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# Compact composition operators on the Dirichlet space and capacity of sets of contact points

Pascal Lefèvre, Daniel Li, Hervé Queffélec,  
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**Abstract.** We prove that for every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap} K = 0$ , there exists a Schur function  $\varphi$  both in the disk algebra  $A(\mathbb{D})$  and in the Dirichlet space  $\mathcal{D}_*$  such that the composition operator  $C_\varphi$  is in all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ , and for which  $K = \{e^{it}; |\varphi(e^{it})| = 1\} = \{e^{it}; \varphi(e^{it}) = 1\}$ . We show that for every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  and every  $\xi \in \partial\mathbb{D}$ , the logarithmic capacity of  $\{e^{it}; \varphi(e^{it}) = \xi\}$  is 0. We show that every compact composition operator  $C_\varphi$  on  $\mathcal{D}_*$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  and on the Hardy-Orlicz space  $H^{\Psi_2}$ ; in particular,  $C_\varphi$  is in every Schatten class  $S_p$ ,  $p > 0$ , both on the Hardy space  $H^2$  and on the Bergman space  $\mathfrak{B}^2$ . On the other hand, there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , but which is not even bounded on  $\mathcal{D}_*$ . We prove that for every  $p > 0$ , there exists a symbol  $\varphi$  such that  $C_\varphi \in S_p(\mathcal{D}_*)$ , but  $C_\varphi \notin S_q(\mathcal{D}_*)$  for any  $q < p$ , that there exists another symbol  $\varphi$  such that  $C_\varphi \in S_q(\mathcal{D}_*)$  for every  $q < p$ , but  $C_\varphi \notin S_p(\mathcal{D}_*)$ . Also, there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but in no Schatten class  $S_p(\mathcal{D}_*)$ .

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**Key-words.** Bergman space – Bergman-Orlicz space – composition operator – Dirichlet space – Hardy space – Hardy-Orlicz space – logarithmic capacity – Schatten classes

## 1 Introduction, notation and background

### 1.1 Introduction

Recall that a Schur function is an analytic self-map of the open unit disk  $\mathbb{D}$ . Every Schur function  $\varphi$  generates a bounded composition operator  $C_\varphi$  on the

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Hardy space  $H^2$ , given by  $C_\varphi(f) = f \circ \varphi$ . Let us also introduce the set  $E_\varphi$  of contact points of the symbol with the unit circle (equipped with its normalized Haar measure  $m$ ), namely:

$$(1.1) \quad E_\varphi = \{e^{it}; |\varphi^*(e^{it})| = 1\}.$$

In terms of  $E_\varphi$ , a well-known necessary condition for compactness of  $C_\varphi$  on  $H^2$  is that  $m(E_\varphi) = 0$ . This set  $E_\varphi$  is otherwise more or less arbitrary. Indeed, it was proved in [7] that there exist compact composition operators  $C_\varphi$  on  $H^2$  such that the Hausdorff dimension of  $E_\varphi$  is 1. This was generalized in [5]: for every Lebesgue-negligible compact set  $K$  of the unit circle  $\mathbb{T}$ , there is a Hilbert-Schmidt composition operator  $C_\varphi$  on  $H^2$  such that  $E_\varphi = K$ , and in [18]:

**Theorem 1.1 ([18])** *For every Lebesgue-negligible compact set  $K$  of the unit-circle  $\mathbb{T}$  and every vanishing sequence  $(\varepsilon_n)$  of positive numbers, there is a composition operator  $C_\varphi$  on  $H^2$  such that  $E_\varphi = K$  and such that its approximation numbers satisfy  $a_n(C_\varphi) \leq C e^{-n\varepsilon_n}$ .*

We are interested here in a different Hilbert space of analytic functions, on which not every Schur function defines a bounded composition operator, namely the Dirichlet space  $\mathcal{D}$ . Recall its definition: the Dirichlet space  $\mathcal{D}$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$(1.2) \quad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , one has:

$$(1.3) \quad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.$$

Then  $\|\cdot\|_{\mathcal{D}}$  is a norm on  $\mathcal{D}$ , making  $\mathcal{D}$  a Hilbert space. Whereas every Schur function  $\varphi$  generates a bounded composition operator  $C_\varphi$  on the Hardy space  $H^2$ , it is no longer the case for the Dirichlet space (see [21], Proposition 3.12, for instance).

In [6], the study of compact composition operators on the Dirichlet space  $\mathcal{D}$  associated with a Schur function  $\varphi$  in connection with the set  $E_\varphi$  was initiated. In particular, it is proved there that if the composition operator  $C_\varphi$  is Hilbert-Schmidt on  $\mathcal{D}$ , then the logarithmic capacity  $\text{Cap } E_\varphi$  of  $E_\varphi$  is 0, but, on the other hand, there are compact composition operators on  $\mathcal{D}$  for which this capacity is positive. The optimality of this theorem was later proved in [5] under the following form:

**Theorem 1.2 (O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi)**  
*For every compact set  $K$  of the unit circle  $\mathbb{T}$  with logarithmic capacity  $\text{Cap } K$  equal to 0, there exists a Hilbert-Schmidt composition operator  $C_\varphi$  on  $\mathcal{D}$  such that  $E_\varphi = K$ .*

In this paper, we shall improve on this last result. We prove in Section 4 (Theorem 4.1) that for every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_*$  such that the composition operator  $C_\varphi$  is in all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ , and for which  $E_\varphi = K$  (and moreover  $E_\varphi = \{e^{it}; \varphi(e^{it}) = 1\}$ ). On the other hand, in Section 2, we show (Theorem 2.1) that for every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  and every  $\xi \in \partial\mathbb{D}$ , the logarithmic capacity of  $E_\varphi(\xi) = \{e^{it}; \varphi(e^{it}) = \xi\}$  is 0.

In link with Hardy and Bergman spaces, we prove, in Section 2 yet, that every compact composition operator  $C_\varphi$  on  $\mathcal{D}_*$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  and on the Hardy-Orlicz space  $H^{\Psi_2}$ . In particular,  $C_\varphi$  is in every Schatten class  $S_p$ ,  $p > 0$ , both on the Hardy space  $H^2$  and on the Bergman space  $\mathfrak{B}^2$  (Theorem 2.5). However, there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , but which is not even bounded on  $\mathcal{D}_*$  (Theorem 2.6).

In Section 3, we give a characterization of the membership of composition operators in the Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$  (actually in  $S_p(\mathcal{D}_{\alpha,*})$ , where  $\mathcal{D}_{\alpha,*}$  is the weighted Dirichlet space). We deduce that for every  $p > 0$ , there exists a symbol  $\varphi$  such that  $C_\varphi \in S_p(\mathcal{D}_*)$ , but  $C_\varphi \notin S_q(\mathcal{D}_*)$  for any  $q < p$ , and that there exists another symbol  $\varphi$  such that  $C_\varphi \in S_q(\mathcal{D}_*)$  for every  $q < p$ , but  $C_\varphi \notin S_p(\mathcal{D}_*)$  (Theorem 3.3). We also show that there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but in no Schatten class  $S_p(\mathcal{D}_*)$  (Theorem 3.4).

## 1.2 Notation and background.

We denote by  $\mathbb{D}$  the unit open disk of the complex plane and by  $\mathbb{T} = \partial\mathbb{D}$  the unit circle.  $A$  is the normalized area measure  $dx dy/\pi$  of  $\mathbb{D}$  and  $m$  the normalized Lebesgue measure  $dt/2\pi$  on  $\mathbb{T}$ .

As said before, a Schur function is an analytic self-map of  $\mathbb{D}$  and the associated composition operator is defined, formally, by  $C_\varphi(f) = f \circ \varphi$ . The function  $\varphi$  is called the symbol of  $C_\varphi$ .

The Dirichlet space  $\mathcal{D}$  is defined above. We shall actually work, for convenience, with its subspace  $\mathcal{D}_*$  of functions  $f \in \mathcal{D}$  such that  $f(0) = 0$ . In this paper, we call  $\mathcal{D}_*$  the *Dirichlet space*.

An orthonormal basis of  $\mathcal{D}_*$  is formed by  $e_n(z) = z^n/\sqrt{n}$ ,  $n \geq 1$ . The reproducing kernel on  $\mathcal{D}_*$ , defined by  $f(a) = \langle f, K_a \rangle$  for every  $f \in \mathcal{D}_*$ , is given by  $K_a(z) = \sum_{n=1}^{\infty} \overline{e_n(a)} e_n(z)$ , so that:

$$(1.4) \quad K_a(z) = \log \frac{1}{1 - \bar{a}z}.$$

Compactness of composition operators on  $\mathcal{D}$  was characterized in terms of Carleson measure by D. Stegenga ([24]) and by B. McCluer and J. Shapiro in terms of angular derivative ([21]). Another characterization, more useful for us here, was given by N. Zorboska ([29], page 2020): for  $\varphi \in \mathcal{D}$ ,  $C_\varphi$  is bounded on  $\mathcal{D}$  if and only:

$$(1.5) \quad \sup_{h \in (0,2)} \sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(w) dA(w) < \infty,$$

where  $W(\xi, h) = \{w \in \mathbb{D}; 1 - |w| \leq h \text{ and } |\arg(w\bar{\xi})| \leq \pi h\}$  is the Carleson window of size  $h \in (0, 2)$  center at  $\xi \in \mathbb{T}$  and  $n_\varphi$  is the counting function of  $\varphi$ :

$$(1.6) \quad n_\varphi(w) = \sum_{\varphi(z)=w} 1, \quad w \in \varphi(\mathbb{D}),$$

(we set  $n_\varphi(w) = 0$  for  $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$ ). In particular, every Schur function with bounded valence defines a bounded composition operator on  $\mathcal{D}$ .

Moreover,  $C_\varphi$  is compact if and only if:

$$(1.7) \quad \sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(w) dA(w) \xrightarrow{h \rightarrow 0} 0.$$

For further informations on the Dirichlet space, one may consult the two surveys [1] and [23], for example.

### 1.2.1 Logarithmic capacity

The notion of logarithmic capacity is tied to the study of the Dirichlet space by the following seminal and sharp result of Beurling ([2]; see also [9]).

**Theorem 1.3 (Beurling)** *For every function  $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{D}$ , there exists a set  $E \subseteq \partial\mathbb{D}$ , with logarithmic capacity 0, such that, if  $t \in \mathbb{T} \setminus E$ , then the radial limit  $f^*(e^{it}) := \lim_{r \rightarrow 1^-} f(re^{it})$  exists (in  $\mathbb{C}$ ). Moreover, the result is optimal: if a compact set  $E \subseteq \mathbb{T}$  has zero logarithmic capacity, there exists  $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{D}$  such that  $f^*(e^{it})$  does not exist on  $E$ .*

Let us recall some definitions (see [9], Chapitre III, [4], Chapter 21, § 7, or [23], Section 4, for example).

Let  $\mu$  be a probability measure supported by a compact subset  $K$  of  $\mathbb{T}$ . The potential  $U_\mu$  of  $\mu$  is defined, for every  $z \in \mathbb{C}$ , by:

$$U_\mu(z) = \int_K \log \frac{e}{|z - w|} d\mu(w).$$

The energy  $I_\mu$  of  $\mu$  is defined by:

$$I_\mu = \int_K U_\mu(z) d\mu(z) = \iint_{K \times K} \log \frac{e}{|z - w|} d\mu(w) d\mu(z).$$

The logarithmic capacity of a Borel set  $E \subseteq \mathbb{T}$  is:

$$\text{Cap } E = \sup_{\mu} e^{-I_\mu},$$

where the supremum is over all Borel probability measures  $\mu$  with compact support contained in  $E$ . Hence  $E$  is of logarithmic capacity 0 (which is the case we are interested in) if and only if  $I_\mu = \infty$  for all probability measures compactly carried by  $E$ . The fact that  $\text{Cap } E = 0$  implies that  $E$  has null Lebesgue

measure ([9], Chapitre III, Théorème I) (hence  $\text{Cap } E > 0$  if  $E$  is a non-void open subset of  $\mathbb{T}$ ), but the converse is wrong, as shown by Cantor's middle-third set  $\mathfrak{C}$ . A compact set  $K$  such that  $\text{Cap } K = 0$  is totally disconnected ([4], Corollary 21.7.7).

If  $E$  is a compact set with  $\text{Cap } E > 0$ , there is a unique probability measure compactly carried by  $E$  that minimizes the energy  $I_\mu$  ([4], Theorem 21.10.2, or [9], Chapitre III, Proposition 4). Such a measure is called the *equilibrium measure* of  $E$ .

If  $\mu$  is the equilibrium measure of the compact set  $K$ , we have Frostman's Theorem ([4], Theorem 21.7.12, or [9], Chapitre III, Proposition 5 and Proposition 6):  $U_\mu(z) \leq I_\mu$  for every  $z \in \mathbb{C}$  and

$$(1.8) \quad U_\mu(z) = I_\mu \quad \text{for almost all } z \in K.$$

Suppose that the compact set  $K$  has zero logarithmic capacity. For  $\varepsilon > 0$ , let  $K_\varepsilon = \{z \in \mathbb{T}; \text{dist}(z, K) \leq \varepsilon\}$ ,  $\mu_\varepsilon$  its equilibrium measure, and  $I_{\mu_\varepsilon}$  its energy. Then ([4], Proposition 21.7.15):

$$(1.9) \quad \lim_{\varepsilon \rightarrow 0} I_{\mu_\varepsilon} = \infty.$$

## 2 Bounded and compact composition operators

In [6], E. A. Gallardo-Gutiérrez and M. J. González showed that for every Hilbert-Schmidt composition operator  $C_\varphi$  on  $\mathcal{D}_*$ , the logarithmic capacity of the set  $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D}; |\varphi(e^{i\theta})| = 1\}$  is zero. On the other hand, they showed that there are compact composition operators on  $\mathcal{D}_*$  for which  $E_\varphi$  has positive logarithmic capacity. We shall see that if we replace  $|\varphi|$  by  $\varphi$  in the definition of  $E_\varphi$ , the result is very different.

**Theorem 2.1** *For every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  and every  $\xi \in \partial\mathbb{D}$ , the logarithmic capacity of  $E_\varphi(\xi) = \{e^{it}; \varphi(e^{it}) = \xi\}$  is 0.*

We first state the following characterization of Hilbert-Schmidt composition operators on  $\mathcal{D}_*$ . This result is stated in [6], but not entirely proved.

**Lemma 2.2** *Let  $\varphi \in \mathcal{D}_*$  be an analytic self-map of  $\mathbb{D}$ . Then  $C_\varphi$  is Hilbert-Schmidt on  $\mathcal{D}_*$  if and only if*

$$(2.1) \quad \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty.$$

**Proof.** Let  $e_n(z) = z^n/\sqrt{n}$ ; then  $(e_n)_{n \geq 1}$  is an orthonormal basis of  $\mathcal{D}_*$  and

$$\sum_{n=1}^{\infty} \|C_\varphi(e_n)\|^2 = \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n} = \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z).$$

Hence (2.1) is satisfied if  $C_\varphi$  is Hilbert-Schmidt. To get the converse, we need to show that (2.1) implies that  $C_\varphi$  is bounded on  $\mathcal{D}_*$ . Let  $f \in \mathcal{D}_*$  and write  $f(z) = \sum_{n=1}^{\infty} c_n z^n$ . Then  $C_\varphi f = \sum_{n=1}^{\infty} c_n \varphi^n$  and

$$\begin{aligned} \|C_\varphi f\| &\leq \sum_{n=1}^{\infty} |c_n| \|\varphi^n\| \leq \left( \sum_{n=1}^{\infty} n |c_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n} \right)^{1/2} \\ &= \left( \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} dA(z) \right)^{1/2} \|f\|. \end{aligned}$$

Then (2.1) implies that  $C_\varphi$  is Hilbert-Schmidt.  $\square$

Now Theorem 2.1 will follow from the next proposition.

**Proposition 2.3** *There exists an analytic self-map  $\sigma$  of  $\mathbb{D}$ , belonging to  $\mathcal{D}_*$  and to the disk algebra  $A(\mathbb{D})$ , such that  $\sigma(1) = 1$  and  $|\sigma(\xi)| < 1$  for  $\xi \in \partial\mathbb{D} \setminus \{1\}$  and such that the associated composition operator  $C_\sigma$  is Hilbert-Schmidt on  $\mathcal{D}_*$ .*

Taking this proposition for granted for a while, we can prove the theorem.

**Proof of Theorem 2.1.** Making a rotation, we may, and do, assume that  $\xi = 1$ . Then, if  $\sigma$  is the map of Proposition 2.3,  $C_\varphi C_\sigma = C_{\sigma \circ \varphi}$  is Hilbert-Schmidt. By [6], the set  $E_{\sigma \circ \varphi}$  has zero logarithmic capacity. But  $\sigma$  has modulus 1 only at 1; hence  $e^{i\theta} \in E_{\sigma \circ \varphi}$  if and only if  $e^{i\theta} \in E_\varphi(1)$ .  $\square$

To prove Proposition 2.3, it will be convenient to use the following criteria, where  $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ .

**Lemma 2.4** *Let  $f \in \mathcal{D}$  such that  $\Re f \geq 1$ . Then if  $\sigma = \varphi_a \circ e^{-1/f}$ , where  $a = e^{-1/f(0)}$ , the composition operator  $C_\sigma$  is Hilbert-Schmidt on  $\mathcal{D}_*$ .*

**Proof.** Let  $\sigma_0 = e^{-1/f}$ . If  $u = \Re f$  and  $v = \Im f$ , one has:

$$|\sigma_0|^2 = \exp\left(-\frac{2u}{u^2+v^2}\right) \quad \text{and} \quad |\sigma'_0|^2 = \frac{u'^2+v'^2}{(u^2+v^2)^2} \exp\left(-\frac{2u}{u^2+v^2}\right).$$

Then  $|\sigma_0| < 1$  and so  $\sigma_0$  is a self-map of  $\mathbb{D}$ . Since  $u \geq 1 > 0$ , one has  $|\sigma'_0|^2 \leq (u'^2+v'^2)/(u^2+v^2)^2 \leq u'^2+v'^2 = |f'|^2$ ; hence  $\sigma_0 \in \mathcal{D}$ .

For  $0 \leq x \leq 2$ , one has  $1 - e^{-x} \geq x/4$ . Therefore, since  $u \geq 1$  implies  $2u/(u^2+v^2) \leq 2/u \leq 2$ , one has:

$$1 - |\sigma_0|^2 \geq \frac{u}{2(u^2+v^2)}.$$

It follows that:

$$\frac{|\sigma'_0|^2}{(1-|\sigma_0|^2)^2} \leq \frac{u'^2+v'^2}{(u^2+v^2)^2} \frac{4(u^2+v^2)^2}{u^2} \leq 4(u'^2+v'^2) = 4|f'|^2.$$

Since  $f \in \mathcal{D}$ ,  $|f'|^2$  has a finite integral and therefore (2.1) is satisfied. It follows that  $C_{\sigma_0}$  is Hilbert-Schmidt on  $\mathcal{D}$  and hence  $C_\sigma = C_{\sigma_0} \circ C_{\varphi_a}$  is Hilbert-Schmidt on  $\mathcal{D}_*$ , since  $\sigma(0) = 0$ .  $\square$

**Proof of Proposition 2.3.** Let  $\Omega$  be the domain defined by:

$$\Omega = \{z \in \mathbb{C}; \Re z > 1 \text{ and } |\Im z| < 1/(\Re z)^2\}.$$

Let  $f$  be a conformal map from  $\mathbb{D}$  onto  $\Omega$  such that  $f(1) = \infty$ . Since  $A(\Omega) < \infty$ , we have  $f \in \mathcal{D}$ . By Lemma 2.4, the function  $\sigma = e^{-1/f}$  has the required properties.  $\square$

For the next result, recall that an Orlicz function  $\Psi$  is a nondecreasing convex function such that  $\Psi(0) = 0$  and  $\Psi(x)/x \rightarrow \infty$  as  $x$  goes to infinity. We refer to [12] for the definition of Hardy-Orlicz and Bergman-Orlicz spaces. In the following result, one set  $\Psi_2(x) = \exp(x^2) - 1$ .

**Theorem 2.5** *Every compact composition operator  $C_\varphi$  on  $\mathcal{D}_*$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  and on the Hardy-Orlicz space  $H^{\Psi_2}$ . In particular,  $C_\varphi$  is in every Schatten class  $S_p$ ,  $p > 0$ , both on the Hardy space  $H^2$  and on the Bergman space  $\mathfrak{B}^2$ .*

**Proof.** Consider the normalized reproducing kernels  $\tilde{K}_a = K_a/\|K_a\|$ ,  $a \in \mathbb{D}$ . When  $|a|$  goes to 1, they tends to 0 uniformly on compact sets of  $\mathbb{D}$ ; hence  $\|C_\varphi^*(\tilde{K}_a)\|$  tends to 0, by compactness of the adjoint operator  $C_\varphi^*$ . But  $C_\varphi^*(K_a) = K_{\varphi(a)}$  and  $\|K_a\|^2 = \langle K_a, K_a \rangle = \log \frac{1}{1-|a|^2}$ , so we get:

$$(2.2) \quad \lim_{|a| \rightarrow 1} \frac{\log \frac{1}{1-|\varphi(a)|^2}}{\log \frac{1}{1-|a|^2}} = 0.$$

This condition means that  $C_\varphi$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  ([12], page 69) and implies that  $C_\varphi$  is in all Schatten classes  $S_p(\mathfrak{B}^2)$ ,  $p > 0$  ([15]).

In the same way, it suffices to show that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , because that implies that  $C_\varphi$  is in all Schatten classes  $S_p(H^2)$  ([11], Theorem 5.2).

Compactness of  $C_\varphi$  on  $H^\Psi$  is equivalent to say ([12], Theorem 4.18) that:

$$\begin{aligned} \rho_\varphi(h) &:= \sup_{|\xi|=1} m(\{e^{it}; \varphi(e^{it}) \in W(\xi, h)\}) \\ &= o_{h \rightarrow 0} \left[ \frac{1}{\Psi(A\Psi^{-1}(1/h))} \right] \quad \text{for every } A > 0. \end{aligned}$$

When  $\Psi = \Psi_2$ , this means that  $\rho_\varphi(h) = o(h^A)$  for every  $A > 0$ . Now, by [14], Theorem 4.2, this is also equivalent to say that:

$$(2.3) \quad \sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} N_\varphi(w) dA(w) = o(h^A) \quad \text{for every } A > 0,$$



where  $N_\varphi$  is the Nevanlinna counting function of  $\varphi$ :

$$(2.4) \quad N_\varphi(w) = \sum_{\varphi(z)=w} (1 - |z|^2), \quad w \in \varphi(\mathbb{D}),$$

and  $N_\varphi(w) = 0$  otherwise.

But (2.2) is equivalent to the fact that for every  $\varepsilon > 0$  there exists  $\delta_\varepsilon > 0$  such that:

$$(2.5) \quad 1 - |\varphi(z)| \geq \delta_\varepsilon (1 - |z|)^\varepsilon, \quad \forall z \in \mathbb{D}.$$

Since  $\varphi(0) = 0$ , we have  $|\varphi(z)| \leq |z|$ , by Schwarz's lemma; hence one has  $N_\varphi(w) \leq 2\delta_\varepsilon^{-1}(1 - |w|)^{1/\varepsilon} n_\varphi(w)$ . It follows that (since  $1 - |w| \leq h$  for  $w \in W(\xi, h)$ ):

$$\frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} N_\varphi(w) dA(w) \leq 2\delta_\varepsilon^{-1} h^{1/\varepsilon} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(w) dA(w),$$

which is  $o(h^{1/\varepsilon})$ , uniformly for  $|\xi| = 1$ , by (1.7).  $\square$

**Remarks.** 1) One may argue that compactness of  $C_\varphi$  on  $H^{\Psi_2}$  implies its compactness on  $\mathfrak{B}^{\Psi_2}$  ([15], Proposition 4.1, or [17], Theorem 9). One may also use the forthcoming Corollary 3.2 saying that  $C_\varphi \in S_p(H^2)$  implies that  $C_\varphi \in S_p(\mathfrak{B}^2)$ .

2) To show the compactness of  $C_\varphi$  on  $H^{\Psi_2}$ , we used its compactness on  $\mathcal{D}_*$  twice. However, due to the fact that  $\varepsilon > 0$  is arbitrary, we may replace  $o(h^{1/\varepsilon})$  by  $O(h^{1/\varepsilon})$ ; hence to end the proof, we only have to use (1.5), *i.e.* the boundedness of  $C_\varphi$  on  $\mathcal{D}_*$ , instead of (1.7).

Note that (2.2) does not suffice to have compactness on  $H^{\Psi_2}$  (in [12], Proposition 5.5, we construct a Blaschke product satisfying (2.2)).

In the opposite direction, we have the following result.

**Theorem 2.6** *There exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , but which is not even bounded on  $\mathcal{D}_*$ .*

To prove this theorem, we first begin with the following key lemma.

**Lemma 2.7** *There exists a constant  $\kappa_1 > 0$  such that for any  $f \in \mathcal{H}(\mathbb{D})$  having radial limits  $f^*$  a.e. and which satisfies, for some  $\alpha \in \mathbb{R}$ :*

$$(2.6) \quad \begin{cases} \Im f(0) < \alpha & \text{and} \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C}; 0 < \Re z < \pi\} \cup \{z \in \mathbb{C}; \Im z < \alpha\}, \end{cases}$$

*we have, for all  $y \geq \alpha$ :*

$$m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\}) \leq \kappa_1 e^{\alpha - y}.$$

**Proof.** Suppose that  $f$  satisfies (2.6), and define  $f_1(z) = -if(z) + \frac{\pi}{2}i - \alpha$ . Then either  $\Re[f_1(z)] < 0$ , or  $-\frac{\pi}{2} < \Im[f_1(z)] < \frac{\pi}{2}$  for every  $z \in \mathbb{D}$ . Therefore, defining  $h(z) = 1 + \exp[f_1(z)]$ , we have  $h: \mathbb{D} \rightarrow \mathbb{H}$ , that is  $\Re[h(z)] > 0$  for every  $z \in \mathbb{D}$ .

Finally define  $h_1(z) = h(z) - i\Im[h(0)]$ . Then  $h_1: \mathbb{D} \rightarrow \mathbb{H}$  and  $h_1(0) \in \mathbb{R}$  (and so  $h_1(0) > 0$ ). Kolmogorov's inequality yields that, for some absolute constant  $C_1$ , one has, for every  $\lambda > 0$ :

$$(2.7) \quad m(\{z \in \mathbb{T}; |h_1^*(z)| \geq \lambda\}) \leq C_1 \frac{h_1(0)}{\lambda}.$$

Observe that, since  $\Im[f(0)] < \alpha$ , we have  $\Re[f_1(0)] < 0$ , and then:

$$(2.8) \quad |\Im[h(0)]| < 1 \quad \text{and} \quad h_1(0) = \Re[h(0)] < 2.$$

Suppose now that, for  $y > \alpha$  and  $z \in \mathbb{D}$ , we have  $\Im[f(z)] > y$ ; then  $\exp[f_1(z)] \in \mathbb{H}$ , and  $|h(z)| \geq |\exp[f_1(z)]| > e^{y-\alpha}$ . Taking radial limits we get, up to a set of null Lebesgue-measure:

$$\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\} \subseteq \{z \in \mathbb{T}; |h^*(z)| \geq e^{y-\alpha}\}.$$

We consider two cases:  $e^{y-\alpha} \geq 2$  and  $e^{y-\alpha} < 2$ . When  $e^{y-\alpha} \geq 2$ , then  $|h^*(z)| \geq e^{y-\alpha}$  yields:

$$|h_1^*(z)| \geq e^{y-\alpha} - |\Im[h(0)]| > e^{y-\alpha} - 1 \geq \frac{1}{2}e^{y-\alpha},$$

by the first part of (2.8). Then, using (2.7) and the second part of (2.8), we have:

$$\begin{aligned} m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\}) &\leq m(\{z \in \mathbb{T}; |h_1^*(z)| > (1/2)e^{y-\alpha}\}) \\ &\leq \frac{2C_1 h_1(0)}{e^{y-\alpha}} \leq \frac{4C_1}{e^{y-\alpha}}, \end{aligned}$$

and, in this case, the lemma is proved, if one takes  $\kappa_1 \geq 4C_1$ .

When  $e^{y-\alpha} < 2$ , then  $e^{\alpha-y} > 1/2$ , and, because:

$$m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\}) \leq 1 < \kappa_1 e^{\alpha-y},$$

since  $\kappa_1 > 2$ , the lemma is proved.  $\square$

Now, we give a general construction of Schur functions with suitable properties.

**Proposition 2.8** *Let  $\mathbf{g}: (0, \infty) \rightarrow (0, \infty)$  be a continuous non-increasing function such that:*

$$\lim_{t \rightarrow 0^+} \mathbf{g}(t) = +\infty, \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathbf{g}(t) = 0.$$

Let  $\mathfrak{h}: (0, \infty) \rightarrow (0, \infty]$  be a lower semicontinuous function such that  $M := \sup\{\mathfrak{h}(t); t \geq \pi\} < +\infty$  and consider the simply connected domain:

$$\Omega = \{x + iy; x \in (0, \infty) \text{ and } \mathfrak{g}(x) < y < \mathfrak{g}(x) + \mathfrak{h}(x)\}.$$

Let  $\mathfrak{f}: \overline{\mathbb{D}} \rightarrow \overline{\Omega} \cup \{\infty\}$  be a conformal mapping from  $\mathbb{D}$  onto  $\Omega$  such that  $\mathfrak{f}(0) = \pi + i(\mathfrak{g}(\pi) + \mathfrak{h}(\pi)/2)$ .

Then the symbol  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  defined by  $\varphi(z) = \exp[-\mathfrak{f}(z)]$ , for every  $z \in \mathbb{D}$ , satisfies, for some  $\varepsilon_0, k_0 > 0$ :

1) For all  $h \in (0, \varepsilon_0)$ :

$$(2.9) \quad m(\{z \in \mathbb{T}; |\varphi^*(z)| > 1 - h\}) \leq k_0 \exp(-\mathfrak{g}(2h)).$$

2) Assume that, for some  $r \in (0, \infty]$  and integers  $0 \leq n < N \leq \infty$ , one has  $\{\mathfrak{h}(t); t \leq r\} \subseteq (2n\pi, 2N\pi]$ . Then, for all  $z \in \mathbb{D}$ , such that  $|z| > e^{-r}$ , we have  $n \leq n_\varphi(z) \leq N$ .

In particular,  $\{z \in \mathbb{D}; |z| > e^{-r}\} \subseteq \varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus \{0\}$ , when  $n \geq 1$ .

#### Remarks.

1. When  $N = 1$ , the map  $\varphi$  is univalent.
2. When  $r = \infty$  and  $n \geq 1$ , we have  $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$ .
3. With  $\mathfrak{g}(t) = 1/t$ , the operator  $C_\varphi$  is compact on  $H^{\Psi_2}$ , therefore belongs to all Schatten classes  $S_p(H^2)$ ,  $p > 0$ .
4. When  $N < \infty$ , the operator  $C_\varphi$  is bounded on the Dirichlet space.
5. When  $n \geq 1$ , the operator  $C_\varphi$  is not compact on the Dirichlet space (since the averages on the windows of the function  $n_\varphi$  cannot uniformly vanish).

**Proof of Proposition 2.8.** We shall apply Lemma 2.7 with  $\alpha = M + \mathfrak{g}(\pi)$ .

Suppose that, for  $z \in \mathbb{T}$  and  $0 < h < 1$ , we have  $|\varphi^*(z)| > 1 - h$ . Then, if  $h$  is small enough,

$$e^{-2h} < 1 - h < |\varphi^*(z)| = \exp(-\Re[f^*(z)]),$$

and therefore  $2h > \Re[f^*(z)]$ . But observe that  $\mathfrak{f}^*(z) \in \overline{\Omega} \cup \{\infty\}$ , and so, if  $2h > \Re[f^*(z)]$ , we necessarily have  $\Im[f^*(z)] \geq \mathfrak{g}(2h)$ . Again, if  $h$  is small enough, we have  $y = \mathfrak{g}(2h) > \alpha$ , and may apply the lemma to obtain:

$$m(\{z \in \mathbb{T}; |\varphi^*(z)| > 1 - h\}) \leq m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq \mathfrak{g}(2h)\}) \leq \kappa_1 e^{\alpha - \mathfrak{g}(2h)}.$$

We get (2.9).

On the other hand, let  $Z \in \mathbb{D}$  such that  $|Z| > e^{-r}$ , we can write  $Z = e^{-x} e^{i\theta}$  with  $x < r$ . We can find  $\theta_j$ 's such that  $\mathfrak{g}(x) < \theta_1 < \dots < \theta_s < \mathfrak{g}(x) + \mathfrak{h}(x)$  and  $\theta_j \equiv \theta[2\pi]$  with  $n \leq s \leq N$ . For each  $j$ , there exists a unique  $z_j \in \mathbb{D}$ , such that  $\Re \mathfrak{f}(z_j) = x$  and  $\Im \mathfrak{f}(z_j) = \theta_j$ ; hence  $\varphi(z_j) = Z$ . Moreover no other  $z \in \mathbb{D}$  can satisfy  $\varphi(z) = Z$ . Hence  $n_\varphi(Z) = s$ .  $\square$

**Proof of Theorem 2.6.** As said before, if one takes  $\mathbf{g}(t) = 1/t$  in Proposition 2.8, then  $C_\varphi$  is compact on  $H^{\Psi_2}$  and hence is in all Schatten classes  $S_p(H^2)$ ,  $p > 0$ . On the other hand, if one choose also  $\mathbf{h}(t) = 1/t$ , then, for every  $r > 0$ ,  $\{\mathbf{h}(t); t \leq r\} = [1/r, \infty)$  and for  $|z| > e^{-r}$ , we get that  $n_\varphi(z) \geq [1/(2\pi r)]$  (the integer part of  $1/(2\pi r)$ ). It follows that, for some constant  $c > 0$ , one has, with  $e^{-r} = 1 - h$ :

$$\frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(z) dA(z) \geq c \frac{1}{\log[1/(1-h)]} \xrightarrow{h \rightarrow 0} \infty.$$

Therefore,  $C_\varphi$  is not bounded on  $\mathcal{D}_*$ , by (1.5).  $\square$

**Remarks.** 1. Actually, as we may take  $\mathbf{g}$  growing as we wish, the proof shows, using [12], Theorem 4.18, that for every Orlicz function  $\Psi$ , one can find a Schur function  $\varphi$  such that  $C_\varphi$  is not bounded on  $\mathcal{D}_*$ , though compact on the Hardy-Orlicz space  $H^\Psi$ .

2. This construction also allows to produce a *univalent* map  $\varphi$ , with an arbitrary small Carleson function  $\rho_\varphi(h) = \sup_{|\xi|=1} m(\{e^{it}; \varphi^*(e^{it}) \in W(\xi, h)\})$ , and such that  $C_\varphi$  is not compact on the Dirichlet space (note we cannot replace “compact” by “bounded” since any Schur function with a bounded valence is bounded on the Dirichlet space).

Indeed, take  $\mathbf{h}(t) = 2\pi$  and  $\mathbf{g}$  be  $\mathcal{C}^1$ :  $\mathbf{g}(t) = 1/t$  for instance. We have  $N = 1$  and so  $\varphi$  is univalent. Now it suffices to notice that the range of the curve

$$\Gamma = \{e^{-x-ig(x)}; x \in (0, \infty)\} = \{(t \cos(1/\ln(t)), t \sin(1/\ln(t))); t \in (0, 1)\} \subseteq \mathbb{D}$$

has a null area measure. The range of  $\varphi$  is  $\mathbb{D} \setminus (\Gamma \cup \{0\})$  and for each  $w \notin \Gamma$ , we have  $n_\varphi(w) = 1$ . Then, for  $h \in (0, 1)$ , we have:

$$\begin{aligned} \frac{1}{h^2} \int_{W(1, h)} n_\varphi(w) dA(w) &= \frac{1}{h^2} \int_{W(1, h) \setminus \Gamma} dA(w) = \frac{1}{h^2} A[W(1, h) \setminus \Gamma] \\ &= \frac{1}{h^2} A[W(1, h)] \approx 1, \end{aligned}$$

and so  $C_\varphi$  is not compact on  $\mathcal{D}_*$ , by (1.7).  $\square$

## 3 Composition operators in Schatten classes

### 3.1 Characterization

In this section, we give a characterization of the membership in the Schatten classes of composition operators on  $\mathcal{D}_*$ . This characterization will be deduced from Luecking’s one for composition operators on the Bergman space. Actually, we shall give it for weighted Dirichlet spaces  $\mathcal{D}_{\alpha, *}$ . Boundedness and compactness has been characterized by B. McCluer and J. Shapiro in [21] and, in other terms, by N. Zorboska in [29].

Recall that for  $\alpha > -1$ , the weighted Dirichlet space  $\mathcal{D}_\alpha$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$(3.1) \quad \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

This is a Hilbert space for the norm given by:

$$(3.2) \quad \|f\|_\alpha^2 = |f(0)|^2 + (\alpha + 1) \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

The standard Dirichlet space  $\mathcal{D}$  corresponds to  $\alpha = 0$ ; the Hardy space  $H^2$  to  $\alpha = 1$  and the standard Bergman space to  $\alpha = 2$ . For more general weights, see [10].

We denote by  $\mathcal{D}_{\alpha,*}$  the subspace of the  $f \in \mathcal{D}_\alpha$  such that  $f(0) = 0$ .

If  $\varphi$  is a Schur function, one defines its *weighted Nevanlinna counting function*  $N_{\varphi,\alpha}$  at  $w \in \Omega := \varphi(\mathbb{D})$  as the number of pre-images of  $w$  with the weight  $(1 - |z|)^\alpha$ :

$$(3.3) \quad N_{\varphi,\alpha}(w) = \sum_{\varphi(z)=w} (1 - |z|^2)^\alpha.$$

For  $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$ , we set  $N_{\varphi,\alpha}(w) = 0$ . One has  $N_{\varphi,1} = N_\varphi$  and  $N_{\varphi,0} = n_\varphi$ .

With this notation, recall the change of variable formula:

$$(3.4) \quad \int_{\mathbb{D}} F[\varphi(z)] |\varphi'(z)|^2 (1 - |z|^2)^\alpha dA(z) = \int_{\Omega} F(w) N_{\varphi,\alpha}(w) dA(w).$$

Denote by  $R_{n,j}$ ,  $n \geq 0$ ,  $0 \leq j \leq 2^n - 1$ , the Hastings-Luecking windows:

$$R_{n,j} = \left\{ z \in \mathbb{D}; 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \quad \text{and} \quad \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \right\}.$$

We can now state.

**Theorem 3.1** *Let  $\alpha > -1$ . Let  $\varphi$  be a Schur function and  $p > 0$ . Then  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  if and only if:*

$$(3.5) \quad \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) dA(w) \right]^{p/2} < \infty.$$

If  $\varphi$  is univalent, (3.5) can be replaced by the purely geometric condition:

$$(3.6) \quad \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} [2^{n(\alpha+2)} A_\alpha(R_{n,j} \cap \Omega)]^{p/2} < \infty,$$

where  $A_\alpha$  is the weighted measure  $dA_\alpha(w) = (\alpha + 1) (1 - |w|^2)^\alpha dA(w)$ .

**Remark.** Of course, every operator in a Schatten class is compact, but we may note that condition (3.5) implies the compactness of  $C_\varphi$ , by [29], Theorem 1 (and [13], Proposition 3.3).

**Proof of Theorem 3.1.** First, we compute  $C_\varphi^* C_\varphi$ . Let us fix  $f$  and  $g$  in the Dirichlet space  $\mathcal{D}_{\alpha,*}$ . We have:

$$\begin{aligned} (\alpha + 1) \int_{\mathbb{D}} ((C_\varphi^* C_\varphi)(f))'(z) \overline{g'(z)} (1 - |z|^2)^\alpha dA(z) &= \langle f \circ \varphi, g \circ \varphi \rangle_{\mathcal{D}_{\alpha,*}} \\ &= (\alpha + 1) \int_{\mathbb{D}} (f' \circ \varphi)(z) \overline{(g' \circ \varphi)(z)} |\varphi'(z)|^2 (1 - |z|^2)^\alpha dA(z). \end{aligned}$$

By the change of variable formula, we get:

$$\int_{\mathbb{D}} ((C_\varphi^* C_\varphi)(f))'(z) \overline{g'(z)} (1 - |z|^2)^\alpha dA = \int_{\mathbb{D}} f'(w) \overline{g'(w)} N_{\varphi,\alpha}(w) dA(w),$$

which is equivalent to:

$$\int_{\mathbb{D}} ((C_\varphi^* C_\varphi)(f))'(z) \overline{G(z)} (1 - |z|^2)^\alpha dA(z) = \int_{\mathbb{D}} f'(w) \overline{G(w)} N_{\varphi,\alpha}(w) dA(w)$$

for every function  $G$  belonging to the weighted Bergman space  $\mathfrak{B}_\alpha^2$ .

That means that  $((C_\varphi^* C_\varphi)(f))' - f' \cdot N_{\varphi,\alpha} / (1 - |w|^2)^\alpha$  is orthogonal to the weighted Bergman space  $\mathfrak{B}_\alpha^2$ . But  $((C_\varphi^* C_\varphi)(f))' \in \mathfrak{B}_\alpha^2$ . Hence  $((C_\varphi^* C_\varphi)(f))'$  is the orthogonal projection onto  $\mathfrak{B}_\alpha^2$  of the function  $f' \cdot N_{\varphi,\alpha} / (1 - |w|^2)^\alpha$ . Thus (see [27], § 6.4.1), we obtain that for every  $z \in \mathbb{D}$ :

$$\begin{aligned} ((C_\varphi^* C_\varphi)(f))'(z) &= (\alpha + 1) \int_{\mathbb{D}} \frac{f'(w)}{(1 - \bar{w}z)^{\alpha+2}} \frac{N_{\varphi,\alpha}(w)}{(1 - |w|^2)^\alpha} (1 - |w|^2)^\alpha dA(w) \\ &= (\alpha + 1) \int_{\mathbb{D}} \frac{f'(w)}{(1 - \bar{w}z)^{\alpha+2}} d\mu(w) \\ &= (\alpha + 1) T_\mu(f')(z), \end{aligned}$$

where  $\mu$  is the positive measure  $A$  with weight  $N_{\varphi,\alpha}$  and  $T_\mu$  is the Toeplitz operator on  $\mathfrak{B}_\alpha^2$  is introduced in [19] (let us point out that  $\alpha$  in [19] corresponds to  $-(\alpha + 1)$  in our work).

In other words, introducing the map  $\Delta(h) = h'$ , which is an isometry from  $\mathcal{D}_{\alpha,*}$  onto  $\mathfrak{B}_\alpha^2$ , we have  $\Delta \circ (C_\varphi^* C_\varphi) = T_\mu \circ \Delta$ . We have the following diagram:

$$\begin{array}{ccc} \mathcal{D}_{\alpha,*} & \xrightarrow{C_\varphi^* C_\varphi} & \mathcal{D}_{\alpha,*} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathfrak{B}_\alpha^2 & \xrightarrow{T_\mu} & \mathfrak{B}_\alpha^2 \end{array}$$

Hence the approximation numbers of  $T_\mu$  (viewed as an operator on  $\mathfrak{B}_\alpha^2$ ) and the ones of  $C_\varphi^* C_\varphi$  (viewed as an operator on  $\mathcal{D}_{\alpha,*}$ ) are the same. In particular,

the membership in the Schatten classes are the same and the final result follows from the main theorem in [19]:  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  if and only if  $C_\varphi^* C_\varphi \in S_{p/2}(\mathcal{D}_{\alpha,*})$  and that holds if and only if:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} [2^{n(\alpha+2)} \mu(R_{n,j})]^{p/2} < \infty.$$

Hence  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  if and only if:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) dA(w) \right]^{p/2} < \infty,$$

and that ends the proof of Theorem 3.1.  $\square$

**Remark.** In the same way, we can obtain other characterizations for  $\mathcal{D}_{\alpha,*}$  by using the ones for  $\mathfrak{B}_\alpha^2$  given in [20] and [28]:  $C_\varphi \in S_p(\mathfrak{B}_\alpha^2)$  if and only if  $N_{\varphi,\alpha+2}(z)/(\log(1/|z|))^{\alpha+2} \in L^{p/2}(\lambda)$ , where  $d\lambda(z) = (1 - |z|^2)^{-2} dA(z)$  is the Möbius invariant measure on  $\mathbb{D}$ , and, when  $\varphi$  has bounded valence and  $p \geq 2$ , if and only if  $(1 - |z|^2)/(1 - |\varphi(z)|^2) \in L^{p(\alpha+2)/2}(\lambda)$ . Such a result can be found in [26].

## 3.2 Applications

We give several applications of the previous theorem.

**Corollary 3.2** *Let  $-1 < \alpha \leq \beta$ ,  $p > 0$ , and  $\varphi$  be a Schur function. Then  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  implies that  $C_\varphi \in S_p(\mathcal{D}_{\beta,*})$ .*

*In particular,  $C_\varphi \in S_p(\mathcal{D}_*)$  implies that  $C_\varphi \in S_p(H^2)$ , which in turn implies that  $C_\varphi \in S_p(\mathfrak{B}^2)$ .*

**Proof.** Assume that  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ . Then

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) dA(w) \right]^{p/2} < \infty.$$

Since, thanks to Schwarz's lemma,  $N_{\varphi,\beta}(w) \leq N_{\varphi,\alpha}(w)(1 - |w|^2)^{\beta-\alpha}$ , we have

$$N_{\varphi,\beta}(w) \leq (2 \cdot 2^{-n})^{\beta-\alpha} N_{\varphi,\alpha}(w) \quad \text{for } w \in R_{n,j}.$$

It follows that

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\beta+2)} \int_{R_{n,j}} N_{\varphi,\beta}(w) dA(w) \right]^{p/2} < \infty,$$

and that proves Corollary 3.2.  $\square$

It is known ([13]) that composition operators on  $H^2$  separate Schatten classes, but the difficulty is that we must not only control the shape of  $\varphi(\partial\mathbb{D})$ , but also the parametrization  $t \mapsto \varphi(e^{it})$ , even if  $\varphi$  is univalent. In the case of the Dirichlet space, this difficulty disappears, because only the areas come into play, and we can easily prove the following result.

**Theorem 3.3** *The composition operators on  $\mathcal{D}_*$  separate Schatten classes, in the following sense. Let  $0 < p_1 < \infty$ . Then, there exists a symbol  $\varphi$  such that:*

$$C_\varphi \in \left( \bigcap_{p > p_1} S_p(\mathcal{D}_*) \right) \setminus S_{p_1}(\mathcal{D}_*).$$

Similarly, there exists a symbol  $\varphi$  such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \left( \bigcup_{p < p_1} S_p(\mathcal{D}_*) \right).$$

In particular, for every  $0 < p_1 < p_2 < \infty$ , there exists  $\varphi$  such that  $C_\varphi \in S_{p_2}(\mathcal{D}_*) \setminus S_{p_1}(\mathcal{D}_*)$ .

**Proof.** Let  $(h_n)_{n \geq 1}$ , with  $0 < h_n < 1$ , be a sequence of real numbers with limit 0 to be adjusted, and  $J$  the Jordan curve formed by the segment  $[0, 1]$  and the north and (truncated) north-east sides of the curvilinear rectangles

$$\{1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\} \times \{0 \leq \arg z < 2^{-n} h_n\}.$$

Let  $\Omega_0$  be the interior of  $J$  and  $\Omega = \Omega_0 \cup D(0, 1/8)$ . Let  $\varphi: \mathbb{D} \rightarrow \Omega$  be a Riemann map such that  $\varphi(0) = 0$ . Since  $\varphi$  is univalent and bounded, it defines a symbol on  $\mathcal{D}_*$ , and the necessary and sufficient condition (3.6) for membership in  $S_p(\mathcal{D}_*)$  reads:

$$(3.7) \quad \sum_{n=0}^{\infty} [4^n 4^{-n} h_n]^{p/2} = \sum_{n=0}^{\infty} h_n^{p/2} < \infty.$$

Indeed, it is clear that, for fixed  $n$ , the Hastings-Luecking windows  $R_{n,j}$  satisfy:

$$R_{n,0} \cap \Omega \neq \emptyset; \quad R_{n,j} \cap \Omega = \emptyset \text{ for } 1 \leq j < 2^n.$$

Therefore, only the Hastings-Luecking windows  $R_{n,0}$  matter. Since:

$$A(R_{n,0} \cap \Omega) = \iint_{1-2^{-n} \leq r < 1-2^{-n-1}, 0 \leq \theta < 2^{-n} h_n} r \, dr \, d\theta \approx 4^{-n} h_n,$$

we can test the criterion (3.7). Now, it is enough to take  $h_n = (n+1)^{-2/p_1}$  to get:

$$C_\varphi \in \left( \bigcap_{p > p_1} S_p(\mathcal{D}_*) \right) \setminus S_{p_1}(\mathcal{D}_*).$$

Similarly, the choice  $h_n = (n+1)^{-2/p_1} [\log(n+2)]^{-4/p_1}$ , gives a symbol  $\varphi$  such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \left( \bigcup_{p < p_1} S_p(\mathcal{D}_*) \right).$$

This ends the proof. □



T. Carroll and C. Cowen ([3] proved, but only for  $\alpha > 0$ , that there exist compact composition operators on  $\mathcal{D}_\alpha$  which are in no Schatten class (see also [8]). In the next result, we shall see that this still true for  $\alpha = 0$ .

**Theorem 3.4** *There exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but in no Schatten class  $S_p(\mathcal{D}_*)$ .*

**Proof.** It suffices to use the proof of Theorem 3.3 and to take, instead of the above  $h_n$ ,  $h_n = 1/\ln(n+2)$ .  $\square$

For the next application, which will be used in Section 4, we need to recall the definition of the cusp map  $\chi$ , introduced in [15], and later used, with a slightly different definition in [18]. Actually, we have to modify it slightly again in order to have  $\chi(0) = 0$ . We first define:

$$\chi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1},$$

then:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where  $a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)$  is chosen in order that  $\chi(0) = 0$ . The image  $\Omega$  of the (univalent) cusp map is formed by the intersection of the inside of the disk  $D(\frac{a}{2}, \frac{a}{2})$  and the outside of the two disks  $D(\frac{ia}{2}, \frac{a}{2})$  and  $D(-\frac{ia}{2}, \frac{a}{2})$ .

**Corollary 3.5** *If  $\chi$  is the cusp map, then  $C_\chi$  belongs to all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ .*

**Proof.** Since  $\chi$  is univalent,  $\chi(0) = 0$ , and  $\Omega = \chi(\mathbb{D})$  has finite area, we have  $\chi \in \mathcal{D}_*$ . A little elementary geometry shows that, for some constant  $C$ , we have:

$$(3.8) \quad w \in \Omega, \quad 0 < h < 1 \text{ and } |w| \geq 1 - h \quad \implies \quad |\Im w| \leq Ch^2.$$

It follows (changing  $C$  if necessary) that  $R_{n,j} \cap \Omega$  is contained in a rectangle of sizes  $2^{-n}$  and  $C4^{-n}$  and with area  $C8^{-n}$ . Hence, for a given  $n$ , at most  $C$  of the Hastings-Luecking windows  $R_{n,j}$  can intersect  $\Omega$ . Therefore, the series in Theorem 3.1 reduces, up to constants, to the series:

$$\sum_{n=0}^{\infty} (4^n 8^{-n})^{p/2} = \sum_{n=0}^{\infty} 2^{-np},$$

which converges for every  $p > 0$ .  $\square$

## 4 Logarithmic capacity and set of contact points

In view of the result of [6] mentioned in the introduction, if  $\text{Cap } K > 0$ , there is no hope to find a symbol  $\varphi$  such that  $E_\varphi = K$  and  $C_\varphi$  is Hilbert-Schmidt on  $\mathcal{D}_*$ . But as was later proved in [5],  $\text{Cap } K > 0$  is the only obstruction. We can improve on the results from [5] as follows: our composition operator is not only Hilbert-Schmidt, but in any Schatten class; moreover, we can replace  $E_\varphi = K$  by  $E_\varphi = E_\varphi(1) = K$ .

**Theorem 4.1** *For every compact set  $K$  of the unit circle  $\mathbb{T}$  with logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $\varphi$  with the following properties:*

- 1)  $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_* := A$ , the ‘‘Dirichlet algebra’’;
- 2)  $E_\varphi = E_\varphi(1) = K$ ;
- 3)  $C_\varphi \in \bigcap_{p>0} S_p(\mathcal{D}_*)$ .

*In fact, the approximation numbers of  $C_\varphi$  satisfy  $a_n(C_\varphi) \leq a \exp(-b\sqrt{n})$ .*

This theorem actually results of the particular following case and the properties of the cusp map seen in Section 3.2.

**Theorem 4.2** *For every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $q \in A(\mathbb{D}) \cap \mathcal{D}_*$  which peaks on  $K$  and such that the composition operator  $C_q: \mathcal{D}_* \rightarrow \mathcal{D}_*$  is bounded (and even Hilbert-Schmidt).*

Recall that a function  $q \in A(\mathbb{D})$ , the disk algebra, is said to *peak* on a compact subset  $K \subseteq \partial\mathbb{D}$  (and is called a *peaking function*) if:

$$q(z) = 1 \text{ if } z \in K; \quad |q(z)| < 1 \text{ if } z \in \overline{\mathbb{D}} \setminus K.$$

**Proof of Theorem 4.1.** We simply take for  $\varphi$  the composed map  $\varphi = \chi \circ q$ , where  $\chi$  is the cusp map and  $q$  our peaking function. Recall that  $\chi \in A(\mathbb{D})$  and that  $\chi$  peaks on  $\{1\}$ . We take advantage of this fact by composing with  $q$ , for which  $C_q: \mathcal{D}_* \rightarrow \mathcal{D}_*$  is bounded as well as  $C_\chi$  (since  $\chi$  is univalent). We clearly have  $\varphi \in A(\mathbb{D})$ ,  $\varphi(z) = \chi(1) = 1$  for  $z \in K$ , and  $|\varphi(z)| < 1$  for  $z \notin K$ , since then  $|q(z)| < 1$ . Therefore  $E_\varphi(1) = K$ . Moreover,  $C_\varphi$  being bounded on  $\mathcal{D}_*$ , we have in particular  $\varphi = C_\varphi(z) \in \mathcal{D}_*$ . Since  $C_\varphi = C_q \circ C_\chi$ , we get 3), by Corollary 3.5.

In [16], we prove that  $a_n(C_\chi) \leq a \exp(-b\sqrt{n})$ . Since  $a_n(C_\varphi) \leq \|C_q\| a_n(C_\chi)$ , by the ideal property of approximation numbers, this ends the proof of Theorem 4.1.  $\square$

In turn, the proof of Theorem 4.2 relies on the following crucial lemma.

**Lemma 4.3** *Let  $K \subseteq \partial\mathbb{D}$  be a compact set such that  $\text{Cap } K = 0$ . Then, there exists a function  $U: \overline{\mathbb{D}} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ , such that:*

- 1)  $U(z) = \infty$  if and only if  $z \in K$ ;
- 2)  $U \geq 1$  on  $\overline{\mathbb{D}}$ ;
- 3)  $U$  is continuous on  $\overline{\mathbb{D}} \setminus K$ , harmonic in  $\mathbb{D}$  and  $\int_{\mathbb{D}} |\nabla U|^2 dA < \infty$ ;
- 4)  $\lim_{z \rightarrow K, z \in \overline{\mathbb{D}}} U(z) = \infty$ ;
- 5) the conjugate function  $V = \tilde{U}$  is continuous on  $\overline{\mathbb{D}} \setminus K$ .

**Proof of Theorem 4.2.** Taking this lemma for granted, let us end the proof of the theorem. We set  $f = U + iV$ ,  $a = e^{-1/f(0)}$  and  $q = \varphi_a \circ e^{-1/f}$ , where  $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ . In view of the third and fourth items of the lemma, we have  $q \in A(\mathbb{D})$ . Since  $U \geq 1$ , Lemma 2.4 shows that  $C_q$  is Hilbert-Schmidt on  $\mathcal{D}_*$ . Moreover, for  $z \in K$ , one has  $f(z) = \infty$  and hence  $q(z) = 1$  since  $\varphi_a(1) = 1$  because  $a \in \mathbb{R}$  (since  $f(0) = U(0)$ ). On the other hand, when  $z \notin K$ , one has  $|f(z)| < \infty$  and hence  $|q(z)| < 1$ . Therefore  $q$  peaks on  $K$ .  $\square$

**Proof of Lemma 4.3.** This proof is strongly influenced by that of Theorem III, page 47, in [9]. Let:

$$(4.1) \quad L(z) = \log \left( \frac{e}{1-z} \right) = P(z) + iQ(z),$$

with

$$P(z) = \log \frac{e}{|1-z|} \text{ and } Q(z) = -\arg(1-z), \quad |Q(z)| \leq \frac{\pi}{2}, \quad z \in \overline{\mathbb{D}} \setminus \{1\},$$

and write:

$$P(z) \sim \sum_{n \in \mathbb{Z}} \gamma_n z^n,$$

with

$$\gamma_n = 1/(2|n|) \quad \text{if } n \neq 0, \quad \text{and } \gamma_0 = 1.$$

For  $0 < \varepsilon < 1/2$ , let  $K_\varepsilon = \{z \in \mathbb{T}; \text{dist}(z, K) \leq \varepsilon\}$ ,  $\mu_\varepsilon$  its equilibrium measure, and  $U_\varepsilon$  the logarithmic potential of  $\mu_\varepsilon$ , that is:

$$U_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{e}{|z-w|} d\mu_\varepsilon(w),$$

that we could as well write (since  $K_\varepsilon \subseteq \mathbb{T}$ ):

$$U_\varepsilon(z) = \int_{K_\varepsilon} P(z\bar{w}) d\mu_\varepsilon(w).$$

Let us set:

$$(4.2) \quad f_\varepsilon(z) = \int_{K_\varepsilon} L(z\bar{w}) d\mu_\varepsilon(w) = U_\varepsilon(z) + iV_\varepsilon(z),$$

with

$$V_\varepsilon(z) = \int_{K_\varepsilon} Q(z \bar{w}) d\mu_\varepsilon(w).$$

Then, if  $I_\varepsilon$  is the energy of  $\mu_\varepsilon$ , one has (see [23], Section 4)  $I_\varepsilon = 1 + \sum_{n=1}^{\infty} \frac{|\widehat{\mu_\varepsilon}(n)|^2}{n}$ , where  $\widehat{\mu_\varepsilon}(n) = \int_{\mathbb{T}} \bar{w}^n d\mu_\varepsilon(w)$  is the  $n$ -th Fourier coefficient of  $\mu_\varepsilon$ , and:

$$(4.3) \quad f_\varepsilon \in \mathcal{D} \quad \text{and} \quad \|f_\varepsilon\|_{\mathcal{D}}^2 = I_\varepsilon.$$

Note that  $\|f_\varepsilon\|_{\mathcal{D}} \geq 1$ .

We claim that there exist  $\delta > 0$  and  $0 < r < 1$  such that:

$$(4.4) \quad z \in \overline{\mathbb{D}} \text{ and } \text{dist}(z, K) \leq \delta \implies U_\varepsilon(rz) \geq I_\varepsilon/2$$

Indeed, let  $P_a(t) = \frac{1-|a|^2}{|e^{it}-a|^2}$  be the Poisson kernel at  $a \in \mathbb{D}$ . Since  $U_\varepsilon$  is harmonic in  $\mathbb{D}$  and integrable on  $\mathbb{T}$  ([4], Proposition 19.5.2), one has, for every  $z \in \mathbb{D}$ :

$$(4.5) \quad U_\varepsilon(z) = \int_{-\pi}^{\pi} U_\varepsilon(e^{it}) P_z(t) \frac{dt}{2\pi}.$$

Let now  $\delta \leq \varepsilon/4$ , to be adjusted later, and take  $1 - \delta \leq r < 1$ . Suppose that  $\text{dist}(z, K) \leq \delta$ , with  $z \in \overline{\mathbb{D}}$ , and let  $u \in K$  such that  $|z - u| \leq \varepsilon/4$ . Note that then  $|rz - u| \leq (1 - r) + |z - u| \leq \varepsilon/2$ . It follows from (4.5) that:

$$I_\varepsilon - U_\varepsilon(rz) = \int_{-\pi}^{\pi} [I_\varepsilon - U_\varepsilon(e^{it})] P_{rz}(t) \frac{dt}{2\pi}$$

(it is useful to recall that  $U_\varepsilon(z) \leq I_\varepsilon$  for every  $z \in \mathbb{C}$ ). Set:

$$J_1 = \int_{|e^{it}-rz| \leq \varepsilon/2} [I_\varepsilon - U_\varepsilon(e^{it})] P_{rz}(t) \frac{dt}{2\pi}$$

and

$$J_2 = \int_{|e^{it}-rz| > \varepsilon/2} [I_\varepsilon - U_\varepsilon(e^{it})] P_{rz}(t) \frac{dt}{2\pi}.$$

For the integral  $J_1$ , we have:

$$|e^{it} - u| \leq |e^{it} - rz| + |rz - u| \leq \varepsilon;$$

therefore  $e^{it} \in K_\varepsilon$ . Since  $U_\varepsilon = I_\varepsilon$  Lebesgue-almost everywhere on  $K_\varepsilon$ , by Frostman's Theorem, we get  $J_1 = 0$ .

For the integral  $J_2$ , we have:

$$P_{rz}(t) \leq \frac{2(1-r|z|)}{(\varepsilon/2)^2} \leq 2 \frac{(1-r) + r(1-|z|)}{(\varepsilon/2)^2} \leq \frac{4\delta}{(\varepsilon/2)^2} = \frac{16\delta}{\varepsilon^2};$$

hence (since  $U_\varepsilon(e^{it}) \geq 0$ ):

$$J_2 \leq \frac{16\delta}{\varepsilon^2} I_\varepsilon.$$

Therefore, if we choose  $0 < \delta \leq \varepsilon^2/32$ , we get:

$$0 \leq I_\varepsilon - U_\varepsilon(rz) \leq I_\varepsilon/2,$$

which gives (4.4). □

Now, as  $\text{Cap } K = 0$ , we know from (1.9) that  $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon = \infty$ , and we can adjust a sequence  $\varepsilon_j \rightarrow 0^+$  so that:

$$(4.6) \quad I_{\varepsilon_j} \geq 4j^6.$$

Using (4.4), we find two sequences  $(\delta_j)_j$  and  $(r_j)_j$ , with  $0 < \delta_j \rightarrow 0$  and  $1 > r_j \rightarrow 1$ , such that, for every  $j \geq 1$ ,

$$(4.7) \quad z \in \overline{\mathbb{D}} \text{ and } \text{dist}(z, K) \leq \delta_j \implies U_{\varepsilon_j}(r_j z) \geq I_{\varepsilon_j}/2.$$

Finally, let us set:

$$(4.8) \quad f_j(z) = f_{\varepsilon_j}(r_j z)$$

and

$$(4.9) \quad f = U + iV = 1 + \sum_{j=1}^{\infty} j^{-2} \frac{f_j}{\|f_j\|_{\mathcal{D}}}.$$

The series defining  $f$  is absolutely convergent in  $\mathcal{D}$ . Note that  $f(0)$  is real.

We now have:

1)  $f$  is continuous on  $\overline{\mathbb{D}} \setminus K$ .

Indeed, let  $z \in \overline{\mathbb{D}} \setminus K$ . Then,  $\text{dist}(z, K) > 0$  and there exists a neighbourhood  $\omega$  of  $z$  in  $\overline{\mathbb{D}}$ , an integer  $j_0 = j_0(z)$  and a positive number  $\delta > 0$  such that:

$$w \in \omega \text{ and } j \geq j_0 \implies \text{dist}(r_j w, K_{\varepsilon_j}) \geq \delta.$$

We then have, for  $w \in \omega$  and  $j \geq j_0$ :

$$\begin{aligned} |f_{\varepsilon_j}(w)| &= \left| \int_{K_{\varepsilon_j}} \log \frac{e}{r_j w - u} d\mu_{\varepsilon_j}(u) \right| \\ &\leq \int_{K_{\varepsilon_j}} \left( \log \frac{e}{|r_j w - u|} + \frac{\pi}{2} \right) d\mu_{\varepsilon_j}(u) \leq \log \frac{e}{\delta} + \frac{\pi}{2} := C, \end{aligned}$$

since  $\mu_{\varepsilon_j}$  is a probability measure supported by  $K_{\varepsilon_j}$ . Therefore, the series defining  $f$  is normally convergent on  $\omega$  since its general term is dominated by  $j^{-2}C$  on  $\omega$ . Since the functions  $f_j$  are continuous on  $\overline{\mathbb{D}}$ , this shows that  $f$  is continuous at  $z$ .

2)  $U(z) := \Re f(z) \geq 1$ .

This is obvious since, for every  $z \in \overline{\mathbb{D}}$ ,

$$U_\varepsilon(z) := \Re f_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{e}{|z - u|} d\mu_\varepsilon(u) \geq 0.$$

3)  $\lim_{z \rightarrow K, z \in \mathbb{D}} U(z) = \infty$ .

Indeed, let  $A > 0$ . Take an integer  $j \geq A$  and suppose that  $\text{dist}(z, K) \leq \delta_j$ . Then, using the positivity of the  $U_{\varepsilon_k}$ 's as well as (4.3), (4.6) and (4.7), we have:

$$U(z) \geq j^{-2} \frac{U_{\varepsilon_j}(r_j z)}{\|f_{\varepsilon_j}\|_{\mathcal{D}}} \geq j^{-2} \frac{I_{\varepsilon_j}/2}{\sqrt{I_{\varepsilon_j}}} \geq j \geq A.$$

This ends the proof of our claims, and of Lemma 4.3.  $\square$

To end this paper, let us mention the following version of the classical Rudin-Carleson Theorem. Though it is not the main subject of this paper, it has the same flavor as Theorem 4.2. We do not give a proof, but only mention that it can be obtained by mixing the proofs of Theorems III.E.2 and III.E.6 in [25] (see pages 181–187).

**Theorem 4.4** *Let  $K$  be a compact subset of  $\mathbb{T}$  with  $\text{Cap } K = 0$ . Given any continuous strictly positive function  $s \in C(\mathbb{T})$  equal to 1 on  $K$ , we can find, for every  $h \in C(K)$  and every  $\varepsilon > 0$ , a function  $f \in A(\mathbb{D}) \cap \mathcal{D}$  such that  $f|_K = h$  and:*

$$|f(\theta)| \leq (1 + \varepsilon) \|h\|_{\infty} s(\theta), \quad \forall \theta \in \mathbb{T}; \quad \|f\|_{\mathcal{D}} \leq (1 + \varepsilon) \|h\|_{\infty}.$$

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