \operatorname{Hom}(\Omega^*_{\overline{q},c}((U\cap V)/\mathcal{F}); \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

The Five Lemma gives the Mayer-Vietoris property:

(8) $$\mathfrak{P}(U), \mathfrak{P}(U) \text{ and } \mathfrak{P}(U \cap V) \Longrightarrow \mathfrak{P}(U \cup V).$$

**8.2 Bredon's trick.** The Mayer-Vietoris property allows to make computations when the manifold is covered by suitable covering. The passage local-global may be done using an adapted version of the Bredon's trick of [6, pag. 289]:

Let $X$ be a paracompact topological space and let $\{U_\alpha\}$ be an open covering, closed for finite intersection. Suppose that $Q(U)$ is a statement about open subsets of $X$, satisfying the following three properties:

(BT1) $Q(U_\alpha)$ is true for each $\alpha$;

(BT2) $Q(U)$, $Q(z)$ and $Q(U \cap V) \Longrightarrow Q(U \cup V)$, where $U$ and $V$ are open subsets of $X$;

(BT3) $Q(U_i) \Longrightarrow Q\left(\bigcup_i U_i\right)$, where $\{U_i\}$ is a disjoint family of open subsets of $X$.

Then $Q(X)$ is true.

Consider a singular closed stratum $S$ of the stratification $\mathsf{S}_{K,M}$. Since $S$ is a $K$-invariant sub manifold of $M$ and then it possesses a $K$-invariant tubular neighborhood $(T, \tau, S, \mathbb{R}^n)$. Notice the the restriction of the $G$-action on $T$ is still orientable. The associated Killing foliation is $\mathcal{F}_T$. We fix a base point $x \in S$. The isotropy subgroup $K_x$ acts orthogonally and effectively on the fiber $\mathbb{R}^n = \tau^{-1}(x)$. So, the induced action $\Lambda_x \colon G_x \times \mathbb{R}^n \to \mathbb{R}^n$ is a tame action and also an orientable action since $(G_y)_x = G_y$ for each $y \in \tau^{-1}(x)$. The associated Killing foliation is $\mathcal{F}_{\mathbb{R}^n}$. We have:

**Proposition 8.3** *If $\mathcal{F}_{\mathbb{R}^n}$ verifies the Poincaré duality property then $\mathcal{F}_T$ also verifies the Poincaré duality property .*

*Proof.* The canonical projection $\pi \colon S \to S/K$ is a $K/K_x$-homogeneous bundle over the manifold $S/K$ (cf. [5, Chapter 5, Theorem 5.8])[4]. We can find a good covering $\{U_\alpha\}$ of it (cf. [4, Theorem 5.1]). For each open subset $V \subset S/K$ we define the statement: $Q(z) = \mathfrak{P}(\tau^{-1}\pi^{-1}(z))$. We prove (BT1)-(BT3) and the Bredon's trick gives $Q(S/K)$, ending the proof.

[4]This fact is a direct consequence of the compactness of $K$ and the main reason to introduce the stratification $\mathsf{S}_{K,M}$.

(BT1) Since $U_\alpha$ is contractible then $\pi^{-1}(U_\alpha) = U_\alpha \times K/K_x$ and therefore $\tau^{-1}\pi^{-1}(U_\alpha) = U_\alpha \times (K \times_{K_x} \mathbb{R}^n)$. Notice that the action of $G$ on $K \times_{K_x} \mathbb{R}^n$ is orientable. The contractibility of $U_\alpha$ implies that we can suppose that $U_\alpha$ is a point. So, property $Q(U_\alpha)$ becomes $\mathfrak{P}(K \times_{K_x} \mathbb{R}^n)$, which is true from Proposition 7.2.

(BT2) Mayer-Vietoris (8).

(BT3) Straightforward. ♣

## 9. Poincaré duality .

We present the main result of this work.

**Theorem 9.1** *Any orientable Killing foliation $\mathcal{F}$ verifies the Poincaré duality property.*

*Proof.* We assume that the Killing foliation $\mathcal{F}$ is induced by an orientable action $\Phi\colon G \times M \to M$ with $G$ connected. We fix a tamer $K$. We prove $\mathfrak{P}(M)$ by induction on depth $\mathsf{S}_{K,M}$.

*First step:* depth $\mathsf{S}_{K,M} = 0$.

We know that the the foliation $\mathcal{F}$ is a regular Riemannian foliation induced by the action of a group of isometries. So, its central sheaf is trivial (cf. [16, Lemma III]). Since the action $\Phi$ is orientable we get that the manifold $M$ is orientable and the foliation $\mathcal{F}$ is tangentially orientable (existence of tangent volume form $\eta$). When the manifold $M$ is compact then $\mathfrak{P}(M)$ comes from [2, Theorem 1.1] and [12, Theorem 3.1] (see also [32, Remarque 2.5 (ii), Corollaire I] and [9, Théorème 4.10]).

In the case of non-compact $M$, we notice that the canonical projection $\pi\colon M \to M/K$ is a $K/K_x$-homogeneous bundle over the manifold $M/K$ (cf. [5, Chapter 5, Theorem 5.8]). For each open subset $V \subset M/K$ we formulate the statement: $Q(z) = \mathfrak{P}(\pi^{-1}(z))$ (see (8)). We prove (BT1)-(BT3) and the Bredon's trick gives $Q(S)$ , ending the proof.

(BT1) Since $U_\alpha$ is contractible then $\pi^{-1}(U_\alpha) = U_\alpha \times K/K_x$. Notice that the action of $G$ on $K/K_x$ is orientable. The contractibility of $U_\alpha$ implies that we can suppose that $U_\alpha$ is a point. So, the property $Q(U_\alpha)$ becomes $\mathfrak{P}(K/K_x)$, which is true by the previous argument since $K/K_x$ is compact

(BT2) Mayer-Vietoris (8).

(BT3) Straightforward.

*Second step: Conical case.*

We suppose that $M = \mathbb{R}^m$ and the Killing foliation $\mathcal{F}$ is defined by an orthogonal, orientable action $\Phi\colon G \times M \to M$ having the origin as unique fixed point. We prove $\mathfrak{P}(\mathbb{R}^m)$.

We denote by $\mathcal{G}$ the induced Killing foliation on $\mathbb{S}^{m-1}$ defined by the restriction an orthogonal, orientable action $\Phi\colon G \times \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ having no fixed point. Since depth $\mathsf{S}_{K,\mathbb{S}^{m-1}}$ < depth $\mathsf{S}_{K,\mathbb{R}^m}$ , we have $\mathfrak{P}(\mathbb{S}^{m-1})$.

From [31, Proposition 4] we have

$$\mathbb{H}^{i}_{\overline{p}}(\mathbb{R}^m/\mathcal{F}) = \begin{cases} \mathbb{H}^{i}_{\overline{p}}\left(\mathbb{S}^{m-1}/\mathcal{G}\right) & \text{if } i \leq \overline{p}(\vartheta) \\ 0 & \text{if } i \geq \overline{p}(\vartheta)+1. \end{cases} \tag{9}$$

Here, $\mathbb{R}^{m+1} = c\mathbb{S}^{m-1} = \mathbb{S}^{m-1}\times]0,\infty[/\mathbb{S}^{m-1}\times\{0\}$ and $\{\vartheta\}$ is the vertex of the cone. Since $\overline{p}$ and $\overline{q}$ are complementary perversities on $\mathbb{R}^m$ then we have $\overline{p}(\{\vartheta\})+\overline{q}(\{\vartheta\}) = \overline{t}(\{\vartheta\}) = m-w-2$, From [30, Proposition 3.7.2] we get

$$\mathbb{H}^{m-w-i}_{\overline{q},c}(\mathbb{R}^m/\mathcal{F}) = \begin{cases} \mathbb{H}^{m-w-i-1}_{\overline{q}}\left(\mathbb{S}^{m-1}/\mathcal{G}\right) & \text{if } i \leq \overline{p}(\{\vartheta\}) \\ 0 & \text{if } i \geq \overline{p}(\{\vartheta\})+1. \end{cases} \tag{10}$$

Now, $\mathfrak{P}(\mathbb{R}^m)$ comes from $\mathfrak{P}(\mathbb{S}^{m-1})$ and the two following facts

(i) The perversities $\overline{p}$ and $\overline{q}$ are complementary on $\mathbb{S}^{m-1}$ (see definition (1)).

A perversity $\overline{r}$ on $\mathbb{R}^m$ induces the perversity $\overline{r}$ on $\mathbb{S}^{m-1}$ by: $\overline{r}(S) = \overline{r}(S\times]0,\infty[)$ for any stratum $S \in \mathsf{S}_{\mathcal{G}}$. The restriction of the foliation $\mathcal{F}$ to $\mathbb{S}^{m-1}\times]0,\infty[$ is the product $\mathcal{G}\times\mathcal{I}$ where $\mathcal{I}$ is the point-wise foliation of $]0,\infty[$. Then we have

$$\begin{aligned} \overline{p}(S)+\overline{q}(S) &= \overline{p}(S\times]0,\infty[)+\overline{q}(S\times]0,\infty[) = \overline{t}(S\times]0,\infty[) = \operatorname{codim}_{\mathbb{R}^m}\mathcal{F} - \operatorname{codim}_{S\times]0,\infty[}(\mathcal{G}\times\mathcal{I})-2 \\ &= \operatorname{codim}_{\mathbb{S}^{m-1}}\mathcal{G} - \operatorname{codim}_S\mathcal{G} - 2 = \overline{t}(S). \end{aligned}$$

(ii) The pairing $P_{\mathbb{R}^m}$ becomes the pairing $P_{\mathbb{S}^{m-1}}$ through the isomorphisms induced by (9) and (10).

First notice that if $\eta$ is a tangent volume form of $\mathcal{G}$ then $\chi^*\eta$ is a tangent volume form for $\mathcal{F}_{\mathbb{R}^m}$ since $G_{(x,t)} = G_x$ for each $(x,t) \in R_{\mathcal{F}} = R_{\mathcal{G}}\times]0,\infty[$. Here $\chi\colon R_{\mathcal{F}} \to R_{\mathcal{G}}$ is the canonical projection. The operator $\aleph\colon \mathbb{H}^{*}_{\overline{p}}\left(\mathbb{S}^{m-1}/\mathcal{G}\right) \to \mathbb{H}^{*}_{\overline{p}}(\mathbb{R}^m/\mathcal{F})$ defining (9) is $\aleph([\alpha]) = [\chi^*\alpha]$, (see [30, Proposition 3.5.2]. The operator $\aleph'\colon \mathbb{H}^{*}_{\overline{q}}\left(\mathbb{S}^{m-1}/\mathcal{G}\right) \to \mathbb{H}^{*}_{\overline{q},c}(\mathbb{R}^m/\mathcal{F})$ defining (10) is $\aleph'([\beta]) = [g\,dt\wedge\chi^*\beta]$, where $g \in C^\infty([0,\infty[)$ with $g \equiv 1$ on $[0,1/4]$, $g \equiv 0$ on $[3/4,1[$ and $\int_0^1 g = 1$ [30, Proposition 3.7.2]. Now, for $[\alpha] \in \mathbb{H}^{i}_{\overline{p}}\left(\mathbb{S}^{m-1}/\mathcal{G}\right)$ and $[\beta] \in \mathbb{H}^{m-1-w-i}_{\overline{q},c}\left(\mathbb{S}^{m-1}/\mathcal{G}\right)$ we have

$$P_{\mathbb{R}^m}(\aleph[\alpha],\aleph'[\beta]) = \int_{R_{\mathcal{G}}\times]0,\infty[} g\,\chi^*\alpha\wedge dt\wedge\chi^*\beta\wedge\chi^*\eta = \left(\int_{R_{\mathcal{G}}}\alpha\wedge\beta\wedge\eta\right)\left(\int_0^1 g\,dt\right) = P_{\mathbb{S}^{m-1}}([\alpha],[\beta]).$$

*Third step: General case.*

Let us suppose that depth $\mathsf{S}_{K,M} > 0$. Since the family $\{M\backslash S_{min}, T_{min}\}$ is a basic covering of $M$ and then we get : $\mathfrak{P}(M\backslash S_{min}), \mathfrak{P}(T_{min}), \mathfrak{P}(T_{min}\backslash S_{min}) \Longrightarrow \mathfrak{P}(M)$ (cf. (8)). The depth of the restriction to $M\backslash S_{min}$ and $T_{min}\backslash S_{min}$ is strictly smaller than depth $\mathsf{S}_{\mathcal{F}}$. So, we get $\mathfrak{P}(T_{min}) \Longrightarrow \mathfrak{P}(M)$. It remains to prove $\mathfrak{P}(T_{min})$. This result comes from Proposition 8.3 and the Second step. ♣

## 10. Appendix: Molino's blow up [17],[31, Section 4.2].

We consider in this Appendix a manifold $M$ endowed with a Killing foliation $\mathcal{F}$ induced by an orientable action $\Phi\colon G\times M \to M$, $G$ connected. We fix a tamer $K$. The compact Lie group[5] $K$ acts smoothly on $M$ defining the isotropy type stratification $\mathsf{S}_{K,M}$. It is given by the equivalence relation

$$x \sim y \Leftrightarrow K_x \text{ is conjugate to } K_y.$$

Since $G$ is a normal subgroup of $K$ then $x\sim y$ implies that $G_x$ is conjugate to $G_y$ on $K$ and therefore $\dim(G_x)_0 = \dim(G_y)_0$. So, condition depth $\mathsf{S}_{K,M} = 0$ implies that the foliation $\mathcal{F}$ is regular foliation and then $M = R_{\mathcal{F}}$.

The Molino's blow up is a technical tool we use to desingularize the Killing foliation $\mathcal{F}$ when depth $\mathsf{S}_{K,M} > 0$. It is a continuous map $\mathcal{L}\colon (\widehat{M},\widehat{\mathcal{F}}) \to (M,\mathcal{F})$ verifying:

- the foliation $\widehat{\mathcal{F}}$ is a Killing foliation defined by an orientable action $\widehat{\Phi}\colon G\times \widehat{M} \to \widehat{M}$ having also $K$ as a tamer group,
- depth $\mathsf{S}_{K,\widehat{M}} <$ depth $\mathsf{S}_{K,M}$,
- the map $\mathcal{L}$ is $G$-equivariant,
- the restriction $\mathcal{L}\colon \widehat{M}\backslash\mathcal{L}^{-1}(S_{min}) \to M\backslash S_{min}$, where $S_{min}$ is the union of closed strata of depth $\mathsf{S}_{K,M}$, is a $K$-equivariant smooth trivial 2-covering,
- depth $\mathsf{S}_{K,M\backslash S_{min}} <$ depth $\mathsf{S}_{K,M}$,
- there exists a commutative diagram

$$\begin{tikzcd} D_{min}\times]-1,1[ \arrow[r, hook] \arrow[d, "\nabla_{min}"'] & \widehat{M} \arrow[d, "\mathcal{L}"] \\ T_{min} \arrow[r, hook] & M \end{tikzcd}$$

  where

  + $T_{min}$ is a $K$-equivariant tubular neighborhood of $S_{mn}$,
  + $D_{min} \subset T_{min}$ are the points whose distance to $S_{min}$ is 1/2,
  + $\nabla_{min}(x,t) = 2|t|\cdot x$.

- depth $\mathsf{S}_{K,D_{min}} <$ depth $\mathsf{S}_{K,M}$. ♣

Hau amaiera da.

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