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# Infinitesimal Carleson property for weighted measures induced by analytic self-maps of the unit disk

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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**Abstract.** We prove that, for every  $\alpha > -1$ , the pull-back measure  $\varphi(\mathcal{A}_\alpha)$  of the measure  $d\mathcal{A}_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha d\mathcal{A}(z)$ , where  $\mathcal{A}$  is the normalized area measure on the unit disk  $\mathbb{D}$ , by every analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  is not only an  $(\alpha + 2)$ -Carleson measure, but that the measure of the Carleson windows of size  $\varepsilon h$  is controlled by  $\varepsilon^{\alpha+2}$  times the measure of the corresponding window of size  $h$ . This means that the property of being an  $(\alpha + 2)$ -Carleson measure is true at all infinitesimal scales. We give an application by characterizing the compactness of composition operators on weighted Bergman-Orlicz spaces.

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**Key-words.** Calderón-Zygmund decomposition ; Carleson measure ; weighted Bergman space

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## 1 Introduction and notation

It is well-known that every analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  induces a bounded composition operator  $f \mapsto C_\varphi(f) = f \circ \varphi$  from the Bergman space  $\mathfrak{B}^2$  into itself. By Hastings's version of the Carleson inclusion theorem ([4]), that means that the pull-back measure  $\mathcal{A}_\varphi$  of the normalized area measure  $\mathcal{A}$  by  $\varphi$  is a 2-Carleson measure, that is, for some constant  $C > 0$ ,

$$\mathcal{A}(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, \varepsilon)\}) \leq C \varepsilon^2$$

for every  $\varepsilon \in (0, 1)$  and every  $\xi \in \mathbb{T}$ , where  $W(\xi, \varepsilon)$  is the Carleson window centered at  $\xi$  and of size  $\varepsilon$ . It was proved in [6], Theorem 3.1, that one actually

has an infinitesimal version of this property, namely, for some constant  $C > 0$ :

$$(1.1) \quad \mathcal{A}(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, \varepsilon h)\}) \leq C \mathcal{A}(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, h)\}) \varepsilon^2,$$

for every  $\varepsilon \in (0, 1)$  and  $h > 0$  small enough.

Now, consider, for  $\alpha > -1$ , the *weighted Bergman space*  $\mathfrak{B}_\alpha^2$ . By Littlewood's subordination principle, every analytic self-map  $\varphi$  of  $\mathbb{D}$  induces a bounded composition operator  $C_\varphi$  from  $\mathfrak{B}_\alpha^2$  into itself (see [8], Proposition 3.4). By Stegenga's version of the Carleson theorem ([9], Theorem 1.2), that means that the pull-back measure of  $\mathcal{A}_\alpha$  (see (1.3) below) by  $\varphi$  is an  $(\alpha + 2)$ -Carleson measure. Our goal in this paper is to show the analog of (1.1) in the following form.

**Theorem 1.1** *For each  $\alpha > -1$ , there exists a constant  $C_\alpha > 0$  such that, for every analytic self-map of the unit disk  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , every  $\varepsilon \in (0, 1)$  and every  $h > 0$  small enough, one has, for every  $\xi \in \mathbb{T}$ :*

$$(1.2) \quad \mathcal{A}_\alpha(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, \varepsilon h)\}) \leq C_\alpha \varepsilon^{\alpha+2} \mathcal{A}_\alpha(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, h)\}).$$

It should be stressed that the heart of the proof given in [6] in the case  $\alpha = 0$  cannot be directly used for the other  $\alpha > -1$ , and we have to change it, justifying the current paper. Moreover, the present proof is simpler than that of [6]. We also pointed out that the result holds in the limiting case  $\alpha = -1$ , corresponding to the Hardy space  $H^2$  ([5], Theorem 4.19), but the proof is different, due to the fact that one uses the normalized Lebesgue measure on  $\mathbb{T}$  and the boundary values of  $\varphi$  instead of measures on  $\mathbb{D}$  and the function  $\varphi$  itself.

We end the paper by an application to the compactness of composition operators on weighted Bergman-Orlicz spaces.

Another application of Theorem 1.1 is given in [7].

**Notation.** In this paper,  $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$  denotes the open unit disk of the complex plane  $\mathbb{C}$ , and  $\mathbb{T} = \partial\mathbb{D}$  is the unit circle. The normalized area measure  $\frac{dx dy}{\pi}$  is denoted by  $\mathcal{A}$ .

For  $\alpha > -1$ , the *weighted Bergman space*  $\mathfrak{B}_\alpha^2$  is the space of all analytic functions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  on  $\mathbb{D}$  such that

$$\|f\|_\alpha^2 := \int_{\mathbb{D}} |f(z)|^2 d\mathcal{A}_\alpha(z) < +\infty,$$

where  $\mathcal{A}_\alpha$  is the weighted measure

$$(1.3) \quad d\mathcal{A}_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha d\mathcal{A}(z).$$

The *Carleson window* centered at  $\xi \in \mathbb{T}$  and of size  $h$ ,  $0 < h < 1$ , is the set

$$W(\xi, h) = \{z \in \mathbb{D}; |z| \geq 1 - h \text{ and } |\arg(z\bar{\xi})| \leq h\}.$$

A measure  $\mu$  on  $\mathbb{D}$  is called an  $\alpha$ -Carleson measure ( $\alpha \geq -1$ ) if

$$\sup_{|\xi|=1} \mu[W(\xi, h)] = O_{h \rightarrow 0}(h^\alpha).$$

Actually, instead of the Carleson window  $W(\xi, h)$ , we shall merely use the sets

$$S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\},$$

which have essentially the same size, so  $\mu$  is an  $\alpha$ -Carleson measure if and only if  $\sup_{|\xi|=1} \mu[S(\xi, h)] = O_{h \rightarrow 0}(h^\alpha)$ .

We denote by  $\Pi^+$  the right-half plane

$$(1.4) \quad \Pi^+ = \{z \in \mathbb{C}; \Re z > 0\}.$$

To avoid any misunderstanding, we denote by  $A$  the area measure on  $\Pi^+$ , and *not* this measure divided by  $\pi$ .

Let  $T: \mathbb{D} \rightarrow \Pi^+$  be the conformal map defined by:

$$(1.5) \quad T(z) = \frac{1-z}{1+z};$$

we denote by  $\tau_\alpha = T(\mathcal{A}_\alpha)$  the pull-back measure defined by:

$$(1.6) \quad \tau_\alpha(B) = \mathcal{A}_\alpha[T^{-1}(B)]$$

for every Borel set  $B$  of  $\Pi^+$ . This is a probability measure on  $\Pi^+$ .

We also need another measure  $\mu_\alpha$  on  $\Pi^+$ , defined by:

$$(1.7) \quad d\mu_\alpha = x^\alpha dx dy.$$

Given two measures  $\mu$  and  $\nu$ , we shall write  $\mu \sim \nu$  when the Radon-Nikodým derivative  $\frac{d\mu}{d\nu}$  is bounded from above and from below.

The *pseudo-hyperbolic distance*  $\rho'$  on  $\mathbb{D}$  is given by

$$(1.8) \quad \rho'(z, w) = \left| \frac{z-w}{1-\bar{z}w} \right|, \quad z, w \in \mathbb{D}.$$

For every  $z \in \mathbb{D}$  and  $r \in (0, 1)$ ,

$$\Delta'(z, r) = \{w \in \mathbb{D}; \rho'(w, z) < r\}$$

is called the *pseudo-hyperbolic disk* with center  $z$  and radius  $r$ . It is (see [1], [3], or [10], for example) the image of the Euclidean disk  $D(0, r)$  by the automorphism

$$\varphi_z(\zeta) = \frac{z-\zeta}{1-\bar{z}\zeta}.$$

The pseudo-hyperbolic distance  $\rho$  on  $\Pi^+$  is deduced by transferring the pseudo-hyperbolic distance  $\rho'$  on  $\mathbb{D}$  with the conformal map  $T$ :

$$(1.9) \quad \rho(a, b) = \rho'(T^{-1}a, T^{-1}b) = \left| \frac{a - b}{\bar{a} + b} \right|,$$

and, for every  $w \in \Pi^+$  and  $r \in (0, 1)$ ,

$$\Delta(w, r) = \{z \in \Pi^+; \rho(z, w) < r\}$$

is the *pseudo-hyperbolic disk* of  $\Pi^+$  with center  $w$  and radius  $r$ .

Finally, we shall use the following notation:

$$(1.10) \quad \Omega = (0, 2) \times (-1, 1).$$

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## 2 Transfer to the right half plane

As in [6], we only have to give the proof for  $\xi = 1$  and, by considering  $g = h/(1 - \varphi)$ , we are boiled down to prove:

**Theorem 2.1** *Let  $\alpha > -1$ . There exist constants  $K_0 > 0$ ,  $c_0 > 0$  and  $\lambda_0 > 1$  such that every analytic function  $g: \mathbb{D} \rightarrow \Pi^+$  with  $|g(0)| \leq c_0$  satisfies, for every  $\lambda \geq \lambda_0$ :*

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \frac{K_0}{\lambda^{\alpha+2}} \mathcal{A}_\alpha(\{|g| > 1\}).$$

As said in the Introduction, this result is an infinitesimal version of the fact that the pull-back measure  $\mathcal{A}_{\alpha, \varphi}$  of  $\mathcal{A}_\alpha$  by any analytic self-map  $\varphi$  of  $\mathbb{D}$  is an  $(\alpha + 2)$ -Carleson measure. In fact, one has the following result.

**Proposition 2.2** *There is some constant  $C = C_\alpha > 0$  such that*

$$(2.1) \quad \mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \frac{C}{\lambda^{\alpha+2}} |g(0)|^{\alpha+2}$$

*for every analytic function  $g: \mathbb{D} \rightarrow \Pi^+$  and every  $\lambda > 0$ .*

The goal is hence to replace in the right-hand side the quantity  $|g(0)|^{\alpha+2}$  by  $\mathcal{A}_\alpha(\{|g| > 1\})$ .

**Proof of Proposition 2.2.** We may assume that  $|g(0)| = 1$ . Hence we may assume that  $\lambda > 2$ , taking  $C \geq 2^{\alpha+2}$ , because  $\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq 1$ .

Set  $\varphi(z) = [g(z) - g(0)]/[g(z) + \overline{g(0)}]$ . Then  $|g(z)| > \lambda$  implies that

$$|\varphi(z) - 1| = 2|\Re g(0)|/|g(z) + \overline{g(0)}| \leq 2/(\lambda - 1) \leq 4/\lambda.$$

But  $\varphi$  maps  $\mathbb{D}$  into itself, so the measure  $\mathcal{A}_{\alpha, \varphi}$  is an  $(\alpha + 2)$ -Carleson measure and (see the proof of [9], Theorem 1.2)

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \mathcal{A}_{\alpha, \varphi}[S(1, 4/\lambda)] \leq C'_\alpha \|C_\varphi\|^2 / (\lambda/4)^{\alpha+2},$$

where  $\|C_\varphi\|$  is the norm of the composition operator  $C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ . But  $\varphi(0) = 0$  and hence  $\|C_\varphi\| = 1$ , by using Littlewood's subordination principle and integrating.  $\square$

For technical reasons, that we shall explain after Lemma 3.4, we need to work with functions defined on  $\Pi^+$ . Proposition 2.2 becomes:

**Proposition 2.3** *There exists a constant  $C = C_\alpha > 0$  such that, for every analytic function  $f: \Pi^+ \rightarrow \Pi^+$ , one has:*

$$(2.2) \quad \tau_\alpha(\{|f| > \lambda\}) \leq \frac{C}{\lambda^{\alpha+2}} |f(1)|^{\alpha+2}.$$

**Proof.** Set  $E_f(\lambda) = \{|f| > \lambda\}$  and define similarly  $E_g(\lambda) = \{|g| > \lambda\}$  where  $g = f \circ T: \mathbb{D} \rightarrow \Pi^+$ . We have  $g(0) = f(1)$  as well as the simple but useful equation:

$$(2.3) \quad T^{-1}[E_f(\lambda)] = E_g(\lambda).$$

So that, by Proposition 2.2:

$$\begin{aligned} \tau_\alpha[E_f(\lambda)] &= \mathcal{A}_\alpha[T^{-1}(E_f(\lambda))] = \mathcal{A}_\alpha[E_g(\lambda)] \\ &\leq \frac{C}{\lambda^{\alpha+2}} |g(0)|^{\alpha+2} = \frac{C}{\lambda^{\alpha+2}} |f(1)|^{\alpha+2}, \end{aligned}$$

and Proposition 2.3 is proved.  $\square$

Now, to prove Theorem 2.1, it suffices to prove that, when one localizes  $f$  on  $\Omega$ , one may replace the quantity  $|f(1)|$  in the right-hand side of (2.2) by  $\tau_\alpha(\{|f| > 1\} \cap \Omega)$ . This is what is claimed in the next result.

**Theorem 2.4** *There exist constants  $K = K_\alpha > 0$ ,  $c_1 > 0$  and  $\lambda_1 > 1$  such that every analytic function  $f: \Pi^+ \rightarrow \Pi^+$  such that  $|f(1)| \leq c_1$  satisfies, for every  $\lambda \geq \lambda_1$ :*

$$\tau_\alpha(\{|f| > \lambda\} \cap \Omega) \leq \frac{K}{\lambda^{\alpha+2}} \tau_\alpha(\{|f| > 1\} \cap \Omega).$$

We shall prove Theorem 2.4 in the next section, but before, let us see why it gives Theorem 2.1 and hence our main result, Theorem 1.1.

**Proof of Theorem 2.1.** Let  $E: \Pi^+ \rightarrow \mathbb{D}$  be the exponential map defined by

$$(2.4) \quad E(z) = e^{-\pi z},$$

which (up to a radius) maps bijectively  $\Omega$  onto the annulus

$$(2.5) \quad U = \{z \in \mathbb{D}; |z| > e^{-2\pi}\}.$$

For every  $g: \mathbb{D} \rightarrow \Pi^+$  with  $|g(0)| \leq (1 - \beta)/(1 + \beta)$  and  $0 < \beta < 1$ , one has, by Schwarz's lemma (see [6], eq. (3.9)):

$$|g(z)| > 1 \quad \implies \quad |z| > \beta.$$

Therefore we only have to work on the annulus  $U$ , taking  $c_0 \leq \tanh \pi$  in Theorem 2.1.

Let  $L = E^{-1}$  be the inverse map of the restriction of  $E$  to  $\Omega$ , and

$$(2.6) \quad \sigma_\alpha = L(\mathcal{A}_\alpha)$$

be the pull-back measure of  $\mathcal{A}_\alpha$  by  $L$ . This measure is carried by  $\Omega$  and we have:

**Lemma 2.5** *On  $\Omega$ , one has:  $\sigma_\alpha \sim \mu_\alpha \sim \tau_\alpha$ .*

Taking this lemma for granted for a while, let us finish the proof of Theorem 2.1 (the measure  $\mu_\alpha$  does not come into play here). Let  $g: \mathbb{D} \rightarrow \Pi^+$  be an analytic function and  $f = g \circ E: \Pi^+ \rightarrow \Pi^+$  (so that  $g = f \circ L$  on  $E(\Omega)$ ). We have  $|f(1)| \leq c_1$  if  $|g(0)| \leq c_0$ , with  $c_0 > 0$  small enough. In fact, the analytic function  $h = T \circ g$  maps  $\mathbb{D}$  into itself and hence, by the Schwarz-Pick inequality,  $h$  is a contraction for the pseudo-hyperbolic distance on  $\mathbb{D}$  (see [1], eq. (3.3), page 18, for example); hence  $\rho'[h(e^{-\pi}), h(0)] \leq \rho'(e^{-\pi}, 0) = e^{-\pi}$ , that is  $\left| \frac{g(e^{-\pi}) - g(0)}{g(e^{-\pi}) + g(0)} \right| \leq e^{-\pi}$ . It follows that  $|g(e^{-\pi})| - |g(0)| \leq e^{-\pi} [|g(e^{-\pi})| + |g(0)|]$ , i.e.  $|g(e^{-\pi})| \leq \frac{1}{\tanh \pi} |g(0)|$ . Therefore  $|f(1)| = |g(e^{-\pi})| \leq c_1$  if  $|g(0)| \leq c_0$  with  $c_0 \leq c_1 \tanh \pi$ .

Set:

$$E_g(\lambda) = \{|g| > \lambda\} \cap U \quad \text{and} \quad E_f(\lambda) = \{|f| > \lambda\} \cap \Omega.$$

Observe that, as in (2.3),

$$L^{-1}[E_f(\lambda)] = E_g(\lambda) \quad \text{and} \quad E^{-1}[E_g(1)] = E_f(1).$$

Hence, in view of Theorem 2.4 and Lemma 2.5:

$$\begin{aligned} \mathcal{A}_\alpha[E_g(\lambda)] &= \mathcal{A}_\alpha(L^{-1}[E_f(\lambda)]) = \sigma_\alpha[E_f(\lambda)] \\ &\leq \frac{K'_\alpha}{\lambda^{\alpha+2}} \sigma_\alpha[E_f(1)] = \frac{K'_\alpha}{\lambda^{\alpha+2}} \sigma_\alpha(E^{-1}[E_g(1)]) \\ &= \frac{K'_\alpha}{\lambda^{\alpha+2}} (E\sigma_\alpha)[E_g(1)] = \frac{K'_\alpha}{\lambda^{\alpha+2}} \mathcal{A}_\alpha[E_g(1)], \end{aligned}$$

which is exactly what we wanted to prove.  $\square$

**Proof of Lemma 2.5.** Let us compute  $\sigma_\alpha$  with the change of variable  $w = E^{-1}(z)$ . One has  $z = E(w)$  and

$$d\mathcal{A}(z) = |E'(w)|^2 \frac{dA(w)}{\pi} = \frac{1}{\pi} e^{-2\pi\Re w} dA(w).$$

We get:

$$\begin{aligned} \int_{\Omega} h(w) d\sigma_\alpha(w) &= \int_U h(Lz) d\mathcal{A}_\alpha(z) = (\alpha + 1) \int_U h(E^{-1}z) (1 - |z|^2)^\alpha d\mathcal{A}(z) \\ &= \frac{\alpha + 1}{\pi} \int_{\Omega} h(w) e^{-2\pi\Re w} (1 - e^{-2\pi\Re w})^\alpha dA(w), \end{aligned}$$

so that

$$(2.7) \quad d\sigma_\alpha(w) = \frac{\alpha + 1}{\pi} e^{-2\pi\Re w} (1 - e^{-2\pi\Re w})^\alpha \mathbb{1}_\Omega(w) dA(w).$$

Thus, on  $\Omega$ , we have  $\sigma_\alpha \sim \mu_\alpha$ . Indeed, the factor  $e^{-2\Re w}$  is bounded from below and from above, and  $(1 - e^{-2\Re w})^\alpha \sim (\Re w)^\alpha$  as  $\Re w$  goes to 0. This proves the first equivalence of Lemma 2.5.

To prove the second equivalence, we use the change of variable formula  $z = Tw$  in

$$\int_{\Omega} h(u) d\tau_\alpha(u) = \int_U h(Tz) d\mathcal{A}_\alpha(z);$$

it gives  $d\tau_\alpha(w) = |T'(w)|^2 (1 - |T(w)|^2)^\alpha (\alpha + 1) dA(w) / \pi$ , i.e.:

$$(2.8) \quad d\tau_\alpha(w) = \frac{4^{\alpha+1}(\alpha + 1)}{\pi} \frac{(\Re w)^\alpha}{|1 + w|^{2(\alpha+2)}} \mathbb{1}_\Omega(w) dA(w),$$

showing that  $\mu_\alpha \sim \tau_\alpha$  on  $\Omega$ . □

### 3 Proof of Theorem 2.4

Let us split, up to a set of measure 0, the square  $\Omega$  into dyadic sub-squares

$$(3.1) \quad Q_l = \left( \frac{2j}{2^n}, \frac{2(j+1)}{2^n} \right) \times \left( \frac{2k}{2^n} - 1, \frac{2(k+1)}{2^n} - 1 \right)$$

of center

$$(3.2) \quad c_l = \frac{2j+1}{2^n} + i \left( \frac{2k+1}{2^n} - 1 \right),$$

with  $n \geq 0$ ,  $0 \leq j, k \leq 2^n - 1$  and where  $l = (n, j, k)$ .

Note that  $\Omega = Q_{(0,0,0)}$ . We are going to use the special form of the measure  $\tau_\alpha$ , taken in (2.8), to get a localized version of Proposition 2.3 as follows.



**Proposition 3.1** *There is a constant  $C_\alpha > 0$  such that, for any analytic function  $f: \Pi^+ \rightarrow \Pi^+$  and any dyadic sub-square  $Q_l$  of  $\Omega$ , one has, for any  $\lambda > 0$ :*

$$(3.3) \quad \tau_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(Q_l) |f(c_l)|^{\alpha+2}.$$

**Proof.** Using Lemma 2.5, we may replace the measure  $\tau_\alpha$  by  $d\mu_\alpha = x^\alpha dx dy$ . This measure is no longer a probability measure, but it has the advantage of being invariant under vertical translations, and, especially, to react to a dilation of positive ratio  $\lambda$  by multiplying the result by the factor  $\lambda^{\alpha+2}$ .

We first need a simple lemma.

**Lemma 3.2** *For every  $0 \leq s < 1$ , there exists a constant  $M_s > 0$  such that, for any analytic function  $f: \Pi^+ \rightarrow \Pi^+$  and any pseudo-hyperbolic disk  $\Delta(c, s)$  in  $\Pi^+$ , we have, for every  $z \in \Delta(c, s)$ :*

$$(3.4) \quad 1/M_s \leq |f(z)|/|f(c)| \leq M_s.$$

**Proof.** By the classical Schwarz-Pick inequality, any analytic map  $f: \Pi^+ \rightarrow \Pi^+$  contracts the pseudo-hyperbolic distance  $\rho$  of  $\Pi^+$  (see [1], Section 6), so that if  $z \in \Delta(c, s)$ , one has:

$$|u| := \left| \frac{f(z) - f(c)}{f(z) + \overline{f(c)}} \right| \leq \left| \frac{z - c}{z + \bar{c}} \right| \leq s.$$

Inverting that relation, we get  $f(z) = \frac{u\overline{f(c)} + f(c)}{1-u}$ , whence

$$|f(z)| \leq |f(c)| \frac{1+|u|}{1-|u|} \leq |f(c)| \frac{1+s}{1-s}$$

and, similarly,  $|f(z)| \geq |f(c)| \frac{1-s}{1+s}$ . The lemma follows, with  $M_s = \frac{1+s}{1-s}$ .  $\square$

Let us now continue the proof of Proposition 3.1.

**Lemma 3.3** *Inequality (3.3) holds when the square  $Q_l$ , of the  $n$ -th generation, does not touch the boundary of  $\Pi^+$ , namely when  $l = (n, j, k)$  with  $j \geq 1$ . More precisely, we have  $Q_l \subseteq \Delta(c_l, s)$  where  $s < 1$  is a numerical constant.*

**Proof.** Recall that  $c_l$  is the center of  $Q_l$ . We claim that we can find some numerical  $s < 1$  such that  $Q_l \subset \Delta(c_l, s)$ . To show that claim, let  $l = (n, j, k)$  and  $z, w \in Q_l$ . We have:

$$1 - \rho(z, w)^2 = 1 - \left| \frac{z - w}{z + \bar{w}} \right|^2 = 4 \frac{\Re z \Re w}{|z + \bar{w}|^2}.$$

But one has  $2j/2^n \leq \Re z, \Re w \leq 2(j+1)/2^n$  whereas  $|\Im(z + \bar{w})| \leq 2^{-n+1}$ ; hence  $\Re z \Re w \geq 4j^2 4^{-n}$  and  $|z + \bar{w}|^2 = (\Re z + \Re w)^2 + [\Im(z + \bar{w})]^2 \leq 16(j+1)^2 4^{-n} + 4 \cdot 4^{-n} \leq 80j^2 4^{-n}$ , because  $j \geq 1$ . Therefore

$$1 - \rho(z, w)^2 \geq 4 \frac{4j^2 4^{-n}}{80j^2 4^{-n}} = \frac{1}{5},$$

so that  $\rho(z, w) \leq s = \sqrt{4/5}$ . In particular, we have  $Q_l \subseteq \Delta(c_l, s)$ .

Now, to prove (3.3), we may assume, by homogeneity (replace  $f$  by  $f/|f(c_l)|$  and  $\lambda$  by  $\lambda/|f(c_l)|$ ), that  $|f(c_l)| = 1$ . We then have, by Lemma 3.2,  $|f(z)| \leq M_s|f(c_l)| = M_s$  for every  $z \in Q_l$ . Hence (3.3) trivially holds when  $\lambda > M_s$ , since then the set in the left-hand side is empty. So we assume  $\lambda \leq M_s$ . In that case, setting  $C_\alpha = M_s^{\alpha+2}$ , we have :

$$\tau_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \tau_\alpha(Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(Q_l).$$

This is the desired inequality, since we have supposed that  $|f(c_l)| = 1$ .  $\square$

**Lemma 3.4** *Inequality (3.3) holds when the square  $Q_l$ , of the  $n$ -th generation, touches the boundary of  $\Pi^+$ , namely when  $l = (n, j, k)$  with  $j = 0$ .*

**Proof.** This case uses the specific properties of the measure  $\mu_\alpha$ . In view of Lemma 2.5, we have to prove that:

$$(3.5) \quad \mu_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}} \mu_\alpha(Q_l) |f(c_l)|^{\alpha+2},$$

when the square  $Q_l \subseteq \Omega$  is supported by the imaginary axis. We may again assume that  $|f(c_l)| = 1$ , and we proceed in three steps.

1) First, (3.5) holds if  $Q_l = Q_{(0,0,0)} = \Omega$ : this is just what we have proved in Proposition 2.3 with (2.2).

2) For  $h > 0$ , (3.5) holds when  $Q_l = h\Omega = (0, 2h) \times (-h, h)$  is a square meeting the imaginary axis in an interval  $(-h, h)$  centered at 0. Indeed, setting  $E_f(\lambda) = \{|f| > \lambda\}$  as well as  $f_h(z) = f(hz)$ , we easily check that

$$(3.6) \quad E_f(\lambda) \cap h\Omega = h[E_{f_h}(\lambda) \cap \Omega].$$

For example, if  $v \in E_{f_h}(\lambda) \cap \Omega$ , one has  $|f(hv)| > \lambda$  and hence  $w = hv \in E_f(\lambda) \cap h\Omega$ , giving one inclusion in (3.6); the other is proved similarly. Using the already mentioned  $(\alpha+2)$ -homogeneity of the measure  $\mu_\alpha$ , we obtain, using (2.2) for  $f_h$ :

$$\begin{aligned} \mu_\alpha[E_f(\lambda) \cap h\Omega] &= \mu_\alpha[h(E_{f_h}(\lambda) \cap \Omega)] = h^{\alpha+2} \mu_\alpha[E_{f_h}(\lambda) \cap \Omega] \\ &\leq h^{\alpha+2} \frac{C_\alpha}{\lambda^{\alpha+2}} |f_h(1)|^{\alpha+2} = \mu_\alpha(Q_l) \frac{C'_\alpha}{\lambda^{\alpha+2}} |f(c_l)|^{\alpha+2}, \end{aligned}$$

with  $C'_\alpha = 4^{-(\alpha+2)}(\alpha+1)C_\alpha$ , since the center  $c_l$  of  $Q_l = h\Omega$  is  $c_l = h$ .

3) Finally, (3.5) holds if  $Q_l$  is any square supported by the imaginary axis. Indeed, this  $Q_l$  is a vertical translate of the second case, and the measure  $\mu_\alpha$  is invariant under vertical translations, which exchange centers.

This ends the proof of the crucial Lemma 3.4 and thereby that of Proposition 3.1.  $\square$

**Remark.** We see here why it is better to work with functions  $f: \Pi^+ \rightarrow \Pi^+$  instead of functions  $g: \mathbb{D} \rightarrow \Pi^+$ ; if the invariance of  $\mu_\alpha$  under vertical translations corresponds to the rotation invariance of  $\mathcal{A}_\alpha$ , the homogeneity of  $\mu_\alpha$ , used in part 2) of the proof, corresponds to an invariance by the automorphisms  $\varphi_a$  of  $\mathbb{D}$ , with real  $a \in \mathbb{D}$ , which is not shared by  $\mathcal{A}_\alpha$ , and writing a measure equivalent to  $\mathcal{A}_\alpha$  having these properties is not so simple.

In order to exploit this proposition, we need the following precisions.

**Lemma 3.5** *There exist constants  $c > 0$  and  $\delta_0 > 0$ , depending only on  $\alpha$ , such that for every  $l$ , there exists  $R_l \subseteq Q_l$  with  $\tau_\alpha(R_l) \geq c \tau_\alpha(Q_l)$  and, for every analytic map  $f: \Pi^+ \rightarrow \Pi^+$ ,*

$$(3.7) \quad |f(z)| > \delta_0 |f(c_l)| \quad \text{for every } z \in R_l.$$

**Proof.** By Lemma 2.5, it suffices to prove this lemma with  $\mu_\alpha$  instead of  $\tau_\alpha$ . Let us consider two cases.

1) If  $l = (n, j, k)$  with  $j \geq 1$ , we can simply take  $R_l = Q_l$ , in view of Lemma 3.2 and Lemma 3.3.

2) If  $l = (n, j, k)$  with  $j = 0$ , we may assume that  $Q_l = \Omega = (0, 2) \times (-1, 1)$ , since either vertical translations or dilations of positive ratio are isometries for the pseudo-hyperbolic distance on  $\Pi^+$  and, on the other hand, multiply the  $\mu_\alpha$ -measure by 1 or  $h^{\alpha+2}$  respectively. It follows that  $c_l = 1$ . We are going to check that  $\Delta(1, 1/4) \subseteq \Omega = Q_l$ , so that we can take  $R_l = \Delta(c_l, 1/4)$ . Indeed, set  $t = 1/4$ ; if  $|u| := \left| \frac{z-1}{z+1} \right| \leq t$ , we have  $z = \frac{1+u}{1-u}$  and

$$\begin{aligned} 0 < \Re z &= \frac{1 - |u|^2}{|1 - u|^2} \leq \frac{1 + t}{1 - t} < 2; \\ |\Im z| &= \frac{2 |\Im u|}{|1 - u|^2} \leq \frac{2t}{(1 - t)^2} = \frac{8}{9} < 1. \end{aligned}$$

Moreover, in view of Lemma 3.2, (3.7) holds with  $\delta_0 = M_t^{-1} = 3/5$ .

Finally, the claim on the measures holds with  $c = \mu_\alpha[\Delta(1, 1/4)]/\mu_\alpha(\Omega)$ .  $\square$

Now, we want to control mean values of  $f$  on some of the  $Q_l$ 's. In order to get that, we have to do a Calderón-Zygmund decomposition.

To that end, we need to know that the mean of  $|f|$  on  $\Omega$  is small, namely less than 1, if  $|f(1)|$  is small enough. This is the aim of the next proposition.

**Proposition 3.6** *There exists a constant  $C > 0$  such that, for every analytic function  $f: \Pi^+ \rightarrow \Pi^+$ , one has:*

$$(3.8) \quad |f(1)| \leq \iint_{\Omega} |f(x + iy)| \frac{dx dy}{\pi} \leq C |f(1)|.$$

Moreover, if  $c$  is the center of an open square  $Q$  contained in  $\Pi^+$ , then:

$$(3.9) \quad \frac{\pi}{4} |f(c)| \leq \frac{1}{A(Q)} \int_Q |f(z)| dA(z) \leq C \frac{\pi}{4} |f(c)|.$$

**Proof.** Let us see first that (3.9) follows from (3.8). Let  $c = a + ib$  ( $a > 0$  and  $b \in \mathbb{R}$ ) be the center of the square  $Q = (a - h, a + h) \times (b - h, b + h)$ , with  $0 < h \leq a$ . Consider the function  $f_1$  defined by:

$$f_1(z) = f[\phi(z)], \quad \text{where} \quad \phi(z) = hz - h + a + ib.$$

Observe that  $\phi: \Pi^+ \rightarrow \Pi^+$  is an affine transformation sending 1 onto  $c$  and that  $\phi(\Omega) = Q$ . Applying (3.8) to  $f_1$  gives:

$$\frac{\pi}{4} |f_1(1)| \leq \frac{1}{A(\Omega)} \int_{\Omega} |f_1(z)| dA(z) \leq C \frac{\pi}{4} |f_1(1)|.$$

This yields (3.9) using an obvious change of variable and  $f_1(1) = f(c)$ .

The left-hand side inequality in (3.8) is due to subharmonicity: consider the open disk  $D$  of center 1 and radius 1; then  $D \subseteq \Omega$  and,  $|f|$  being subharmonic, we have:

$$|f(1)| \leq \frac{1}{\pi} \iint_D |f(x + iy)| dx dy \leq \frac{1}{\pi} \iint_{\Omega} |f(x + iy)| dx dy.$$

We now prove the right-hand side inequality. Using Lemma 2.3 and the fact that  $\mu_0 \sim \tau_0$  on  $\Omega$  (note that  $\mu_0$  is just the area measure  $A$  on  $\Pi^+$ ), we have the existence of a constant  $\kappa > 0$  such that, for all  $\lambda > 0$ :

$$(3.10) \quad \mu_0(\{|f| > \lambda\} \cap \Omega) \leq \frac{\kappa}{\lambda^2} |f(1)|^2.$$

From this estimate (3.10), we can control the integral of  $|f|$  over  $\Omega$  (recall that  $\mu_0(\Omega) = 4$ ):

$$\begin{aligned} \int_{\Omega} |f| d\mu_0 &= \int_0^{+\infty} \mu_0(\{|f| > \lambda\} \cap \Omega) d\lambda \\ &\leq 4 |f(1)| + \int_{|f(1)|}^{+\infty} \frac{\kappa |f(1)|^2}{\lambda^2} d\lambda = (4 + \kappa) |f(1)|. \end{aligned}$$

The proposition follows.  $\square$

**Remark.** We do not know if the constant  $\pi/4$  in the left-hand side of (3.9) can be replaced by a better constant; however, it is not possible to replace this factor  $\pi/4$  by 1. Let us see an example.

Let us define  $f(z) = \exp((Tz)^4)$  where  $Tz = (1 - z)/(1 + z)$ . Recall that  $T$  sends  $\Pi^+$  to the unit disk  $\mathbb{D}$ , and therefore  $f(z) \in \Pi^+$ , for every  $z \in \Pi^+$  because  $|\arg(\exp w)| < 1 < \pi/2$ , for all  $w \in \mathbb{D}$ .

Now let  $Q$  be the unit square  $Q = (-1, 1) \times (-1, 1)$ . For  $0 < t \leq 1/2$ , let  $Q_t$  be the square, centered in 1,  $Q_t = (1 - t, 1 + t) \times (-t, t)$ , which is contained in  $\Pi^+$ , and define

$$\sigma(t) = \frac{1}{A(Q_t)} \int_{Q_t} |f(z)| dA(z) = \frac{1}{4t^2} \int_{-t}^t \left[ \int_{1-t}^{1+t} |f(x + iy)| dx \right] dy.$$

Using a change of variable we have:

$$\sigma(t) = \frac{1}{4} \iint_Q |f(1 + tx + ity)| dx dy.$$

We are going to prove that there exists  $t$  such that  $\sigma(t) < 1 = |f(1)|$  and the average of  $|f|$  in the cube  $Q_t$  is smaller than  $|f(1)|$ . Now observe that

$$f(z) = \frac{1}{16}(z-1)^4 + O((z-1)^5), \quad z \rightarrow 1.$$

Consequently, there exists a constant  $C > 0$  such that, for  $z \in Q_{1/2}$ ,

$$\Re(f(z)) \leq \frac{1}{16} \Re((z-1)^4) + C|z-1|^5$$

and then, there exists  $C_1 > 0$ , such that for every  $x + iy \in Q$  and  $t \in (0, 1/2)$ ,

$$\begin{aligned} |f(1 + tx + ity)| &\leq \exp\left[\frac{1}{16} \Re(t^4(x + iy)^4) + C_1 t^5\right] \\ &= \exp\left[\frac{t^4}{16}(x^4 + y^4 - 6x^2y^2) + C_1 t^5\right]. \end{aligned}$$

Integrating over  $Q$ , putting:

$$\tau(s) = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \exp((s/16)(x^4 + y^4 - 6x^2y^2) + C_1 s^{5/4}) dx dy,$$

we get that  $\sigma(t) \leq \tau(t^4)$ , for  $t \in (0, 1/2]$ . We just need to prove that, for  $s > 0$  close enough to 0, we have  $\tau(s) < 1$ . But this is easy because  $\tau(0) = 1$ , and

$$\tau'(0) = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \frac{1}{16} (x^4 + y^4 - 6x^2y^2) dx dy = \frac{1}{64} \left( \frac{4}{5} + \frac{4}{5} - \frac{8}{3} \right) = -\frac{1}{60} < 0.$$

Return now to the proof of Theorem 2.4.

Consider, for every  $n \geq 0$ , the conditional expectation of the restriction to  $\Omega$  of  $|f|$  with respect to the algebra  $\mathcal{Q}_n$  generated by the squares  $Q_{(n,j,k)}$ ,  $0 \leq j, k \leq 2^n - 1$  (note that  $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$ ):

$$(3.11) \quad (\mathbb{E}_n |f|)(z) = \sum_{j,k=0}^{2^n-1} \left( \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA \right) \mathbb{1}_{Q_{(n,j,k)}}(z),$$

and the maximal function  $Mf$  is defined by:

$$(3.12) \quad Mf(z) = \sup_n (\mathbb{E}_n |f|)(z).$$

One has

$$(3.13) \quad M(f)(z) = \sup_{z \in Q_{(n,j,k)}} \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA.$$

Since  $f$  is continuous on  $\Omega$ , one has  $\lim_{n \rightarrow \infty} \mathbb{E}_n |f|(z) = |f(z)|$  for every  $z \in \Omega$ , and it follows that:

$$(3.14) \quad \{|f| > 1\} \subseteq \{Mf > 1\}.$$

Now, the set  $\{Mf > 1\} \cap \Omega$  can be split into a disjoint union

$$\{Mf > 1\} \cap \Omega = \bigsqcup_{n \geq 1} Z_n,$$

where

$$Z_n = \{z \in \Omega; (\mathbb{E}_n |f|)(z) > 1 \text{ and } (\mathbb{E}_j |f|)(z) \leq 1 \text{ if } j < n\}.$$

(note that, by Proposition 3.6,  $\mathbb{E}_0 |f| \leq 1$  if  $|f(1)|$  is small enough).

Since  $\mathbb{E}_n |f|$  is constant on the sets  $Q \in \mathcal{Q}_n$ , each  $Z_n$  can be in its turn decomposed, up to a set of measure 0, into a disjoint union  $E_n = \bigsqcup_{(j,k) \in J_n} Q_{(n,j,k)}$ .

By definition, for  $z \in Z_n$ , one has  $(\mathbb{E}_n |f|)(z) \geq 1$  and hence, for  $(j, k) \in J_n$ ,

$$\frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA \geq 1 \quad \text{for } z \in Q_{(n,j,k)}.$$

But, on the other hand,  $(\mathbb{E}_{n-1} |f|)(z) \leq 1$  for  $z \in Z_n$ , and we have, if  $z \in Q_{(n,j,k)}$ :

$$\begin{aligned} (\mathbb{E}_n |f|)(z) &= \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA \leq \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n-1,j',k')}} |f| dA \\ &\leq 4 \frac{1}{A(Q_{(n-1,j',k')})} \int_{Q_{(n-1,j',k')}} |f| dA \leq 4, \end{aligned}$$

where  $Q_{(n-1,j',k')}$  is the square of rank  $(n-1)$  containing  $Q_{(n,j,k)}$ .

Finally, we can write  $\{Mf > 1\} \cap \Omega$  as a disjoint union, up to a set of measure 0,

$$(3.15) \quad \{Mf > 1\} \cap \Omega = \bigsqcup_{l \in L} Q_l,$$

where  $L$  is a subset of all the indices  $(n, j, k)$ , for which:

$$(3.16) \quad 1 \leq \frac{1}{A(Q_l)} \int_{Q_l} |f| dA \leq 4.$$

Equations (3.14), (3.15) and (3.16) define the Calderón-Zygmund decomposition of the function  $f$ .

We are now ready to end the proof of Theorem 2.4.

For  $\lambda \geq 1$ , set  $E_\lambda = \{|f| > \lambda\}$ ; one has, by (3.15), Proposition 3.1 and (3.9):

$$\begin{aligned} \tau_\alpha(E_\lambda \cap \Omega) &= \tau_\alpha(E_\lambda \cap \{Mf > 1\} \cap \Omega) = \sum_{l \in L} \tau_\alpha(E_\lambda \cap Q_l) \\ &\leq \frac{K_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) |f(c_l)|^{\alpha+2} \\ &\leq \frac{K_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) \left(\frac{16}{\pi}\right)^{\alpha+2} = \frac{C_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) \\ &= \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{Mf > 1\} \cap \Omega). \end{aligned}$$

But, on the other hand, the sets  $R_l$  of Lemma 3.5 are disjoint, since  $R_l \subseteq Q_l$  and we have  $|f| > \delta_0 |f(c_l)| > (4/\pi C) \delta_0 := \delta_1$  on  $R_l$ , in view of Lemma 3.5 and Proposition 3.6. Therefore:

$$\begin{aligned} \tau_\alpha(|f| > \delta_1) &\geq \tau_\alpha\left(\bigsqcup_l R_l\right) = \sum_{l \in L} \tau_\alpha(R_l) \geq c \sum_{l \in L} \tau_\alpha(Q_l) = c \tau_\alpha\left(\bigsqcup_l Q_l\right) \\ &= c \tau_\alpha(\{Mf > 1\} \cap \Omega). \end{aligned}$$

We get hence  $\tau_\alpha(E_\lambda \cap \Omega) \leq \frac{C'_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{|f| > \delta_1\})$  for  $\lambda \geq 1$ , with  $C'_\alpha = C_\alpha/c$ . Applying this to  $f/\delta_1$  instead of  $f$ , we get:

$$\tau_\alpha(\{|f| > \lambda\} \cap \Omega) \leq \frac{C''_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{|f| > 1\})$$

for  $\lambda > \lambda_1 := 1/\delta_1$ , and that finishes the proof of Theorem 2.4.

## 4 An application to composition operators

In this section, we give an application of our main result to composition operators on weighted Bergman-Orlicz spaces.

Recall that an Orlicz function  $\Psi: [0, \infty) \rightarrow \mathbb{R}_+$  is a non-decreasing convex function such that  $\Psi(0) = 0$  and  $\Psi(x)/x \rightarrow \infty$  as  $x$  goes to  $\infty$ . The *weighted Bergman-Orlicz space*  $\mathfrak{B}_\alpha^\Psi$  is the space of all analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\int_{\mathbb{D}} \Psi(|f|/C) d\mathcal{A}_\alpha < +\infty$$

for some constant  $C > 0$ . The norm of  $f$  in  $\mathfrak{B}_\alpha^\Psi$  is the infimum of the constants  $C$  for which the above integral is  $\leq 1$ . With this norm,  $\mathfrak{B}_\alpha^\Psi$  is a Banach space.

Now, every analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  defines a bounded linear operator  $C_\varphi: \mathfrak{B}_\alpha^\Psi \rightarrow \mathfrak{B}_\alpha^\Psi$  by  $C_\varphi(f) = f \circ \varphi$ , called the *composition operator of symbol*  $\varphi$ . This is a consequence of the classical Littlewood's subordination principle, using the facts that the measure  $\mathcal{A}_\alpha$  is radial and the function  $\Psi(|f|/C)$  is subharmonic for every analytic function  $f: \mathbb{D} \rightarrow \mathbb{C}$ . Such an operator may be seen as

a Carleson embedding  $J_\mu: \mathfrak{B}_\alpha^\Psi \rightarrow L^\Psi(\mu)$  for the pull-back measure  $\mu = \varphi(\mathcal{A}_\alpha)$ . S. Charpentier ([2]), following [6], has characterized the compactness of such embeddings (actually in the more general setting of the unit ball  $\mathbb{B}_N$  of  $\mathbb{C}^N$  instead of the unit disk  $\mathbb{D}$  of  $\mathbb{C}$ ):

**Theorem 4.1 (S. Charpentier)** *For every finite positive measure  $\mu$  on  $\mathbb{D}$  and for every  $\alpha > -1$ , one has:*

1) *If  $\mathfrak{B}_\alpha^\Psi$  is compactly contained in  $L^\Psi(\mu)$ , then*

$$(4.1) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/\rho_\mu(h))} = 0.$$

2) *Conversely, if*

$$(4.2) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/h^{\alpha+2}K_\mu(h))} = 0,$$

*then  $\mathfrak{B}_\alpha^\Psi$  is compactly contained in  $L^\Psi(\mu)$ .*

Here  $\rho_\mu$  is the Carleson function of  $\mu$ , defined as:

$$(4.3) \quad \rho_\mu(h) = \sup_{|\xi|=1} \mu[W(\xi, h)]$$

and

$$(4.4) \quad K_\mu(h) = \sup_{0 < t \leq h} \frac{\rho_\mu(t)}{t^{\alpha+2}}.$$

When  $\mu = \varphi(\mathcal{A}_\alpha)$  is the pull-back measure of  $\mathcal{A}_\alpha$  by an analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , we denote them by  $\rho_{\varphi, \alpha+2}$  and  $K_{\varphi, \alpha+2}$  respectively.

We gave in [6], in the non-weighted case, examples showing that conditions (4.1) and (4.2) are not equivalent for general measures  $\mu$ . However, Theorem 1.1 implies that  $K_{\varphi, \alpha+2}(h) \lesssim \rho_{\varphi, \alpha+2}(h)/h^{\alpha+2}$  and so conditions (4.1) and (4.2) are equivalent in this case. Therefore, we get:

**Theorem 4.2** *For every  $\alpha > -1$ , every Orlicz function  $\Psi$ , and every analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , the composition operator  $C_\varphi: \mathfrak{B}_\alpha^\Psi \rightarrow \mathfrak{B}_\alpha^\Psi$  is compact if and only if:*

$$(4.5) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/\rho_{\varphi, \alpha+2}(h))} = 0.$$

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