

Finiteness of the basic intersection cohomology of a Killing foliation

Martintxo Saralegi-Aranguren*

Université d'Artois

Robert Wolak†

Uniwersytet Jagiellonski

January 3, 2010

Abstract

We prove that the basic intersection cohomology $H_{\bar{p}}^*(M/\mathcal{F})$, where \mathcal{F} is the singular foliation determined by an isometric action of a Lie group G on the compact manifold M , is finite dimensional.

This paper deals with an action $\Phi: G \times M \rightarrow M$ of a Lie group on a compact manifold preserving a riemannian metric on it. The orbits of this action define a singular foliation \mathcal{F} on M . Putting together the orbits of the same dimension we get a stratification of M . This structure is still very regular. The foliation \mathcal{F} is in fact a conical foliation and we can define the basic intersection cohomology $H_{\bar{p}}^*(M/\mathcal{F})$ (cf. [10]). This invariant becomes the basic cohomology $H^*(M/\mathcal{F})$ when the action Φ is almost free, and the intersection cohomology $H_{\bar{p}}^*(M/G)$ when the Lie group G is compact.

The aim of this work is to prove that this cohomology $H_{\bar{p}}^*(M/\mathcal{F})$ is finite dimensional. This result generalizes [3] (almost free case), [11] (abelian case) and [10] (compact case).

The paper is organized as follows. In Section 1 we present the foliation \mathcal{F} . The basic intersection cohomology $H_{\bar{p}}^*(M/\mathcal{F})$ associated to this foliation is studied in Section two. Twisted products are studied in Section 3. The finiteness of $H_{\bar{p}}^*(M/\mathcal{F})$ is proved in Section 4.

In the sequel M is a connected, second countable, Hausdorff, without boundary and smooth (of class C^∞) manifold of dimension m . All the maps are considered smooth unless something else is indicated.

1 Killing foliations determined by isometric actions.

We study in this work the foliations induced by isometric actions: the *Killing foliations*. These foliations are examples of the conical foliations for which the basic intersection cohomology has been defined (see [10, 11]). We present this geometrical framework in this section.

*Univ Lille Nord de France F-59 000 LILLE, FRANCE. UArtois, Laboratoire de Mathématiques de Lens EA 2462. Fédération CNRS Nord-Pas-de-Calais FR 2956. Faculté des Sciences Jean Perrin. Rue Jean Souvraz, S.P. 18. F-62 300 LENS, FRANCE. saralegi@euler.univ-artois.fr.

†Instytut Matematyki. Uniwersytet Jagiellonski. Stanisława Lojasiewicza 6, 30 348 KRAKOW, POLAND. robert.wolak@im.uj.edu.pl. Partially supported by the KBN grant 2 PO3A 021 25.

1.1 Killing foliations. A smooth action $\Phi: G \times M \rightarrow M$ of a Lie group G on a manifold M is a *isometric action* when there exists a riemannian metric μ on M preserved by G .

The connected components of the orbits of the action Φ determine a partition \mathcal{F} on M . In fact, this partition is a singular riemannian foliation that we shall call *Killing foliation* (cf. [7]). Notice that \mathcal{F} is also a conical foliation in the sense of [10, 11]. Classifying the points of M following the dimension of the leaves of \mathcal{F} one gets the *stratification* $\mathbf{S}_{\mathcal{F}}$ of \mathcal{F} . It is determined by the equivalence relation $x \sim y \Leftrightarrow \dim G_x = \dim G_y$. The elements of $\mathbf{S}_{\mathcal{F}}$ are called *strata*.

In the particular case where the closure of G in the isometry group of (M, μ) is a compact Lie group¹ we shall say that the action Φ is a *tame action*. In fact, a smooth action $\Phi: G \times M \rightarrow M$ is tame if and only if it extends to a smooth action $\Phi: K \times M \rightarrow M$ where K is a compact Lie group containing G (cf. [6]). The group K is not unique, but we always can choose K in such a way that G is dense in K . We shall say that K is a *tamer group*. Here the strata of $\mathbf{S}_{\mathcal{F}}$ are K -invariant closed submanifolds of M .

Since the aim of this work is the study of \mathcal{F} and not the action Φ itself, we can consider that the Lie group G is connected. Let us see that.

Proposition 1.1.1 *Let $\Phi: G \times M \rightarrow M$ is a tame action. Let G_0 be the connected component of G containing the unity element. The Killing foliation defined by the restriction $\Phi: G_0 \times M \rightarrow M$ is also \mathcal{F} .*

Proof. The partition \mathcal{F} is defined by this equivalence relation:

$$x \sim y \iff \exists \text{ continuous path } \alpha: [0, 1] \rightarrow G(x) \text{ such that } \alpha(0) = x \text{ and } \alpha(1) = y.$$

Since the map $\Delta: G \rightarrow G(x)$, defined by $\Delta(g) = \Phi(g, x) = g \cdot x$, is a submersion (see for example [2]) then

$$x \sim y \iff \exists \text{ continuous path } \beta: [0, 1] \rightarrow G \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y,$$

and by definition of G_0

$$x \sim y \iff \exists \text{ continuous path } \beta: [0, 1] \rightarrow G_0 \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y.$$

This gives the result. ♣

When G is connected, the tamer group K has richer properties.

Proposition 1.1.2 *Let G be a connected Lie subgroup of a compact Lie group K . If G is dense in K then $G \triangleleft K$ and the quotient group K/G is commutative.*

Proof. The Lie algebra \mathfrak{g} is Ad_G -invariant and hence, by density, Ad_K -invariant. Then \mathfrak{g} is an ideal of \mathfrak{k} . The connectedness of G gives that G is a normal subgroup of K . Since Ad_G acts trivially on $\mathfrak{k}/\mathfrak{g}$, Ad_K acts trivially, too. Therefore, $\mathfrak{k}/\mathfrak{g}$ is abelian (see for example [8, pag. 628]). ♣

1.2 Particular tame actions. A *trio* is a triple (K, G, H) , with K is a compact Lie group, G a normal subgroup of K and H a closed subgroup of K . We present now some tame actions associated to a trio (K, G, H) . They are going to be intensively used in this work. First of all we need some definitions.

- The action $\Phi_l: K \times K \rightarrow K$ is defined by $\Phi_l(g, k) = g \cdot k$. For each element u of the Lie algebra \mathfrak{k} of K , we shall write X^u the associated (right invariant) vector field. It is defined by $X^u(k) = T_e R_k(u)$ where $R_k: K \rightarrow K$ is given by $R_k(\ell) = \ell \cdot k$.

¹This is always the case when the manifold M is a compact.

- The action $\Phi_r: K \times K \rightarrow K$ is defined by $\Phi_r(g, k) = k \cdot g^{-1}$. For each element $u \in \mathfrak{k}$ of K , we shall write X_u the associated (left invariant) vector field. It is defined by $X_u(k) = -T_e L_k(u)$ where $L_k: K \rightarrow K$ is given by $L_k(\ell) = k \cdot \ell$.
- The action $\Psi: K \times K/H \rightarrow K/H$ is defined by $\Psi(g, kH) = (g \cdot k)H$. For each element $u \in \mathfrak{k}$, we shall write Y_u the associated vector field. Since the canonical projection $\pi: K \rightarrow K/H$ is a K -equivariant map, then we have $\pi_* X^u = Y_u$ for each $u \in \mathfrak{k}$.
- The action $\Gamma: H \times H \rightarrow H$ is defined by $\Gamma(g, h) = g \cdot h$. For each element u of the Lie algebra \mathfrak{h} of H we write Z^u the associated (right invariant) vector field.

The associated actions we are going to use are the following.

(a) *The restriction $\Phi_l: G \times K \rightarrow K$, which induces the regular Killing foliation \mathcal{K} .*

(b) *The restriction $\Phi_r: G \times K \rightarrow K$, which induces the regular Killing foliation \mathcal{K} .*

Since $G \triangleleft K$, the foliation \mathcal{K} is determined by the family of vector fields $\{X^u / u \in \mathfrak{g}\}$, where \mathfrak{g} is the Lie algebra of G , and also by the family $\{X_u / u \in \mathfrak{g}\}$. The orbits $G(k) = Gk = kG$ have the same dimension $\dim G$.

(c) *The restriction $\Psi: G \times K/H \rightarrow K/H$, which induces the regular Killing foliation \mathcal{D} .*

The foliation \mathcal{D} is determined by the family of vector fields $\{Y_u / u \in \mathfrak{g}\}$. The orbits $G(kH)$ have the same dimension $\dim G - \dim(G \cap H)$.

(d) *The restriction $\Gamma: (G \cap H) \times H \rightarrow H$, which induces the regular Killing foliation \mathcal{C} .*

The foliation \mathcal{C} is determined by the family of vector fields $\{Z^u / u \in \mathfrak{g} \cap \mathfrak{h}\}$. The orbits $(G \cap H)(k)$ have the same dimension $\dim(G \cap H)$.

(e) *The restriction $\Phi_r: GH \times K \rightarrow K$, which induces the regular Killing foliation \mathcal{E} .*

Notice that GH is a Lie group since G is normal in K . The foliation \mathcal{E} is, in fact, determined by the vector fields $\{X_u / u \in \mathfrak{g} + \mathfrak{h}\}$. The orbits $(GH)(k)$ have the same dimension $\dim G + \dim H - \dim(G \cap H)$.

1.3 Twisted product. In order to prove the finiteness of the basic intersection cohomology we decompose the manifold in a finite number of simpler pieces. These are the twisted products we introduce now.

We fix a trio (K, G, H) and a smooth action $\Theta: H \times N \rightarrow N$ of H on the manifold N . The *twisted product* is the quotient $K \times_H N$ of $K \times N$ by the equivalence relation $(k, z) \sim (k \cdot h^{-1}, \Theta(h, z) = h \cdot z)$. The element of $K \times_H N$ corresponding to $(k, z) \in K \times N$ is denoted by $\langle k, z \rangle$. This manifold is endowed with the tame action

$$\Phi: G \times (K \times_H N) \longrightarrow (K \times_H N),$$

defined by $\Phi(g, \langle k, z \rangle) = \langle g \cdot k, z \rangle$. We denote by \mathcal{W} the induced Killing foliation.

We also use the following tame action, namely, the restriction

$$\Theta: (G \cap H) \times N \rightarrow N$$

whose induced Killing foliation is denoted by \mathcal{N} .

The canonical projection $\Pi: K \times N \rightarrow K \times_H N$ relates the involved foliations as follows:

(a) $\Pi_*(\mathcal{K} \times \mathcal{I}) = \mathcal{W}$, where \mathcal{I} is the pointwise foliation (since the map Π is G -equivariant).

(b) $\mathcal{S}_{\mathcal{W}} = \{\Pi(K \times S) / S \in \mathcal{S}_N\} = \Pi(\{K\} \times \mathcal{S}_N)$ (since $G_{\langle k, z \rangle} = k(G \cap H)_z k^{-1}$).

2 Basic Intersection cohomology

In this section we recall the definition of the basic intersection² cohomology and we present the main properties we are going to use in this work. For the rest of this section, we fix a conical foliation \mathcal{F} defined on a manifold M . The associated stratification is $\mathbf{S}_{\mathcal{F}}$. The regular stratum of is denoted by $R_{\mathcal{F}}$. We shall write $m = \dim M$, $r = \dim \mathcal{F}$ and $s = m - r = \text{codim}_M \mathcal{F}$.

We are going to deal with differential forms on a product $(\text{manifold}) \times [0, 1]^p$, they are restrictions of differential forms defined on $(\text{manifold}) \times]-1, 1[^p$.

2.1 Perverse forms. Recall that a *conical chart* is a foliated diffeomorphism $\varphi: (\mathbb{R}^{m-n-1} \times c\mathbb{S}^n, \mathcal{H} \times c\mathcal{G}) \rightarrow (U, \mathcal{F}_U)$ where $(\mathbb{R}^{m-n-1}, \mathcal{H})$ is a simple foliation and $(\mathbb{S}^n, \mathcal{G})$ is a conical foliation without 0-dimensional leaves. We also shall denote this chart by (U, φ, S) where S is the stratum of $\mathbf{S}_{\mathcal{F}}$ verifying $\varphi(\mathbb{R}^{m-n-1} \times \{\vartheta\}) = U \cap S$.

The differential complex $\Pi_{\mathcal{F}}^*(M \times [0, 1]^p)$ of *perverse forms* of $M \times [0, 1]^p$ is introduced by induction on depth $\mathbf{S}_{\mathcal{F}}$. When this depth is 0 then

$$\Pi_{\mathcal{F}}^*(M \times [0, 1]^p) = \Omega^*(M \times [0, 1]^p).$$

Consider now the generic case. A perverse form of $M \times [0, 1]^p$ is first of all a differential form $\omega \in \Omega^*(R_{\mathcal{F}} \times [0, 1]^p)$ such that,

$$\begin{cases} \text{the pull-back} & (\varphi \times \mathbb{I}_{[0, 1]^p})^* \omega \in \Omega^*(\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times]0, 1[\times [0, 1]^p) \\ \text{extends to} & \omega_{\varphi} \in \Pi_{\mathcal{H} \times c\mathcal{G}}^*(\mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1]^{p+1}) \end{cases}$$

for any conical chart (U, φ) , where \mathbb{I}_{\cdot} stands for the identity map. Notice that $\Omega^*(M)$ is included on $\Pi_{\mathcal{F}}^*(M)$ ³.

2.2 Perverse degree. The amount of transversality of a perverse form $\omega \in \Pi_{\mathcal{F}}^*(M)$ with respect to a singular stratum $S \in \mathbf{S}_{\mathcal{F}}$ is measured by the perverse degree $\|\omega\|_S$. We recall here the definition of local perverse degree $\|\omega\|_U \in \{-\infty\} \cup \mathbb{N}$ of ω relatively to a conical chart (U, φ, S) :

1. $\|\omega\|_U = -\infty$ when $\omega_{\varphi} \equiv 0$ on $\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times \{0\}$,
2. $\|\omega\|_U \leq p$, with $p \in \mathbb{N}$, when $\omega_{\varphi}(v_0, \dots, v_p, -) \equiv 0$ where the vectors $\{v_0, \dots, v_p\}$ are tangent to the fibers of $P_{\varphi}: \mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times \{0\} \rightarrow U \cap S$ ⁴.

This number does not depend on the choice of the conical chart (cf. [11, Proposition 1.3.1]). Finally, we define the *perverse degree* $\|\omega\|_S$ by

$$\|\omega\|_S = \sup \left\{ \|\omega\|_U / (U, \varphi, S) \text{ conical chart} \right\}.$$

The perverse degree of $\omega \in \Omega^*(M)$ verifies $\|\omega\|_S \leq 0$ for any singular stratum $S \in \mathbf{S}_{\mathcal{F}}$ (cf. 2.1).

²We refer the reader to [10],[11] for details.

³Through the restriction $\omega \mapsto \omega_{R_{\mathcal{F}}}$.

⁴The map $P_{\varphi}: \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1[\rightarrow U$ is defined by $P_{\varphi}(x, y, t) = \varphi(x, [y, t])$.

2.3 Basic cohomology. The basic cohomology of the foliation \mathcal{F} is an important tool to study its transversal structure and plays the rôle of the cohomology of the orbit space M/\mathcal{F} , which can be a wild topological space. A differential form $\omega \in \Omega^*(M)$ is *basic* if $i_X\omega = i_Xd\omega = 0$, for each vector field X on M tangent to the foliation \mathcal{F} . For exemple, a function f is basic iff f is constant on the leaves of \mathcal{F} . We shall write $\Omega^*(M/\mathcal{F})$ for the complex of basic forms. Its cohomology $H^*(M/\mathcal{F})$ is the *basic cohomology* of (M, \mathcal{F}) . We also use the *relative basic cohomology* $H^*((M, \mathcal{F})/\mathcal{F})$, that is, the cohomology computed from the complex of basic forms vanishing on the saturated set $F \subset M$. The basic cohomology does not use the stratification $\mathcal{S}_{\mathcal{F}}$.

2.4 Basic intersection cohomology. A *perversity* is a map $\bar{p}: \mathbf{S}_{\mathcal{F}}^{\sigma} \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$, where $\mathbf{S}_{\mathcal{F}}^{\sigma}$ is the family of singular strata. The *constant perversity* \bar{t} is defined by $\bar{t}(S) = t$, where $t \in \mathbb{Z} \cup \{-\infty, \infty\}$.

The basic intersection cohomology appears when one considers basic perverse forms whose perverse degree is controlled by a perversity. We shall put

$$\Omega_{\bar{p}}^*(M/\mathcal{F}) = \left\{ \omega \in \Pi_{\mathcal{F}}^*(M) \mid \omega \text{ is basic and } \max(\|\omega\|_s, \|d\omega\|_s) \leq \bar{p}(S) \quad \forall S \in \mathbf{S}_{\mathcal{F}}^{\sigma} \right\}$$

the complex of basic perverse forms whose perverse degree (and that of their derivative) is bounded by the perversity \bar{p} . The cohomology $H_{\bar{p}}^*(M/\mathcal{F})$ of this complex is the *basic intersection cohomology*⁵ of (M, \mathcal{F}) relatively to the perversity \bar{p} .

Consider a twisted product $K \times_H N$. Perversities on $K \times_H N$ and $K \times N$ are determinate by perversities on N by the formula (cf. 1.3(b)):

$$(1) \quad \bar{p}(K \times S) = \bar{p}(\Pi(K \times S)) = \bar{p}(S).$$

2.5 Mayer-Vietoris. This is the technique we use in order to decompose the manifold in nicer pieces. An open covering $\{U, V\}$ of M by saturated open subsets is a *basic covering*. It possesses a subordinated partition of the unity made up of basic functions defined on M (see [9]). For a such covering we have the Mayer-Vietoris short sequence

$$0 \rightarrow \Omega_{\bar{p}}^*(M/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*(U/\mathcal{F}) \oplus \Omega_{\bar{p}}^*(V/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*((U \cap V)/\mathcal{F}) \rightarrow 0,$$

where the maps are defined by $\omega \mapsto (\omega, \omega)$ and $(\alpha, \beta) \mapsto \alpha - \beta$. The third map is onto since the elements of the partition of the unity are *controlled functions*, id est, elements of $\Omega_0^*(-)$ (cf. 2.2). Thus, the sequence is exact. This result is not longer true for more general coverings.

We shall use in this work the two following local calculations (see [11, Proposition 3.5.1 and Proposition 3.5.2] for the proofs).

Proposition 2.6 *Let $(\mathbb{R}^k, \mathcal{H})$ be a simple foliation. Consider \bar{p} a perversity on M and define the perversity \bar{p} on $\mathbb{R}^k \times M$ by $\bar{p}(\mathbb{R}^k \times S) = \bar{p}(S)$. The canonical projection $\text{pr}: \mathbb{R}^k \times M \rightarrow M$ induces the isomorphism*

$$H_{\bar{p}}^*(M/\mathcal{F}) \cong H_{\bar{p}}^*(\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F}).$$

Proposition 2.7 *Let \mathcal{G} be a conical foliation without 0-dimensional leaves on the sphere \mathbb{S}^n . A perversity \bar{p} on $c\mathbb{S}^n$ gives the perversity \bar{p} on \mathbb{S}^n defined by $\bar{p}(S) = \bar{p}(S \times]0, 1[)$. The canonical projection $\text{pr}: \mathbb{S}^n \times]0, 1[\rightarrow \mathbb{S}^n$ induces the isomorphism*

$$H_{\bar{p}}^i(c\mathbb{S}^n/c\mathcal{G}) = \begin{cases} H_{\bar{p}}^i(\mathbb{S}^n/\mathcal{G}) & \text{if } i \leq \bar{p}(\{\vartheta\}) \\ 0 & \text{if } i > \bar{p}(\{\vartheta\}). \end{cases}$$

⁵BIC for short.

In the next section we shall need the following technical Lemma.

Lemma 2.8 *Let $\Phi: K \times M \rightarrow M$ be a smooth action, where K is a compact Lie group, and let V be a fundamental vector field of this action. Consider a normal subgroup G of K and write \mathcal{F} the associated conical foliation on M . Then, the interior operator $i_V: \Omega_{\bar{p}}^*(M/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^{*-1}(M/\mathcal{F})$ is well defined, for any perversity \bar{p} .*

Proof. Since the question is a local one, then it suffices to consider where M is a twisted product $K \times_H N$ ⁶. Notice that the blow up $\Pi: K \times N \rightarrow K \times_H N$ is a K -equivariant map relatively to the action $\ell \cdot (k, z) = (\ell \cdot k, z)$. This gives $\Pi_*(X^u, 0) = V$ for some $u \in \mathfrak{k}$. From Lemma 3.1 we know that it suffices to prove that the operator

$$i_{(X^u, 0)}: \Omega_{\bar{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}) \longrightarrow \Omega_{\bar{p}}^{*-1}(K \times N/\mathcal{K} \times \mathcal{N})$$

is well defined. Since $G \triangleleft K$ then the vector field X^u preserves the foliation \mathcal{K} . So, it suffices to prove that the operator

$$i_{(X^u, 0)}: \Omega_{\bar{p}}^*(K \times N) \longrightarrow \Omega_{\bar{p}}^{*-1}(K \times N)$$

is well defined. This comes from the fact that X^u acts on the K -factor while the perversion conditions are measured on the N -factor (cf. (1)). ♣

3 The BIC of a twisted product

We compute now the BIC of a twisted product $K \times_H N$ (cf. 1.3) for a perversity \bar{p} (cf. (1)).

Lemma 3.1 *The natural projection $\Pi: K \times N \rightarrow K \times_H N$ induces the differential monomorphism*

$$(2) \quad \Pi^*: \Omega_{\bar{p}}^*(K \times_H N/\mathcal{W}) \longrightarrow \Omega_{\bar{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}).$$

Moreover, given a differential form ω on $K \times_H N/\mathcal{W}$, we have:

$$(3) \quad \Pi^* \omega \in \Omega_{\bar{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}) \iff \omega \in \Omega_{\bar{p}}^*(K \times_H N/\mathcal{W}).$$

Proof. Notice that the injectivity of Π^* comes from the fact that Π is a surjection. For the rest, we proceed in several steps.

(a) A foliated atlas for $\pi: K \rightarrow K/H$.

Since $\pi: K \rightarrow K/H$ is a H -principal bundle then it possesses an atlas $\mathcal{A} = \{\varphi: \pi^{-1}(U) \rightarrow U \times H\}$ made up with H -equivariant charts: $\varphi(k \cdot h^{-1}) = (\pi(k), h \cdot h_0)$ if $\varphi(k) = (\pi(k), h_0)$. We study the foliation $\varphi_* \mathcal{K}$. This equivariance property gives $\varphi_* X_u = (0, Z^u)$ for each $u \in \mathfrak{g} \cap \mathfrak{h}$. Thus, the trace of the foliation $\varphi_* \mathcal{K}$ on the fibers of the canonical projection $\text{pr}: U \times H \rightarrow U$ is C . On the other hand, since the map π is a G -equivariant map then $\pi_* \mathcal{K} = \mathcal{D}$, which gives $\text{pr}_* \varphi_* \mathcal{K} = \mathcal{D}$. We conclude that $\varphi_* \mathcal{K} \subset \mathcal{D} \times C$. By dimension reasons we get $\varphi_* \mathcal{K} = \mathcal{D} \times C$. The atlas \mathcal{A} is an H -equivariant foliated atlas of π .

(b) A foliated atlas for $\Pi: K \times N \rightarrow K \times_H N$.

⁶In fact, N is an euclidean space \mathbb{R}^a et Θ is an orthogonal action.

We claim that $\mathcal{A}_\# = \{\bar{\varphi}: \pi^{-1}(U) \times_H N \rightarrow U \times N / (U, \varphi) \in \mathcal{A}\}$ is a foliated atlas of $K \times_H N$ where the map $\bar{\varphi}$ is defined by $\bar{\varphi}(k, z) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k), e))^{-1} \cdot k, z)))$. This map is a diffeomorphism whose inverse is $\bar{\varphi}^{-1}(u, z) = (\varphi^{-1}(u, e), z)$. It verifies

$$\bar{\varphi}_* \mathcal{W} \stackrel{1.3(a)}{=} \bar{\varphi}_* \Pi_*(\mathcal{K} \times \mathcal{I}) = \bar{\varphi}_* \Pi_*(\varphi^{-1} \times \mathbb{I}_N)_*(\mathcal{D} \times \mathcal{C} \times \mathcal{I}).$$

A straightforward calculation shows $\bar{\varphi}_* \Pi_*(\varphi^{-1} \times \mathbb{I}_N) = (\mathbb{I}_U \times \Theta)$. Since \mathcal{C} is defined by the action Γ then $\Theta_*(\mathcal{C} \times \mathcal{I}) = \mathcal{N}$. Finally we obtain $\bar{\varphi}_* \mathcal{W} = \mathcal{D} \times \mathcal{N}$.

(c) Last Step.

Given $(U, \varphi) \in \mathcal{A}_\#$, we have the commutative diagram

$$\begin{array}{ccc} U \times H \times N & \xrightarrow{\varphi^{-1} \times \mathbb{I}_N} & K \times N \\ \downarrow Q & & \downarrow \Pi \\ U \times N & \xrightarrow{\bar{\varphi}^{-1}} & K \times_H N \end{array}$$

where $Q(u, h, z) = (u, h^{-1} \cdot z)$, $\Pi^{-1}(\text{Im } \bar{\varphi}^{-1}) = \text{Im } (\varphi^{-1} \times \mathbb{I}_N)$ and the rows are foliated imbeddings. Now, since (2) and (3) are local questions then it suffices to prove that

- $Q^*: \Omega_p^*(U \times N / \mathcal{D} \times \mathcal{N}) \rightarrow \Omega_p^*(U \times H \times N / \mathcal{D} \times \mathcal{C} \times \mathcal{N})$ is well-defined, and
- $Q^* \omega \in \Omega_p^*(U \times H \times N / \mathcal{D} \times \mathcal{C} \times \mathcal{N}) \iff \omega \in \Omega_p^*(U \times N / \mathcal{D} \times \mathcal{N})$, for any $\omega \in \Omega^*(U \times R_N)$.

This comes from the fact that the map

$$\nabla: (U \times H \times N, \mathcal{D} \times \mathcal{C} \times \mathcal{N}) \rightarrow (U \times H \times N, \mathcal{D} \times \mathcal{C} \times \mathcal{N}),$$

defined by $\nabla(u, h, z) = (u, h, h^{-1} \cdot z)$, is a foliated diffeomorphism⁷ and $Q = \text{pr}_0 \circ \nabla$, with $\text{pr}_0: U \times H \times N \rightarrow U \times N$ canonical projection (cf. Proposition 2.6). ♣

3.2 The Lie algebra \mathfrak{k} . We suppose in this paragraph that that G is also dense on K . Choose ν a bi-invariant riemannian metric on K , which exists by compactness. Consider

$$B = \{u_1, \dots, u_a, u_{a+1}, \dots, u_b, u_{b+1}, \dots, u_c, u_{c+1}, \dots, u_f\}$$

an orthonormal basis of the Lie algebra \mathfrak{k} of K with $\{u_1, \dots, u_b\}$ basis of the Lie algebra \mathfrak{g} of G and $\{u_{a+1}, \dots, u_c\}$ basis of the Lie algebra \mathfrak{h} of H . For each indice $1 \leq i \leq f$ we shall write $X_i \equiv X_{u_i}$ and $X^i \equiv X^{u_i}$ (cf. 1.2).

Let $\gamma_i \in \Omega^1(K)$ be the dual form of X_i , that is, $\gamma_i = i_{X_i} \nu$. Notice that $\delta_{ij} = \gamma_j(X_i)$. These forms are invariant by the left action of K . Since the flow of X^j is the multiplication on the left by $\exp(tu_j)$ then $L_{X^j} \gamma_i = 0$ for each $1 \leq j \leq f$.

For the differential, we have the formula $d\gamma_l = \sum_{1 \leq i < j \leq f} C_{ij}^l \gamma_i \wedge \gamma_j$, where $[X_i, X_j] = \sum_{l=1}^f C_{ij}^l X_l$, and $1 \leq i, j, l \leq f$. We have several restrictions on these coefficients. Since $G \triangleleft K$ then \mathfrak{g} is an ideal of \mathfrak{k} and therefore we have

$$C_{ij}^l = 0 \text{ for } i \leq b < l.$$

⁷Since $G \cap H \triangleleft H$.

Since K/G is an abelian group (cf. Proposition 1.1.2) then the induced bracket on $\mathfrak{k}/\mathfrak{g}$ is zero and therefore we have

$$C_{ij}^l = 0 \text{ for } b < i, j, l \leq f.$$

These equations imply that

$$(4) \quad d\gamma_l = 0 \text{ for each } b < l.$$

The \mathcal{E} -basic differential forms in $\bigwedge^*(\gamma_1, \dots, \gamma_f)$ are exactly $\bigwedge^*(\gamma_{c+1}, \dots, \gamma_f)$ since they are cycles and the family $\{X_1, \dots, X_c\}$ generates the foliation \mathcal{E} . This gives

$$(5) \quad H^*(K/\mathcal{E}) = \bigwedge^*(\gamma_{c+1}, \dots, \gamma_f).$$

3.3 Two actions of H/H_0 . The Lie group H preserves the foliation \mathcal{N} since the Lie group $G \cap H$ is a normal subgroup of H . Put H_0 the connected component of H containing the unity element. Since it is a connected compact Lie group then a standard argument shows that

$$(6) \quad \left(H_{\bar{p}}^*(N/\mathcal{N})\right)^{H_0} = H^*\left(\left(\Omega_{\bar{p}}^*(N/\mathcal{N})\right)^{H_0}\right) = H_{\bar{p}}^*(N/\mathcal{N})$$

(cf. [5, Theorem I, Ch. IV, vol. II]). We conclude that the finite group H/H_0 acts naturally on $H_{\bar{p}}^*(N/\mathcal{N})$.

Since H_0 is a connected Lie subgroup of GH then $\left(H^*(K/\mathcal{E})\right)^{H_0} = H^*(K/\mathcal{E})$. We conclude that the finite group H/H_0 acts naturally on $H^*(K/\mathcal{E})$.

Proposition 3.4 *Let (K, G, H) be a trio with G connected and dense in K . Then*

$$H_{\bar{p}}^*(K \times_H N/\mathcal{W}) = \left(H^*(K/\mathcal{E}) \otimes H_{\bar{p}}^*(N/\mathcal{N})\right)^{H/H_0}.$$

Proof. Using the blow up $\Pi: K \times N \rightarrow K \times_H N$, the computation of $H_{\bar{p}}^*(K \times_H N/\mathcal{W})$ can be done by using the complex $\text{Im} \left\{ \Pi^*: \Omega_{\bar{p}}^*(K \times_H N/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}) \right\}$ (cf. Lemma 3.1). We study this complex in several steps. We fix $B = \{u_1, \dots, u_f\}$ an orthonormal basis of \mathfrak{k} as in 3.2.

i Description of $\Omega^*(K \times R_N)$.

A differential form $\omega \in \Omega^*(K \times R_N)$ is of the form

$$(7) \quad \eta + \sum_{1 \leq i_1 < \dots < i_\ell \leq f} \gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \dots, i_\ell},$$

where the forms $\eta, \eta_{i_1, \dots, i_\ell} \in \Omega^*(K \times R_N)$ verify $i_{X_j} \eta = i_{X_j} \eta_{i_1, \dots, i_\ell} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \dots < i_\ell \leq f$.

ii Description of $\Pi_{\mathcal{K} \times \mathcal{N}}^*(K \times N)$.

Since the foliation \mathcal{K} is regular then we always can choose a conical chart of the form $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ where (U_1, φ_1) is a foliated chart of (K, \mathcal{K}) and (U_2, φ_2) is a conical chart of (N, \mathcal{N}) . The local blow up of the chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ is constructed from the second factor without modifying the first one. So, the differential forms γ_i are always perverse forms and a differential form $\omega \in \Pi_{\mathcal{K} \times \mathcal{N}}^*(K \times N)$ is of the form (7) where $\eta, \eta_{i_1, \dots, i_\ell} \in \Pi_{\mathcal{K} \times \mathcal{N}}^*(K \times N)$ verify $i_{X_j} \eta = i_{X_j} \eta_{i_1, \dots, i_\ell} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \dots < i_\ell \leq f$.

⟨iii⟩ Description of $\Omega^*(K \times R_N/\mathcal{K} \times N)$.

Take $\omega \in \Omega^*(K \times R_N/\mathcal{K} \times N)$. Since \mathcal{K} is generated by the family $\{X_j / 1 \leq j \leq b\}$ then $L_{X_j}\omega = 0$ for any $1 \leq j \leq b$, or equivalently, $R_g^*\omega = \omega$ for each $g \in G$ since G is connected. By density, $R_k^*\omega = \omega$ for each $k \in K$ and therefore $L_{X_j}\omega = 0$ for any $1 \leq j \leq f$ since K is connected. We conclude that $L_{X_j}\eta = L_{X_j}\eta_{i_1, \dots, i_\ell} = 0$ for any $1 \leq j \leq f$ and each $1 \leq i_1 < \dots < i_\ell \leq f$. This gives $\omega \in \bigwedge^*(\gamma_1, \dots, \gamma_f) \otimes \Omega^*(R_N)$.

The N -basic differential forms of $\Omega^*(R_N)$ are exactly $\Omega^*(R_N/N)$. The \mathcal{K} -basic differential forms of $\bigwedge^*(\gamma_1, \dots, \gamma_f)$ are exactly $\bigwedge^*(\gamma_{b+1}, \dots, \gamma_f)$ (cf. (4)). From these two facts, we get

$$\Omega^*(K \times R_N/\mathcal{K} \times N) = \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega^*(R_N/N)$$

as differential graduate commutative algebras.

⟨iv⟩ Description of $\Omega_p^*(K \times N/\mathcal{K} \times N)$.

From ⟨ii⟩ and ⟨iii⟩ it suffices to control the perverse degree of the forms

$$\eta + \sum_{b+1 \leq i_1 < \dots < i_\ell \leq f} \gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \dots, i_\ell} \in \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Pi_N^*(N).$$

Consider S a stratum of \mathbf{S}_N . From $\|\gamma_i\|_{K \times S} = 0$ and $\|\eta\|_{K \times S} = \|\eta\|_S$, we get $\|\gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \dots, i_\ell}\|_{K \times S} = \|\eta_{i_1, \dots, i_\ell}\|_S$. We conclude that

$$\Omega_p^*(K \times N/\mathcal{K} \times N) \cong \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_p^*(N/N)$$

(cf. 1.3(b)).

⟨v⟩ Description of $\text{Im} \left\{ \Pi^*: \Omega_p^*(K \times_H N/\mathcal{F}) \longrightarrow \Omega_p^*(K \times N/\mathcal{K} \times N) \right\}$.

We denote by $\{W_{a+1}, \dots, W_c\}$ the fundamental vector fields of the action $\Theta: H \times N \rightarrow N$ associated to the basis $\{u_{a+1}, \dots, u_c\}$. Consider now the action $\Upsilon: H \times (K \times N) \rightarrow (K \times N)$ defined by $\Upsilon(h, (k, z)) = (k \cdot h^{-1}, \Theta(h, z))$. Its fundamental vector fields associated to the basis $\{u_{a+1}, \dots, u_c\}$ are $\{(X_{a+1}, W_{a+1}), \dots, (X_c, W_c)\}$. Given $h \in H$, we take $\Upsilon_h: K \times N \rightarrow K \times N$ the map defined by $\Upsilon_h(k, z) = \Upsilon(h, (k, z))$. Then, we have

$$\text{Im } \Pi^* = \left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_p^*(N/N) \left| \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } a < i \leq c \\ \text{(ii) } L_{X_i}\omega = -L_{W_i}\omega \text{ if } a < i \leq c, \\ \text{(iii) } (\Upsilon_h)^*\omega = \omega \text{ for } h \in H. \end{array} \right. \right\}.$$

Let H_0 be the unity connected component of H . Recall that the subgroup H_0 is normal in H and that the quotient H/H_0 is a finite group. Condition (ii) gives that ω is H_0 -invariant. So, condition (iii) can be replaced by: (iv) $(\Upsilon_h)^*\omega = \omega$ for $h \in H/H_0$. Therefore

$$\text{Im } \Pi^* = \left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_p^*(N/N) \left| \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } a < i \leq c \\ \text{(ii) } L_{X_i}\omega = -L_{W_i}\omega \text{ if } a < i \leq c. \end{array} \right. \right\}^{H/H_0}.$$

Since the group H/H_0 is a finite one, we get that the cohomology $H^*(\text{Im } \Pi^*)$ is isomorphic to $(H^*(A^*))^{H/H_0}$, where A^* is the differential complex

$$\left\{ \omega \in \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_p^*(N/N) \left| \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } a < i \leq c \\ \text{(ii) } L_{X_i}\omega = -L_{W_i}\omega \text{ if } a < i \leq c \end{array} \right. \right\}.$$

So, it remains to compute $H^*(A^*)$. This computation can be simplified by using these three facts:

- $i_{W_i}\omega = L_{W_i}\omega = 0$ for each $a < i \leq b$, since the foliation \mathcal{N} is defined by the action of $G \cap H$.
- $i_{X_i}\gamma_j = \delta_{ij}$ for all i, j (cf. 3.2).
- $d\gamma_j = 0$ for $b < j$ (cf. (4)).

We get that A^* is the differential complex

$$\left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_p^*(N/\mathcal{N}) \left| \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } b < i \leq c \\ \text{(ii) } 0 = L_{W_i}\omega \text{ if } b < i \leq c \end{array} \right. \right\} =$$

$$\bigwedge^* (\gamma_{c+1}, \dots, \gamma_f) \otimes \underbrace{\left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_c) \otimes \Omega_p^*(N/\mathcal{N}) \left| \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } b < i \leq c \\ \text{(ii) } 0 = L_{W_i}\omega \text{ if } b < i \leq c \end{array} \right. \right\}}_{B^*}.$$

A straightforward computation gives that the canonical writing of a form $\omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_c) \otimes \Omega_p^*(N/\mathcal{N})$ verifying (i) is

$$(8) \quad \omega = \omega_0 + \sum_{b < i_1 < \dots < i_\ell \leq c} (-1)^\ell \gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge (i_{W_{i_\ell}} \dots i_{W_{i_1}} \omega_0)$$

for some $\omega_0 \in \Omega_p^*(N/\mathcal{N})$ (cf. Lemma 2.8).

Consider now $b < i, j \leq c$. Since K/G is an abelian group (cf. Proposition 1.1.2) and H is a Lie group then $[W_i, W_j] = \sum_{l=a+1}^b C_{ij}^l W_l$. Then, $i_{[W_i, W_j]}\omega_0 = 0$ since the foliation \mathcal{N} is defined by the action of $G \cap H$.

So, the canonical writing of a form $\omega \in B^*$ is (8) for some $\omega_0 \in \{ \eta \in \Omega_p^*(N/\mathcal{N}) / L_{W_i}\eta = 0 \text{ if } b < i \leq c \} = (\Omega_p^*(N/\mathcal{N}))^{H_0}$.

Then, the operator $\Delta: B^* \rightarrow (\Omega_p^*(N/\mathcal{N}))^{H_0}$, defined by $\Delta(\omega) = \omega_0$, is a differential isomorphism. We conclude that the differential complex A^* is isomorphic to $\bigwedge^* (\gamma_{c+1}, \dots, \gamma_f) \otimes (\Omega_p^*(N/\mathcal{N}))^{H_0}$ and therefore $H^*(A) \cong H^*(K/\mathcal{E}) \otimes H_p^*(N/\mathcal{N})$ (cf. (5) and (6)). Since the operator Δ is (H/H_0) -equivariant (cf. 3.3) then we get

$$H_p^*(K \times_H N/\mathcal{W}) = H^*(\text{Im } \Pi^*) = (H^*(A))^{H/H_0} = (H^*(K/\mathcal{E}) \otimes H_p^*(N/\mathcal{N}))^{H/H_0}.$$

This ends the proof. ♣

3.5 Remarks.

(a) When the Lie group G is commutative then K is also commutative. Differential forms γ_\bullet are K -invariants on the left and on the right, so $(H^*(K/\mathcal{E}))^H = H^*(K/\mathcal{E})$ and therefore

$$H_p^*(K \times_H N/\mathcal{W}) = H^*(K/\mathcal{E}) \otimes (H_p^*(N/\mathcal{N}))^{H/H_0} = H^*(K/\mathcal{E}) \otimes (H_p^*(N/\mathcal{N}))^H$$

as it has been proved in [11, Proposition 3.8.4].

(b) Since the foliation \mathcal{E} is a riemannian foliation defined on a compact manifold then we know that the cohomology $H^*(K/\mathcal{E})$ is finite (cf. [4]). So, the finiteness of $H_p^*(K \times_H N/\mathcal{W})$ depends on the finiteness of $H_p^*(N/\mathcal{N})$.

4 Finiteness of the BIC

We prove in this section that the BIC of a Killing foliation on a compact manifold is finite dimensional. First of all, we present two geometrical tools we shall use in the proof: the isotropy type stratification and the Molino's blow up.

We fix an isometric action $\Phi: G \times M \rightarrow M$ on the compact manifold M . We denote by \mathcal{F} the induced Killing foliation. For the study of \mathcal{F} we can suppose that G is connected (see Lemma 1.1.1). We fix K a tamer group. Notice that the group G is normal in K and the quotient K/G is commutative (cf. Proposition 1.1.2).

4.1 Isotropy type stratification. The *isotropy type stratification* $\mathbf{S}_{K,M}$ of M is defined by the equivalence relation⁸:

$$x \sim y \Leftrightarrow K_x \text{ is conjugated to } K_y.$$

When $\text{depth } \mathbf{S}_{K,M} > 0$, any closed stratum $S \in \mathbf{S}_{K,M}$ is a K -invariant submanifold of M and then it possesses a K -invariant tubular neighborhood $(T, \tau, S, \mathbb{R}^m)$ whose structural group is $O(m)$. Recall that there are the following smooth maps associated with this neighborhood:

- + The *radius map* $\rho: T \rightarrow [0, 1[$ defined fiberwise from the assignation $[x, t] \mapsto t$. Each $t \neq 0$ is a regular value of the ρ . The pre-image $\rho^{-1}(0)$ is S . This map is K -invariant, that is, $\rho(k \cdot z) = \rho(z)$.
- + The *contraction* $H: T \times [0, 1] \rightarrow T$ defined fiberwisely from $([x, t], r) \mapsto [x, rt]$. The restriction $H_t: T \rightarrow T$ is an embedding for each $t \neq 0$ and $H_0 \equiv \tau$. We shall write $H(z, t) = t \cdot z$. This map is K -invariant, that is, $t \cdot (k \cdot z) = k \cdot (t \cdot z)$.

The hyper-surface $D = \rho^{-1}(1/2)$ is the *tube* of the tubular neighborhood. It is a K -invariant submanifold of T . Notice that the map

$$\nabla: D \times [0, 1[\longrightarrow T,$$

defined by $\nabla(z, t) = (2t) \cdot z$ is a K -equivariant smooth map, where K acts trivially on the $[0, 1[$ -factor. Its restriction $\nabla: D \times]0, 1[\longrightarrow T \setminus S$ is a K -equivariant diffeomorphism.

Denote S_{\min} the union of closed (minimal) strata and choose T_{\min} a disjoint family of K -invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by D_{\min} . Notice that the induced map $\nabla_{\min}: D_{\min} \times]0, 1[\longrightarrow T_{\min} \setminus S_{\min}$ is a K -equivariant diffeomorphism.

4.2 Molino's blow up. The Molino' blow up [7] of the foliation \mathcal{F} produces a new foliation $\widehat{\mathcal{F}}$ of the same kind but of smaller depth. We suppose $\text{depth } \mathbf{S}_{K,M} > 0$. The *blow up* of M is the compact manifold

$$\widehat{M} = \left\{ (D_{\min} \times]-1, 1[) \bigsqcup \left((M \setminus S_{\min}) \times \{-1, 1\} \right) \right\} / \sim,$$

where $(z, t) \sim (\nabla_{\min}(z, |t|), t/|t|)$, and the map $\mathcal{L}: \widehat{M} \longrightarrow M$ defined by

$$\mathcal{L}(v) = \begin{cases} \nabla_{\min}(z, |t|) & \text{if } v = (z, t) \in D_{\min} \times]-1, 1[\\ z & \text{if } v = (z, j) \in (M \setminus S_{\min}) \times \{-1, 1\}. \end{cases}$$

Notice that \mathcal{L} is a continuous map whose restriction $\mathcal{L}: \widehat{M} \setminus \mathcal{L}^{-1}(S_{\min}) \rightarrow M \setminus S_{\min}$ is a K -equivariant smooth trivial 2-covering.

⁸For notions related with compact Lie group actions, we refer the reader to [1].

Since the map ∇_{\min} is K -equivariant then Φ induces the action $\widehat{\Phi}: K \times \widehat{M} \rightarrow \widehat{M}$ by saying that the blow-up \mathcal{L} is K -equivariant. The open submanifolds $\mathcal{L}^{-1}(T_{\min})$ and $\mathcal{L}^{-1}(T_{\min} \setminus S_{\min})$ are clearly K -diffeomorphic to $D_{\min} \times]-1, 1[$ and $D_{\min} \times (]-1, 0[\cup]0, 1[)$ respectively.

The restriction $\widehat{\Phi}: G \times \widehat{M} \rightarrow \widehat{M}$ is an isometric action with K as a tamer group. The induced Killing foliation is $\widehat{\mathcal{F}}$. Foliations \mathcal{F} and $\widehat{\mathcal{F}}$ are related by \mathcal{L} which is a foliated map. Moreover, if S is a not minimal stratum of $\mathbf{S}_{K,M}$ then there exists a unique stratum $S' \in \mathbf{S}_{K,\widehat{M}}$ such that $\mathcal{L}^{-1}(S) \subset S'$. The family $\{S' / S \in \mathbf{S}_{K,M}\}$ covers \widehat{M} and verifies the relationship: $S_1 < S_2 \Leftrightarrow S'_1 < S'_2$. We conclude the important property

$$(9) \quad \text{depth } \mathbf{S}_{K,\widehat{M}} < \text{depth } \mathbf{S}_{K,M}.$$

4.3 Finiteness of a tubular neighborhood. We suppose $\text{depth } \mathbf{S}_{K,M} > 0$. Consider a closed stratum $S \in \mathbf{S}_{K,M}$. Take $(T, \tau, S, \mathbb{R}^m)$ a K -invariant tubular neighborhood. We fix a base point $x \in S$. The isotropy subgroup K_x acts orthogonally on the fiber $\mathbb{R}^m = \tau^{-1}(x)$. So, the induced action $\Lambda_x: G_x \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is an isometric action, it gives the Killing foliation \mathcal{N} on \mathbb{R}^m .

Proposition 4.3.1 *If the BIC of $(\mathbb{R}^m, \mathcal{N})$ is finite dimensional then the BIC of (T, \mathcal{F}) is also finite dimensional.*

Proof. We proceed in two steps.

(a) $K_y = K_x$ for each $y \in S$.

The canonical projection $\pi: S \rightarrow S/K$ is an homogeneous bundle with fiber K/K_x . For any open subset $V \subset S/K$ the pull back $\tau^{-1}\pi^{-1}(V)$ is a K -invariant subset of T , then we can apply the Mayer-Vietoris technics to this kind of subsets (cf. 2.5).

Since the manifold S/K is a compact one then we can find a finite good covering $\{U_i / i \in I\}$ of it (cf. [2]). An inductive argument on the cardinality of I reduces the proof of the Lemma to the case where $T = \tau^{-1}\pi^{-1}(V)$, where V is a contractible open subset of S/K .

Here, the manifold T is K -equivalently diffeomorphic to $V \times (K \times_{K_x} \mathbb{R}^m)$, where K does not act on the first factor. So, the natural retraction of V to a point gives a K -equivariant retraction of T to the twisted product $K \times_{K_x} \mathbb{R}^m$. Now the result comes directly from 3.5(b) since (K, G, K_x) is a trio.

(b) General case.

The stratum S is K -equivariantly diffeomorphic to the twisted product $K \times_{N(K_x)} F$ where $N(K_x)$ is the normalizer of K_x on K and $F = S^{K_x}$. So, the tubular neighborhood T is K -equivariantly diffeomorphic to the twisted product $K \times_{N(K_x)} N$ where N is the manifold $\tau^{-1}(F)$. The previous case gives that the BIC of (N, \mathcal{F}_N) is finite dimensional. Now the result comes directly from 3.5(b) since $(K, G, N(K_x))$ is a trio. ♣

The main result of this work is the following

Theorem 4.4 *The BIC of the foliation determined by an isometric action on a compact manifold is finite dimensional.*

Proof. Let \mathcal{F} be a Killing foliation defined on a compact manifold M induced by an isometric action $\Phi: G \times M \rightarrow M$ where G is a Lie group. Without loss of generality we can suppose that the Lie group G is a connected one (cf. Lemma 1.1.1). We fix a tamer group K . We know that G is normal in K and the quotient group K/G is commutative (cf. Proposition 1.1.2).

Let us consider the following statement

$\mathfrak{A}(U, \mathcal{F}) =$ “The BIC $H_{\bar{p}}^*(U/\mathcal{F})$ is finite dimensional for each perversity \bar{p} ,”

where $U \subset M$ is a K -invariant submanifold. We prove $\mathfrak{A}(M, \mathcal{F})$ by induction on $\dim M$. The result is clear when $\dim M = 0$. We suppose $\mathfrak{A}(W, \mathcal{F})$ for any K -invariant compact submanifold W of M with $\dim W < \dim M$ and we prove $\mathfrak{A}(M, \mathcal{F})$. We proceed in several steps.

First step: 0-depth. Let us suppose $\text{depth } \mathbf{S}_{K,M} = 0$. Since $G \triangleleft K$ and K_x is conjugated to K_y , then G_x is conjugated to G_y , $\forall x, y \in M$. We get that the foliation \mathcal{F} is a (regular) riemannian foliation (cf. [7]). Its BIC is just the basic cohomology (cf. 2.3). Then $\mathfrak{A}(M, \mathcal{F})$ comes from [4].

Second step: Inside M . Let us suppose $\text{depth } \mathbf{S}_{K,M} > 0$. The family $\{M \setminus S_{\min}, T_{\min}\}$ is a basic covering of M and the we get the exact sequence (cf. 2.5)

$$0 \rightarrow \Omega_{\bar{p}}^*(M/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*((M \setminus S_{\min})/\mathcal{F}) \oplus \Omega_{\bar{p}}^*(T_{\min}/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*((T_{\min} \setminus S_{\min})/\mathcal{F}) \rightarrow 0.$$

The Five Lemma gives

$$\mathfrak{A}(T_{\min} \setminus S_{\min}, \mathcal{F}), \mathfrak{A}(T_{\min}, \mathcal{F}) \text{ and } \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

Since $T_{\min} \setminus S_{\min}$ is K -diffeomorphic to $D_{\min} \times]0, 1[$ (cf. (cf. 4.1)) then $\mathfrak{A}(D_{\min}, \mathcal{F}) \implies \mathfrak{A}(T_{\min} \setminus S_{\min}, \mathcal{F})$. The inequality $\dim D_{\min} < \dim M$ gives

$$\mathfrak{A}(T_{\min}, \mathcal{F}) \text{ and } \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

In order to prove $\mathfrak{A}(T_{\min}, \mathcal{F})$ it suffices to prove $\mathfrak{A}(T, \mathcal{F})$ where $(T, \tau, S, \mathbb{R}^m)$ a K -invariant tubular neighborhood of closed stratum S of $\mathbf{S}_{K,M}$. Following Proposition 4.3.1 we have

$$\mathfrak{A}(\mathbb{R}^m, \mathcal{N}) \implies \mathfrak{A}(T, \mathcal{F}) \implies \mathfrak{A}(T_{\min}, \mathcal{F}).$$

Consider the orthogonal decomposition $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, where $\mathbb{R}^{m_1} = (\mathbb{R}^m)^{G_x}$. The only fixed point of the restriction $\Lambda_x: G_x \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^{m_2}$ is the origin. So, there exists a Killing foliation⁹ \mathcal{G} on the sphere \mathbb{S}^{m_2-1} with $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \mathcal{F}) = (\mathbb{R}^{m_1} \times c\mathbb{S}^{m_2-1}, \mathcal{I} \times c\mathcal{G})$. Propositions 2.6 and 2.7 give:

$$\mathfrak{A}(\mathbb{S}^{m_2-1}, \mathcal{G}) \implies \mathfrak{A}(\mathbb{R}^{m_1} \times c\mathbb{S}^{m_2-1}, \mathcal{I} \times c\mathcal{G}) \implies \mathfrak{A}(\mathbb{R}^m, \mathcal{N}).$$

Finally, since $\dim \mathbb{S}^{m_2-1} < m \leq \dim T \leq \dim M$ we have

$$(10) \quad \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

Third step: Blow-up. Let us suppose $\text{depth } \mathbf{S}_{K,M} > 0$. The family $\{\mathcal{L}^{-1}(M \setminus S_{\min}), \mathcal{L}^{-1}(T_{\min})\}$ is a basic covering of \widehat{M} and the we get the exact sequence (cf. 2.5)

$$0 \rightarrow \Omega_{\bar{p}}^*(\widehat{M}/\widehat{\mathcal{F}}) \rightarrow \Omega_{\bar{p}}^*(\mathcal{L}^{-1}(M \setminus S_{\min})/\widehat{\mathcal{F}}) \oplus \Omega_{\bar{p}}^*(\mathcal{L}^{-1}(T_{\min})/\widehat{\mathcal{F}}) \rightarrow \Omega_{\bar{p}}^*(\mathcal{L}^{-1}(T_{\min} \setminus S_{\min})/\widehat{\mathcal{F}}) \rightarrow 0.$$

Following 4.2 we have that

- $\mathcal{L}^{-1}(M \setminus S_{\min})$ is K -diffeomorphic to two copies of $M \setminus S_{\min}$,
- $\mathcal{L}^{-1}(T_{\min})$ is K -diffeomorphic to $D_{\min} \times]-1, 1[$,
- $\mathcal{L}^{-1}(T_{\min} \setminus S_{\min})$ is K -diffeomorphic to $D_{\min} \times (]-1, 0[\cup]0, 1[)$.

⁹It is given by the orthogonal action $\Lambda_x: G_x \times \mathbb{S}^{m_2-1} \rightarrow \mathbb{S}^{m_2-1}$.

Now, the Five Lemma gives

$$\mathfrak{A}(D_{\min}, \widehat{\mathcal{F}}) \text{ and } \mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}).$$

But, the inequality $\dim D_{\min} < \dim M$ gives

$$(11) \quad \mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}).$$

Forth step: Final blow-up. When $\text{depth } \mathbf{S}_{K,M} = 0$ we get $\mathfrak{A}(M, \mathcal{F})$ from the First step. Let us suppose $\text{depth } \mathbf{S}_{K,M} > 0$. From (10) and (11) we get

$$\mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M, \mathcal{F}).$$

with $\text{depth } \mathbf{S}_{K,\widehat{M}} < \text{depth } \mathbf{S}_{K,M}$ (cf. (9)). By iterating this procedure we get

$$\mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) = \mathfrak{A}\left(\widehat{\widehat{M}}, \widehat{\widehat{\mathcal{F}}}\right) \implies \dots \implies \mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M, \mathcal{F}),$$

with $\text{depth } \mathbf{S}_{K,\widehat{M}} = 0$. We finish the proof by applying again the First Step. ♣

References

- [1] G. E. Bredon: *Introduction to Compact Transformation Groups* - Pure Appl. Math. **46**. Academic Press. 1972.
- [2] G. E. Bredon: *Topology and Geometry* - Graduate Text in Math. **139**. Springer. 1993.
- [3] A. El Kacimi and G. Hector *Décomposition de Hodge basique pour un feuilletage riemannien.* - Ann. Inst. Fourier **36**(1988), 207-227.
- [4] A. El Kacimi, V. Sergiescu and G. Hector *La cohomologie basique d'un feuilletage riemannien est de dimension finie* - Math. Z. **188**(1985), 593-599.
- [5] W. Greub, S. Halperin and R. Vanstone: *Connections, curvature and cohomology* - Pure and Appl. Math., vol. II, Academic Press, New York and London, 1972.
- [6] S. Kobayashi: *Transformation groups in differential geometry*. Classic in Mathematics. Springer, 1995.
- [7] P. Molino: *Riemannian Foliations* - Progress in Math. **73**. Birkhäuser, 1988.
- [8] D. Poguntke: *Dense Lie group homomorphisms* - J. Algebra **169**(1994), 625-647.
- [9] J.I. Royo Prieto, M. Saralegi-Aranguren and R. Wolak: *Tautness for riemannian foliations on non-compact manifolds.* - Manus. Math. **126** (2008) 177-200.
- [10] M. Saralegi-Aranguren and R. Wolak: *The BIC of a conical fibration.* - Mat. Zametki **77**(2005), 235–257. Translation in Math. Notes **77**(2005), 213-231.
- [11] M. Saralegi-Aranguren and R. Wolak: *The BIC of a singular foliation defined by an abelian group of isometries.* - Annal. Pol. Math. **89** (2006) 203-246.