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Infinitesimal Carleson property for weighted measures induced by analytic self-maps of the unit disk

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We prove that, for every $\alpha > -1$, the pull-back measure $\varphi(\mathcal{A}_\alpha)$ of the measure $d\mathcal{A}_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha d\mathcal{A}(z)$, where \mathcal{A} is the normalized area measure on the unit disk \mathbb{D} , by every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is not only an $(\alpha + 2)$ -Carleson measure, but that the measure of the Carleson windows of size εh is controlled by $\varepsilon^{\alpha+2}$ times the measure of the corresponding window of size h . This means that the property of being an $(\alpha + 2)$ -Carleson measure is true at all infinitesimal scales. We give an application by characterizing the compactness of composition operators on weighted Bergman-Orlicz spaces.

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1 Introduction and notation

It is well-known that every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ induces a bounded composition operator $f \mapsto C_\varphi(f) = f \circ \varphi$ from the Bergman space \mathfrak{B}^2 into itself. By Hastings's version of the Carleson inclusion theorem ([4]), that means that the pull-back measure \mathcal{A}_φ of the normalized area measure \mathcal{A} by φ is a 2-Carleson measure, that is, for some constant $C > 0$,

$$\mathcal{A}(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, \varepsilon)\}) \leq C \varepsilon^2$$

for every $\varepsilon \in (0, 1)$ and every $\xi \in \mathbb{T}$, where $W(\xi, \varepsilon)$ is the Carleson window centered at ξ and of size ε . It was proved in [6], Theorem 3.1, that one actually

has an infinitesimal version of this property, namely, for some constant $C > 0$:

$$(1.1) \quad \mathcal{A}(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, \varepsilon h)\}) \leq C \mathcal{A}(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, h)\}) \varepsilon^2,$$

for every $\varepsilon \in (0, 1)$ and $h > 0$ small enough.

Now, consider, for $\alpha > -1$, the *weighted Bergman space* \mathfrak{B}_α^2 . By Littlewood's subordination principle, every analytic self-map φ of \mathbb{D} induces a bounded composition operator C_φ from \mathfrak{B}_α^2 into itself (see [8], Proposition 3.4). By Stegenga's version of the Carleson theorem ([9], Theorem 1.2), that means that the pull-back measure of \mathcal{A}_α (see (1.3) below) by φ is an $(\alpha + 2)$ -Carleson measure. Our goal in this paper is to show the analog of (1.1) in the following form.

Theorem 1.1 *For each $\alpha > -1$, there exists a constant $C_\alpha > 0$ such that, for every analytic self-map of the unit disk $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, every $\varepsilon \in (0, 1)$ and every $h > 0$ small enough, one has, for every $\xi \in \mathbb{T}$:*

$$(1.2) \quad \mathcal{A}_\alpha(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, \varepsilon h)\}) \leq C_\alpha \varepsilon^{\alpha+2} \mathcal{A}_\alpha(\{z \in \mathbb{D}; \varphi(z) \in W(\xi, h)\}).$$

It should be stressed that the heart of the proof given in [6] in the case $\alpha = 0$ cannot be directly used for the other $\alpha > -1$, and we have to change it, justifying the current paper. Moreover, the present proof is simpler than that of [6]. We also pointed out that the result holds in the limiting case $\alpha = -1$, corresponding to the Hardy space H^2 ([5], Theorem 4.19), but the proof is different, due to the fact that one uses the normalized Lebesgue measure on \mathbb{T} and the boundary values of φ instead of measures on \mathbb{D} and the function φ itself.

We end the paper by an application to the compactness of composition operators on weighted Bergman-Orlicz spaces.

Another application of Theorem 1.1 is given in [7].

Notation. In this paper, $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ denotes the open unit disk of the complex plane \mathbb{C} , and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. The normalized area measure $\frac{dx dy}{\pi}$ is denoted by \mathcal{A} .

For $\alpha > -1$, the *weighted Bergman space* \mathfrak{B}_α^2 is the space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on \mathbb{D} such that

$$\|f\|_\alpha^2 := \int_{\mathbb{D}} |f(z)|^2 d\mathcal{A}_\alpha(z) < +\infty,$$

where \mathcal{A}_α is the weighted measure

$$(1.3) \quad d\mathcal{A}_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha d\mathcal{A}(z).$$

The *Carleson window* centered at $\xi \in \mathbb{T}$ and of size h , $0 < h < 1$, is the set

$$W(\xi, h) = \{z \in \mathbb{D}; |z| \geq 1 - h \text{ and } |\arg(z\bar{\xi})| \leq h\}.$$

A measure μ on \mathbb{D} is called an α -Carleson measure ($\alpha \geq -1$) if

$$\sup_{|\xi|=1} \mu[W(\xi, h)] = O_{h \rightarrow 0}(h^\alpha).$$

Actually, instead of the Carleson window $W(\xi, h)$, we shall merely use the sets

$$S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\},$$

which have essentially the same size, so μ is an α -Carleson measure if and only if $\sup_{|\xi|=1} \mu[S(\xi, h)] = O_{h \rightarrow 0}(h^\alpha)$.

We denote by Π^+ the right-half plane

$$(1.4) \quad \Pi^+ = \{z \in \mathbb{C}; \Re z > 0\}.$$

To avoid any misunderstanding, we denote by A the area measure on Π^+ , and *not* this measure divided by π .

Let $T: \mathbb{D} \rightarrow \Pi^+$ be the conformal map defined by:

$$(1.5) \quad T(z) = \frac{1-z}{1+z};$$

we denote by $\tau_\alpha = T(\mathcal{A}_\alpha)$ the pull-back measure defined by:

$$(1.6) \quad \tau_\alpha(B) = \mathcal{A}_\alpha[T^{-1}(B)]$$

for every Borel set B of Π^+ . This is a probability measure on Π^+ .

We also need another measure μ_α on Π^+ , defined by:

$$(1.7) \quad d\mu_\alpha = x^\alpha dx dy.$$

Given two measures μ and ν , we shall write $\mu \sim \nu$ when the Radon-Nikodým derivative $\frac{d\mu}{d\nu}$ is bounded from above and from below.

The *pseudo-hyperbolic distance* ρ' on \mathbb{D} is given by

$$(1.8) \quad \rho'(z, w) = \left| \frac{z-w}{1-\bar{z}w} \right|, \quad z, w \in \mathbb{D}.$$

For every $z \in \mathbb{D}$ and $r \in (0, 1)$,

$$\Delta'(z, r) = \{w \in \mathbb{D}; \rho'(w, z) < r\}$$

is called the *pseudo-hyperbolic disk* with center z and radius r . It is (see [1], [3], or [10], for example) the image of the Euclidean disk $D(0, r)$ by the automorphism

$$\varphi_z(\zeta) = \frac{z-\zeta}{1-\bar{z}\zeta}.$$

The pseudo-hyperbolic distance ρ on Π^+ is deduced by transferring the pseudo-hyperbolic distance ρ' on \mathbb{D} with the conformal map T :

$$(1.9) \quad \rho(a, b) = \rho'(T^{-1}a, T^{-1}b) = \left| \frac{a - b}{\bar{a} + b} \right|,$$

and, for every $w \in \Pi^+$ and $r \in (0, 1)$,

$$\Delta(w, r) = \{z \in \Pi^+; \rho(z, w) < r\}$$

is the *pseudo-hyperbolic disk* of Π^+ with center w and radius r .

Finally, we shall use the following notation:

$$(1.10) \quad \Omega = (0, 2) \times (-1, 1).$$

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2 Transfer to the right half plane

As in [6], we only have to give the proof for $\xi = 1$ and, by considering $g = h/(1 - \varphi)$, we are boiled down to prove:

Theorem 2.1 *Let $\alpha > -1$. There exist constants $K_0 > 0$, $c_0 > 0$ and $\lambda_0 > 1$ such that every analytic function $g: \mathbb{D} \rightarrow \Pi^+$ with $|g(0)| \leq c_0$ satisfies, for every $\lambda \geq \lambda_0$:*

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \frac{K_0}{\lambda^{\alpha+2}} \mathcal{A}_\alpha(\{|g| > 1\}).$$

As said in the Introduction, this result is an infinitesimal version of the fact that the pull-back measure $\mathcal{A}_{\alpha, \varphi}$ of \mathcal{A}_α by any analytic self-map φ of \mathbb{D} is an $(\alpha + 2)$ -Carleson measure. In fact, one has the following result.

Proposition 2.2 *There is some constant $C = C_\alpha > 0$ such that*

$$(2.1) \quad \mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \frac{C}{\lambda^{\alpha+2}} |g(0)|^{\alpha+2}$$

for every analytic function $g: \mathbb{D} \rightarrow \Pi^+$ and every $\lambda > 0$.

The goal is hence to replace in the right-hand side the quantity $|g(0)|^{\alpha+2}$ by $\mathcal{A}_\alpha(\{|g| > 1\})$.

Proof of Proposition 2.2. We may assume that $|g(0)| = 1$. Hence we may assume that $\lambda > 2$, taking $C \geq 2^{\alpha+2}$, because $\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq 1$.

Set $\varphi(z) = [g(z) - g(0)]/[g(z) + \overline{g(0)}]$. Then $|g(z)| > \lambda$ implies that

$$|\varphi(z) - 1| = 2|\Re g(0)|/|g(z) + \overline{g(0)}| \leq 2/(\lambda - 1) \leq 4/\lambda.$$

But φ maps \mathbb{D} into itself, so the measure $\mathcal{A}_{\alpha, \varphi}$ is an $(\alpha + 2)$ -Carleson measure and (see the proof of [9], Theorem 1.2)

$$\mathcal{A}_\alpha(\{|g| > \lambda\}) \leq \mathcal{A}_{\alpha, \varphi}[S(1, 4/\lambda)] \leq C'_\alpha \|C_\varphi\|^2 / (\lambda/4)^{\alpha+2},$$

where $\|C_\varphi\|$ is the norm of the composition operator $C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$. But $\varphi(0) = 0$ and hence $\|C_\varphi\| = 1$, by using Littlewood's subordination principle and integrating. \square

For technical reasons, that we shall explain after Lemma 3.4, we need to work with functions defined on Π^+ . Proposition 2.2 becomes:

Proposition 2.3 *There exists a constant $C = C_\alpha > 0$ such that, for every analytic function $f: \Pi^+ \rightarrow \Pi^+$, one has:*

$$(2.2) \quad \tau_\alpha(\{|f| > \lambda\}) \leq \frac{C}{\lambda^{\alpha+2}} |f(1)|^{\alpha+2}.$$

Proof. Set $E_f(\lambda) = \{|f| > \lambda\}$ and define similarly $E_g(\lambda) = \{|g| > \lambda\}$ where $g = f \circ T: \mathbb{D} \rightarrow \Pi^+$. We have $g(0) = f(1)$ as well as the simple but useful equation:

$$(2.3) \quad T^{-1}[E_f(\lambda)] = E_g(\lambda).$$

So that, by Proposition 2.2:

$$\begin{aligned} \tau_\alpha[E_f(\lambda)] &= \mathcal{A}_\alpha[T^{-1}(E_f(\lambda))] = \mathcal{A}_\alpha[E_g(\lambda)] \\ &\leq \frac{C}{\lambda^{\alpha+2}} |g(0)|^{\alpha+2} = \frac{C}{\lambda^{\alpha+2}} |f(1)|^{\alpha+2}, \end{aligned}$$

and Proposition 2.3 is proved. \square

Now, to prove Theorem 2.1, it suffices to prove that, when one localizes f on Ω , one may replace the quantity $|f(1)|$ in the right-hand side of (2.2) by $\tau_\alpha(\{|f| > 1\} \cap \Omega)$. This is what is claimed in the next result.

Theorem 2.4 *There exist constants $K = K_\alpha > 0$, $c_1 > 0$ and $\lambda_1 > 1$ such that every analytic function $f: \Pi^+ \rightarrow \Pi^+$ such that $|f(1)| \leq c_1$ satisfies, for every $\lambda \geq \lambda_1$:*

$$\tau_\alpha(\{|f| > \lambda\} \cap \Omega) \leq \frac{K}{\lambda^{\alpha+2}} \tau_\alpha(\{|f| > 1\} \cap \Omega).$$

We shall prove Theorem 2.4 in the next section, but before, let us see why it gives Theorem 2.1 and hence our main result, Theorem 1.1.

Proof of Theorem 2.1. Let $E: \Pi^+ \rightarrow \mathbb{D}$ be the exponential map defined by

$$(2.4) \quad E(z) = e^{-\pi z},$$

which (up to a radius) maps bijectively Ω onto the annulus

$$(2.5) \quad U = \{z \in \mathbb{D}; |z| > e^{-2\pi}\}.$$

For every $g: \mathbb{D} \rightarrow \Pi^+$ with $|g(0)| \leq (1 - \beta)/(1 + \beta)$ and $0 < \beta < 1$, one has, by Schwarz's lemma (see [6], eq. (3.9)):

$$|g(z)| > 1 \quad \implies \quad |z| > \beta.$$

Therefore we only have to work on the annulus U , taking $c_0 \leq \tanh \pi$ in Theorem 2.1.

Let $L = E^{-1}$ be the inverse map of the restriction of E to Ω , and

$$(2.6) \quad \sigma_\alpha = L(\mathcal{A}_\alpha)$$

be the pull-back measure of \mathcal{A}_α by L . This measure is carried by Ω and we have:

Lemma 2.5 *On Ω , one has: $\sigma_\alpha \sim \mu_\alpha \sim \tau_\alpha$.*

Taking this lemma for granted for a while, let us finish the proof of Theorem 2.1 (the measure μ_α does not come into play here). Let $g: \mathbb{D} \rightarrow \Pi^+$ be an analytic function and $f = g \circ E: \Pi^+ \rightarrow \Pi^+$ (so that $g = f \circ L$ on $E(\Omega)$). We have $|f(1)| \leq c_1$ if $|g(0)| \leq c_0$, with $c_0 > 0$ small enough. In fact, the analytic function $h = T \circ g$ maps \mathbb{D} into itself and hence, by the Schwarz-Pick inequality, h is a contraction for the pseudo-hyperbolic distance on \mathbb{D} (see [1], eq. (3.3), page 18, for example); hence $\rho'[h(e^{-\pi}), h(0)] \leq \rho'(e^{-\pi}, 0) = e^{-\pi}$, that is $\left| \frac{g(e^{-\pi}) - g(0)}{g(e^{-\pi}) + g(0)} \right| \leq e^{-\pi}$. It follows that $|g(e^{-\pi})| - |g(0)| \leq e^{-\pi} [|g(e^{-\pi})| + |g(0)|]$, i.e. $|g(e^{-\pi})| \leq \frac{1}{\tanh \pi} |g(0)|$. Therefore $|f(1)| = |g(e^{-\pi})| \leq c_1$ if $|g(0)| \leq c_0$ with $c_0 \leq c_1 \tanh \pi$.

Set:

$$E_g(\lambda) = \{|g| > \lambda\} \cap U \quad \text{and} \quad E_f(\lambda) = \{|f| > \lambda\} \cap \Omega.$$

Observe that, as in (2.3),

$$L^{-1}[E_f(\lambda)] = E_g(\lambda) \quad \text{and} \quad E^{-1}[E_g(1)] = E_f(1).$$

Hence, in view of Theorem 2.4 and Lemma 2.5:

$$\begin{aligned} \mathcal{A}_\alpha[E_g(\lambda)] &= \mathcal{A}_\alpha(L^{-1}[E_f(\lambda)]) = \sigma_\alpha[E_f(\lambda)] \\ &\leq \frac{K'_\alpha}{\lambda^{\alpha+2}} \sigma_\alpha[E_f(1)] = \frac{K'_\alpha}{\lambda^{\alpha+2}} \sigma_\alpha(E^{-1}[E_g(1)]) \\ &= \frac{K'_\alpha}{\lambda^{\alpha+2}} (E\sigma_\alpha)[E_g(1)] = \frac{K'_\alpha}{\lambda^{\alpha+2}} \mathcal{A}_\alpha[E_g(1)], \end{aligned}$$

which is exactly what we wanted to prove. \square

Proof of Lemma 2.5. Let us compute σ_α with the change of variable $w = E^{-1}(z)$. One has $z = E(w)$ and

$$d\mathcal{A}(z) = |E'(w)|^2 \frac{dA(w)}{\pi} = \frac{1}{\pi} e^{-2\pi\Re w} dA(w).$$

We get:

$$\begin{aligned} \int_{\Omega} h(w) d\sigma_\alpha(w) &= \int_U h(Lz) d\mathcal{A}_\alpha(z) = (\alpha + 1) \int_U h(E^{-1}z) (1 - |z|^2)^\alpha d\mathcal{A}(z) \\ &= \frac{\alpha + 1}{\pi} \int_{\Omega} h(w) e^{-2\pi\Re w} (1 - e^{-2\pi\Re w})^\alpha dA(w), \end{aligned}$$

so that

$$(2.7) \quad d\sigma_\alpha(w) = \frac{\alpha + 1}{\pi} e^{-2\pi\Re w} (1 - e^{-2\pi\Re w})^\alpha \mathbb{1}_\Omega(w) dA(w).$$

Thus, on Ω , we have $\sigma_\alpha \sim \mu_\alpha$. Indeed, the factor $e^{-2\Re w}$ is bounded from below and from above, and $(1 - e^{-2\Re w})^\alpha \sim (\Re w)^\alpha$ as $\Re w$ goes to 0. This proves the first equivalence of Lemma 2.5.

To prove the second equivalence, we use the change of variable formula $z = Tw$ in

$$\int_{\Omega} h(u) d\tau_\alpha(u) = \int_U h(Tz) d\mathcal{A}_\alpha(z);$$

it gives $d\tau_\alpha(w) = |T'(w)|^2 (1 - |T(w)|^2)^\alpha (\alpha + 1) dA(w) / \pi$, i.e.:

$$(2.8) \quad d\tau_\alpha(w) = \frac{4^{\alpha+1}(\alpha + 1)}{\pi} \frac{(\Re w)^\alpha}{|1 + w|^{2(\alpha+2)}} \mathbb{1}_\Omega(w) dA(w),$$

showing that $\mu_\alpha \sim \tau_\alpha$ on Ω . □

3 Proof of Theorem 2.4

Let us split, up to a set of measure 0, the square Ω into dyadic sub-squares

$$(3.1) \quad Q_l = \left(\frac{2j}{2^n}, \frac{2(j+1)}{2^n} \right) \times \left(\frac{2k}{2^n} - 1, \frac{2(k+1)}{2^n} - 1 \right)$$

of center

$$(3.2) \quad c_l = \frac{2j+1}{2^n} + i \left(\frac{2k+1}{2^n} - 1 \right),$$

with $n \geq 0$, $0 \leq j, k \leq 2^n - 1$ and where $l = (n, j, k)$.

Note that $\Omega = Q_{(0,0,0)}$. We are going to use the special form of the measure τ_α , taken in (2.8), to get a localized version of Proposition 2.3 as follows.

Proposition 3.1 *There is a constant $C_\alpha > 0$ such that, for any analytic function $f: \Pi^+ \rightarrow \Pi^+$ and any dyadic sub-square Q_l of Ω , one has, for any $\lambda > 0$:*

$$(3.3) \quad \tau_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(Q_l) |f(c_l)|^{\alpha+2}.$$

Proof. Using Lemma 2.5, we may replace the measure τ_α by $d\mu_\alpha = x^\alpha dx dy$. This measure is no longer a probability measure, but it has the advantage of being invariant under vertical translations, and, especially, to react to a dilation of positive ratio λ by multiplying the result by the factor $\lambda^{\alpha+2}$.

We first need a simple lemma.

Lemma 3.2 *For every $0 \leq s < 1$, there exists a constant $M_s > 0$ such that, for any analytic function $f: \Pi^+ \rightarrow \Pi^+$ and any pseudo-hyperbolic disk $\Delta(c, s)$ in Π^+ , we have, for every $z \in \Delta(c, s)$:*

$$(3.4) \quad 1/M_s \leq |f(z)|/|f(c)| \leq M_s.$$

Proof. By the classical Schwarz-Pick inequality, any analytic map $f: \Pi^+ \rightarrow \Pi^+$ contracts the pseudo-hyperbolic distance ρ of Π^+ (see [1], Section 6), so that if $z \in \Delta(c, s)$, one has:

$$|u| := \left| \frac{f(z) - f(c)}{f(z) + \overline{f(c)}} \right| \leq \left| \frac{z - c}{z + \bar{c}} \right| \leq s.$$

Inverting that relation, we get $f(z) = \frac{u\overline{f(c)} + f(c)}{1-u}$, whence

$$|f(z)| \leq |f(c)| \frac{1+|u|}{1-|u|} \leq |f(c)| \frac{1+s}{1-s}$$

and, similarly, $|f(z)| \geq |f(c)| \frac{1-s}{1+s}$. The lemma follows, with $M_s = \frac{1+s}{1-s}$. \square

Let us now continue the proof of Proposition 3.1.

Lemma 3.3 *Inequality (3.3) holds when the square Q_l , of the n -th generation, does not touch the boundary of Π^+ , namely when $l = (n, j, k)$ with $j \geq 1$. More precisely, we have $Q_l \subseteq \Delta(c_l, s)$ where $s < 1$ is a numerical constant.*

Proof. Recall that c_l is the center of Q_l . We claim that we can find some numerical $s < 1$ such that $Q_l \subset \Delta(c_l, s)$. To show that claim, let $l = (n, j, k)$ and $z, w \in Q_l$. We have:

$$1 - \rho(z, w)^2 = 1 - \left| \frac{z - w}{z + \bar{w}} \right|^2 = 4 \frac{\Re z \Re w}{|z + \bar{w}|^2}.$$

But one has $2j/2^n \leq \Re z, \Re w \leq 2(j+1)/2^n$ whereas $|\Im(z + \bar{w})| \leq 2^{-n+1}$; hence $\Re z \Re w \geq 4j^2 4^{-n}$ and $|z + \bar{w}|^2 = (\Re z + \Re w)^2 + [\Im(z + \bar{w})]^2 \leq 16(j+1)^2 4^{-n} + 4 \cdot 4^{-n} \leq 80j^2 4^{-n}$, because $j \geq 1$. Therefore

$$1 - \rho(z, w)^2 \geq 4 \frac{4j^2 4^{-n}}{80j^2 4^{-n}} = \frac{1}{5},$$

so that $\rho(z, w) \leq s = \sqrt{4/5}$. In particular, we have $Q_l \subseteq \Delta(c_l, s)$.

Now, to prove (3.3), we may assume, by homogeneity (replace f by $f/|f(c_l)|$ and λ by $\lambda/|f(c_l)|$), that $|f(c_l)| = 1$. We then have, by Lemma 3.2, $|f(z)| \leq M_s|f(c_l)| = M_s$ for every $z \in Q_l$. Hence (3.3) trivially holds when $\lambda > M_s$, since then the set in the left-hand side is empty. So we assume $\lambda \leq M_s$. In that case, setting $C_\alpha = M_s^{\alpha+2}$, we have :

$$\tau_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \tau_\alpha(Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(Q_l).$$

This is the desired inequality, since we have supposed that $|f(c_l)| = 1$. \square

Lemma 3.4 *Inequality (3.3) holds when the square Q_l , of the n -th generation, touches the boundary of Π^+ , namely when $l = (n, j, k)$ with $j = 0$.*

Proof. This case uses the specific properties of the measure μ_α . In view of Lemma 2.5, we have to prove that:

$$(3.5) \quad \mu_\alpha(\{|f| > \lambda\} \cap Q_l) \leq \frac{C_\alpha}{\lambda^{\alpha+2}} \mu_\alpha(Q_l) |f(c_l)|^{\alpha+2},$$

when the square $Q_l \subseteq \Omega$ is supported by the imaginary axis. We may again assume that $|f(c_l)| = 1$, and we proceed in three steps.

1) First, (3.5) holds if $Q_l = Q_{(0,0,0)} = \Omega$: this is just what we have proved in Proposition 2.3 with (2.2).

2) For $h > 0$, (3.5) holds when $Q_l = h\Omega = (0, 2h) \times (-h, h)$ is a square meeting the imaginary axis in an interval $(-h, h)$ centered at 0. Indeed, setting $E_f(\lambda) = \{|f| > \lambda\}$ as well as $f_h(z) = f(hz)$, we easily check that

$$(3.6) \quad E_f(\lambda) \cap h\Omega = h[E_{f_h}(\lambda) \cap \Omega].$$

For example, if $v \in E_{f_h}(\lambda) \cap \Omega$, one has $|f(hv)| > \lambda$ and hence $w = hv \in E_f(\lambda) \cap h\Omega$, giving one inclusion in (3.6); the other is proved similarly. Using the already mentioned $(\alpha+2)$ -homogeneity of the measure μ_α , we obtain, using (2.2) for f_h :

$$\begin{aligned} \mu_\alpha[E_f(\lambda) \cap h\Omega] &= \mu_\alpha[h(E_{f_h}(\lambda) \cap \Omega)] = h^{\alpha+2} \mu_\alpha[E_{f_h}(\lambda) \cap \Omega] \\ &\leq h^{\alpha+2} \frac{C_\alpha}{\lambda^{\alpha+2}} |f_h(1)|^{\alpha+2} = \mu_\alpha(Q_l) \frac{C'_\alpha}{\lambda^{\alpha+2}} |f(c_l)|^{\alpha+2}, \end{aligned}$$

with $C'_\alpha = 4^{-(\alpha+2)}(\alpha+1)C_\alpha$, since the center c_l of $Q_l = h\Omega$ is $c_l = h$.

3) Finally, (3.5) holds if Q_l is any square supported by the imaginary axis. Indeed, this Q_l is a vertical translate of the second case, and the measure μ_α is invariant under vertical translations, which exchange centers.

This ends the proof of the crucial Lemma 3.4 and thereby that of Proposition 3.1. \square

Remark. We see here why it is better to work with functions $f: \Pi^+ \rightarrow \Pi^+$ instead of functions $g: \mathbb{D} \rightarrow \Pi^+$; if the invariance of μ_α under vertical translations corresponds to the rotation invariance of \mathcal{A}_α , the homogeneity of μ_α , used in part 2) of the proof, corresponds to an invariance by the automorphisms φ_a of \mathbb{D} , with real $a \in \mathbb{D}$, which is not shared by \mathcal{A}_α , and writing a measure equivalent to \mathcal{A}_α having these properties is not so simple.

In order to exploit this proposition, we need the following precisions.

Lemma 3.5 *There exist constants $c > 0$ and $\delta_0 > 0$, depending only on α , such that for every l , there exists $R_l \subseteq Q_l$ with $\tau_\alpha(R_l) \geq c \tau_\alpha(Q_l)$ and, for every analytic map $f: \Pi^+ \rightarrow \Pi^+$,*

$$(3.7) \quad |f(z)| > \delta_0 |f(c_l)| \quad \text{for every } z \in R_l.$$

Proof. By Lemma 2.5, it suffices to prove this lemma with μ_α instead of τ_α . Let us consider two cases.

1) If $l = (n, j, k)$ with $j \geq 1$, we can simply take $R_l = Q_l$, in view of Lemma 3.2 and Lemma 3.3.

2) If $l = (n, j, k)$ with $j = 0$, we may assume that $Q_l = \Omega = (0, 2) \times (-1, 1)$, since either vertical translations or dilations of positive ratio are isometries for the pseudo-hyperbolic distance on Π^+ and, on the other hand, multiply the μ_α -measure by 1 or $h^{\alpha+2}$ respectively. It follows that $c_l = 1$. We are going to check that $\Delta(1, 1/4) \subseteq \Omega = Q_l$, so that we can take $R_l = \Delta(c_l, 1/4)$. Indeed, set $t = 1/4$; if $|u| := \left| \frac{z-1}{z+1} \right| \leq t$, we have $z = \frac{1+u}{1-u}$ and

$$0 < \Re z = \frac{1 - |u|^2}{|1 - u|^2} \leq \frac{1 + t}{1 - t} < 2;$$

$$|\Im z| = \frac{2 |\Im u|}{|1 - u|^2} \leq \frac{2t}{(1 - t)^2} = \frac{8}{9} < 1.$$

Moreover, in view of Lemma 3.2, (3.7) holds with $\delta_0 = M_t^{-1} = 3/5$.

Finally, the claim on the measures holds with $c = \mu_\alpha[\Delta(1, 1/4)]/\mu_\alpha(\Omega)$. \square

Now, we want to control mean values of f on some of the Q_l 's. In order to get that, we have to do a Calderón-Zygmund decomposition.

To that end, we need to know that the mean of $|f|$ on Ω is small, namely less than 1, if $|f(1)|$ is small enough. This is the aim of the next proposition.

Proposition 3.6 *There exists a constant $C > 0$ such that, for every analytic function $f: \Pi^+ \rightarrow \Pi^+$, one has:*

$$(3.8) \quad |f(1)| \leq \iint_{\Omega} |f(x + iy)| \frac{dx dy}{\pi} \leq C |f(1)|.$$

Moreover, if c is the center of an open square Q contained in Π^+ , then:

$$(3.9) \quad \frac{\pi}{4} |f(c)| \leq \frac{1}{A(Q)} \int_Q |f(z)| dA(z) \leq C \frac{\pi}{4} |f(c)|.$$

Proof. Let us see first that (3.9) follows from (3.8). Let $c = a + ib$ ($a > 0$ and $b \in \mathbb{R}$) be the center of the square $Q = (a - h, a + h) \times (b - h, b + h)$, with $0 < h \leq a$. Consider the function f_1 defined by:

$$f_1(z) = f[\phi(z)], \quad \text{where} \quad \phi(z) = hz - h + a + ib.$$

Observe that $\phi: \Pi^+ \rightarrow \Pi^+$ is an affine transformation sending 1 onto c and that $\phi(\Omega) = Q$. Applying (3.8) to f_1 gives:

$$\frac{\pi}{4} |f_1(1)| \leq \frac{1}{A(\Omega)} \int_{\Omega} |f_1(z)| dA(z) \leq C \frac{\pi}{4} |f_1(1)|.$$

This yields (3.9) using an obvious change of variable and $f_1(1) = f(c)$.

The left-hand side inequality in (3.8) is due to subharmonicity: consider the open disk D of center 1 and radius 1; then $D \subseteq \Omega$ and, $|f|$ being subharmonic, we have:

$$|f(1)| \leq \frac{1}{\pi} \iint_D |f(x + iy)| dx dy \leq \frac{1}{\pi} \iint_{\Omega} |f(x + iy)| dx dy.$$

We now prove the right-hand side inequality. Using Lemma 2.3 and the fact that $\mu_0 \sim \tau_0$ on Ω (note that μ_0 is just the area measure A on Π^+), we have the existence of a constant $\kappa > 0$ such that, for all $\lambda > 0$:

$$(3.10) \quad \mu_0(\{|f| > \lambda\} \cap \Omega) \leq \frac{\kappa}{\lambda^2} |f(1)|^2.$$

From this estimate (3.10), we can control the integral of $|f|$ over Ω (recall that $\mu_0(\Omega) = 4$):

$$\begin{aligned} \int_{\Omega} |f| d\mu_0 &= \int_0^{+\infty} \mu_0(\{|f| > \lambda\} \cap \Omega) d\lambda \\ &\leq 4 |f(1)| + \int_{|f(1)|}^{+\infty} \frac{\kappa |f(1)|^2}{\lambda^2} d\lambda = (4 + \kappa) |f(1)|. \end{aligned}$$

The proposition follows. \square

Remark. We do not know if the constant $\pi/4$ in the left-hand side of (3.9) can be replaced by a better constant; however, it is not possible to replace this factor $\pi/4$ by 1. Let us see an example.

Let us define $f(z) = \exp((Tz)^4)$ where $Tz = (1 - z)/(1 + z)$. Recall that T sends Π^+ to the unit disk \mathbb{D} , and therefore $f(z) \in \Pi^+$, for every $z \in \Pi^+$ because $|\arg(\exp w)| < 1 < \pi/2$, for all $w \in \mathbb{D}$.

Now let Q be the unit square $Q = (-1, 1) \times (-1, 1)$. For $0 < t \leq 1/2$, let Q_t be the square, centered in 1, $Q_t = (1 - t, 1 + t) \times (-t, t)$, which is contained in Π^+ , and define

$$\sigma(t) = \frac{1}{A(Q_t)} \int_{Q_t} |f(z)| dA(z) = \frac{1}{4t^2} \int_{-t}^t \left[\int_{1-t}^{1+t} |f(x + iy)| dx \right] dy.$$

Using a change of variable we have:

$$\sigma(t) = \frac{1}{4} \iint_Q |f(1 + tx + ity)| dx dy.$$

We are going to prove that there exists t such that $\sigma(t) < 1 = |f(1)|$ and the average of $|f|$ in the cube Q_t is smaller than $|f(1)|$. Now observe that

$$f(z) = \frac{1}{16}(z-1)^4 + O((z-1)^5), \quad z \rightarrow 1.$$

Consequently, there exists a constant $C > 0$ such that, for $z \in Q_{1/2}$,

$$\Re(f(z)) \leq \frac{1}{16} \Re((z-1)^4) + C|z-1|^5$$

and then, there exists $C_1 > 0$, such that for every $x + iy \in Q$ and $t \in (0, 1/2)$,

$$\begin{aligned} |f(1 + tx + ity)| &\leq \exp\left[\frac{1}{16} \Re(t^4(x + iy)^4) + C_1 t^5\right] \\ &= \exp\left[\frac{t^4}{16}(x^4 + y^4 - 6x^2y^2) + C_1 t^5\right]. \end{aligned}$$

Integrating over Q , putting:

$$\tau(s) = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \exp((s/16)(x^4 + y^4 - 6x^2y^2) + C_1 s^{5/4}) dx dy,$$

we get that $\sigma(t) \leq \tau(t^4)$, for $t \in (0, 1/2]$. We just need to prove that, for $s > 0$ close enough to 0, we have $\tau(s) < 1$. But this is easy because $\tau(0) = 1$, and

$$\tau'(0) = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \frac{1}{16} (x^4 + y^4 - 6x^2y^2) dx dy = \frac{1}{64} \left(\frac{4}{5} + \frac{4}{5} - \frac{8}{3} \right) = -\frac{1}{60} < 0.$$

Return now to the proof of Theorem 2.4.

Consider, for every $n \geq 0$, the conditional expectation of the restriction to Ω of $|f|$ with respect to the algebra \mathcal{Q}_n generated by the squares $Q_{(n,j,k)}$, $0 \leq j, k \leq 2^n - 1$ (note that $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$):

$$(3.11) \quad (\mathbb{E}_n |f|)(z) = \sum_{j,k=0}^{2^n-1} \left(\frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA \right) \mathbb{1}_{Q_{(n,j,k)}}(z),$$

and the maximal function Mf is defined by:

$$(3.12) \quad Mf(z) = \sup_n (\mathbb{E}_n |f|)(z).$$

One has

$$(3.13) \quad M(f)(z) = \sup_{z \in Q_{(n,j,k)}} \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA.$$

Since f is continuous on Ω , one has $\lim_{n \rightarrow \infty} \mathbb{E}_n |f|(z) = |f(z)|$ for every $z \in \Omega$, and it follows that:

$$(3.14) \quad \{|f| > 1\} \subseteq \{Mf > 1\}.$$

Now, the set $\{Mf > 1\} \cap \Omega$ can be split into a disjoint union

$$\{Mf > 1\} \cap \Omega = \bigsqcup_{n \geq 1} Z_n,$$

where

$$Z_n = \{z \in \Omega; (\mathbb{E}_n |f|)(z) > 1 \text{ and } (\mathbb{E}_j |f|)(z) \leq 1 \text{ if } j < n\}.$$

(note that, by Proposition 3.6, $\mathbb{E}_0 |f| \leq 1$ if $|f(1)|$ is small enough).

Since $\mathbb{E}_n |f|$ is constant on the sets $Q \in \mathcal{Q}_n$, each Z_n can be in its turn decomposed, up to a set of measure 0, into a disjoint union $E_n = \bigsqcup_{(j,k) \in J_n} Q_{(n,j,k)}$.

By definition, for $z \in Z_n$, one has $(\mathbb{E}_n |f|)(z) \geq 1$ and hence, for $(j, k) \in J_n$,

$$\frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA \geq 1 \quad \text{for } z \in Q_{(n,j,k)}.$$

But, on the other hand, $(\mathbb{E}_{n-1} |f|)(z) \leq 1$ for $z \in Z_n$, and we have, if $z \in Q_{(n,j,k)}$:

$$\begin{aligned} (\mathbb{E}_n |f|)(z) &= \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n,j,k)}} |f| dA \leq \frac{1}{A(Q_{(n,j,k)})} \int_{Q_{(n-1,j',k')}} |f| dA \\ &\leq 4 \frac{1}{A(Q_{(n-1,j',k')})} \int_{Q_{(n-1,j',k')}} |f| dA \leq 4, \end{aligned}$$

where $Q_{(n-1,j',k')}$ is the square of rank $(n-1)$ containing $Q_{(n,j,k)}$.

Finally, we can write $\{Mf > 1\} \cap \Omega$ as a disjoint union, up to a set of measure 0,

$$(3.15) \quad \{Mf > 1\} \cap \Omega = \bigsqcup_{l \in L} Q_l,$$

where L is a subset of all the indices (n, j, k) , for which:

$$(3.16) \quad 1 \leq \frac{1}{A(Q_l)} \int_{Q_l} |f| dA \leq 4.$$

Equations (3.14), (3.15) and (3.16) define the Calderón-Zygmund decomposition of the function f .

We are now ready to end the proof of Theorem 2.4.

For $\lambda \geq 1$, set $E_\lambda = \{|f| > \lambda\}$; one has, by (3.15), Proposition 3.1 and (3.9):

$$\begin{aligned} \tau_\alpha(E_\lambda \cap \Omega) &= \tau_\alpha(E_\lambda \cap \{Mf > 1\} \cap \Omega) = \sum_{l \in L} \tau_\alpha(E_\lambda \cap Q_l) \\ &\leq \frac{K_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) |f(c_l)|^{\alpha+2} \\ &\leq \frac{K_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) \left(\frac{16}{\pi}\right)^{\alpha+2} = \frac{C_\alpha}{\lambda^{\alpha+2}} \sum_{l \in L} \tau_\alpha(Q_l) \\ &= \frac{C_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{Mf > 1\} \cap \Omega). \end{aligned}$$

But, on the other hand, the sets R_l of Lemma 3.5 are disjoint, since $R_l \subseteq Q_l$ and we have $|f| > \delta_0 |f(c_l)| > (4/\pi C) \delta_0 := \delta_1$ on R_l , in view of Lemma 3.5 and Proposition 3.6. Therefore:

$$\begin{aligned} \tau_\alpha(|f| > \delta_1) &\geq \tau_\alpha\left(\bigsqcup_l R_l\right) = \sum_{l \in L} \tau_\alpha(R_l) \geq c \sum_{l \in L} \tau_\alpha(Q_l) = c \tau_\alpha\left(\bigsqcup_l Q_l\right) \\ &= c \tau_\alpha(\{Mf > 1\} \cap \Omega). \end{aligned}$$

We get hence $\tau_\alpha(E_\lambda \cap \Omega) \leq \frac{C'_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{|f| > \delta_1\})$ for $\lambda \geq 1$, with $C'_\alpha = C_\alpha/c$. Applying this to f/δ_1 instead of f , we get:

$$\tau_\alpha(\{|f| > \lambda\} \cap \Omega) \leq \frac{C''_\alpha}{\lambda^{\alpha+2}} \tau_\alpha(\{|f| > 1\})$$

for $\lambda > \lambda_1 := 1/\delta_1$, and that finishes the proof of Theorem 2.4.

4 An application to composition operators

In this section, we give an application of our main result to composition operators on weighted Bergman-Orlicz spaces.

Recall that an Orlicz function $\Psi: [0, \infty) \rightarrow \mathbb{R}_+$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(x)/x \rightarrow \infty$ as x goes to ∞ . The *weighted Bergman-Orlicz space* \mathfrak{B}_α^Ψ is the space of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\int_{\mathbb{D}} \Psi(|f|/C) d\mathcal{A}_\alpha < +\infty$$

for some constant $C > 0$. The norm of f in \mathfrak{B}_α^Ψ is the infimum of the constants C for which the above integral is ≤ 1 . With this norm, \mathfrak{B}_α^Ψ is a Banach space.

Now, every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ defines a bounded linear operator $C_\varphi: \mathfrak{B}_\alpha^\Psi \rightarrow \mathfrak{B}_\alpha^\Psi$ by $C_\varphi(f) = f \circ \varphi$, called the *composition operator of symbol* φ . This is a consequence of the classical Littlewood's subordination principle, using the facts that the measure \mathcal{A}_α is radial and the function $\Psi(|f|/C)$ is subharmonic for every analytic function $f: \mathbb{D} \rightarrow \mathbb{C}$. Such an operator may be seen as

a Carleson embedding $J_\mu: \mathfrak{B}_\alpha^\Psi \rightarrow L^\Psi(\mu)$ for the pull-back measure $\mu = \varphi(\mathcal{A}_\alpha)$. S. Charpentier ([2]), following [6], has characterized the compactness of such embeddings (actually in the more general setting of the unit ball \mathbb{B}_N of \mathbb{C}^N instead of the unit disk \mathbb{D} of \mathbb{C}):

Theorem 4.1 (S. Charpentier) *For every finite positive measure μ on \mathbb{D} and for every $\alpha > -1$, one has:*

1) *If \mathfrak{B}_α^Ψ is compactly contained in $L^\Psi(\mu)$, then*

$$(4.1) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/\rho_\mu(h))} = 0.$$

2) *Conversely, if*

$$(4.2) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/h^{\alpha+2}K_\mu(h))} = 0,$$

then \mathfrak{B}_α^Ψ is compactly contained in $L^\Psi(\mu)$.

Here ρ_μ is the Carleson function of μ , defined as:

$$(4.3) \quad \rho_\mu(h) = \sup_{|\xi|=1} \mu[W(\xi, h)]$$

and

$$(4.4) \quad K_\mu(h) = \sup_{0 < t \leq h} \frac{\rho_\mu(t)}{t^{\alpha+2}}.$$

When $\mu = \varphi(\mathcal{A}_\alpha)$ is the pull-back measure of \mathcal{A}_α by an analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, we denote them by $\rho_{\varphi, \alpha+2}$ and $K_{\varphi, \alpha+2}$ respectively.

We gave in [6], in the non-weighted case, examples showing that conditions (4.1) and (4.2) are not equivalent for general measures μ . However, Theorem 1.1 implies that $K_{\varphi, \alpha+2}(h) \lesssim \rho_{\varphi, \alpha+2}(h)/h^{\alpha+2}$ and so conditions (4.1) and (4.2) are equivalent in this case. Therefore, we get:

Theorem 4.2 *For every $\alpha > -1$, every Orlicz function Ψ , and every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, the composition operator $C_\varphi: \mathfrak{B}_\alpha^\Psi \rightarrow \mathfrak{B}_\alpha^\Psi$ is compact if and only if:*

$$(4.5) \quad \lim_{h \rightarrow 0} \frac{\Psi^{-1}(1/h^{\alpha+2})}{\Psi^{-1}(1/\rho_{\varphi, \alpha+2}(h))} = 0.$$

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