# Finitness of the basic intersection cohomology of a Killing foliation

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**Abstract**

We prove that the basic intersection cohomology $\mathbb{H}^*_{\overline{p}}(M/\mathcal{F})$, where $\mathcal{F}$ is the singular foliation determined by an isometric action of a Lie group $G$ on the compact manifold $M$, is finite dimensional.

This paper deals with an action $\Phi\colon G \times M \to M$ of a Lie group on a compact manifold preserving a riemannian metric on it. The orbits of this action define a singular foliation $\mathcal{F}$ on $M$. Putting together the orbits of the same dimension we get a stratification of $M$. This structure is still very regular. The foliation $\mathcal{F}$ is in fact a conical foliation and we can define the basic intersection cohomology $\mathbb{H}^*_{\overline{p}}(M/\mathcal{F})$ (cf. [10]). This invariant becomes the basic cohomology $H^*(M/\mathcal{F})$ when the action $\Phi$ is almost free, and the intersection cohomology $\mathbb{H}^*_{\overline{p}}(M/G)$ when the Lie group $G$ is compact.

The aim of this work is to prove that this cohomology $\mathbb{H}^*_{\overline{p}}(M/\mathcal{F})$ is finite dimensional. This result generalizes [3] (almost free case), [11] (abelian case) and [10] (compact case).

The paper is organized as follows. In Section 1 we present the foliation $\mathcal{F}$. The basic intersection cohomology $\mathbb{H}^*_{\overline{p}}(M/\mathcal{F})$ associated to this foliation is studied in Section two. Twisted products are studied in Section 3. The finiteness of $\mathbb{H}^*_{\overline{p}}(M/\mathcal{F})$ is proved in Section 4.

In the sequel $M$ is a connected, second countable, Haussdorff, without boundary and smooth (of class $C^\infty$) manifold of dimension $m$. All the maps are considered smooth unless something else is indicated.

## 1 Killing foliations determined by isometric actions.

We study in this work the foliations induced by isometric actions: the *Killing foliations*. These foliations are examples of the conical foliations for which the basic intersection cohomology has been defined (see [10, 11]). We present this geometrical framework in this section.

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**1.1 Killing foliations**. A smooth action $\Phi\colon G \times M \to M$ of a Lie group $G$ on a manifold $M$ is a *isometric action* when there exists a riemannian metric $\mu$ on $M$ preserved by $G$.

The connected components of the orbits of the action $\Phi$ determine a partition $\mathcal{F}$ on $M$. In fact, this partition is a singular riemannian foliation that we shall call *Killing foliation* (cf. [7]). Notice that $\mathcal{F}$ is also a conical foliation in the sense of [10, 11]. Classifying the points of $M$ following the dimension of the leaves of $\mathcal{F}$ one gets the *stratification* $\mathsf{S}_{\mathcal{F}}$ of $\mathcal{F}$. It is determined by the equivalence relation $x \sim y \Leftrightarrow \dim G_x = \dim G_y$. The elements of $\mathsf{S}_{\mathcal{F}}$ are called *strata*.

In the particular case where the closure of $G$ in the isometry group of $(M, \mu)$ is a compact Lie group[1] we shall say that the action $\Phi$ is a *tame action*. In fact, a smooth action $\Phi\colon G \times M \to M$ is tame if and only if it extends to a smooth action $\Phi\colon K \times M \to M$ where $K$ is a compact Lie group containing $G$ (cf. [6]). The group $K$ is not unique, but we always can choose $K$ in such a way that $G$ is dense in $K$. We shall say that $K$ is a *tamer group*. Here the strata of $\mathsf{S}_{\mathcal{F}}$ are $K$-invariant closed submanifolds of $M$.

Since the aim of this work is the study of $\mathcal{F}$ and not the action $\Phi$ itself, we can consider that the Lie group $G$ is connected. Let us see that.

**Proposition 1.1.1** *Let $\Phi\colon G \times M \to M$ is a tame action. Let $G_0$ be the connected component of $G$ containing the unity element. The Killing folation defined by the restriction $\Phi\colon G_0 \times M \to M$ is also $\mathcal{F}$.*

*Proof.* The partition $\mathcal{F}$ is defined by this equivalence relation:

$$x \sim y \Longleftrightarrow \exists \text{ continuous path } \alpha\colon [0,1] \to G(x) \text{ such that } \alpha(0) = x \text{ and } \alpha(1) = y.$$

Since the map $\Delta\colon G \to G(x)$, defined by $\Delta(g) = \Phi(g, x) = g \cdot x$, is a submersion (see for example [2]) then

$$x \sim y \Longleftrightarrow \exists \text{ continuous path } \beta\colon [0,1] \to G \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y,$$

and by definition of $G_0$

$$x \sim y \Longleftrightarrow \exists \text{ continuous path } \beta\colon [0,1] \to G_0 \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y.$$

This gives the result. ♣

When $G$ is connected, the tamer group $K$ has richer properties.

**Proposition 1.1.2** *Let $G$ be a connected Lie subgroup of a compact Lie group $K$. If $G$ is dense in $K$ then $G \triangleleft K$ and the quotient group $K/G$ is commutative.*

*Proof.* The Lie algebra $\mathfrak{g}$ is $\mathrm{Ad}_G$- invariant and hence, by density, $\mathrm{Ad}_K$-invariant. Then $\mathfrak{g}$ is an ideal of $\mathfrak{k}$. The connectedness of $G$ gives that $G$ is a normal subgroup of $K$. Since $\mathrm{Ad}_G$ acts trivially on $\mathfrak{k}/\mathfrak{g}$, $\mathrm{Ad}_K$ acts trivially, too. Therefore, $\mathfrak{k}/\mathfrak{g}$ is abelian (see for example [8, pag. 628]). ♣

**1.2 Particular tame actions.** A *trio* is a triple $(K, G, H)$, with $K$ is a compact Lie group, $G$ a normal subgroup of $K$ and $H$ a closed subgroup of $K$. We present now some tame actions associated to a trio $(K, G, H)$. They are going to be intensively used in this work. First of all we need some definitions.

- The action $\Phi_l\colon K \times K \to K$ is defined by $\Phi_l(g, k) = g \cdot k$. For each element $u$ of the Lie algebra $\mathfrak{k}$ of $K$, we shall write $X^u$ the associated (right invariant) vector field. It is defined by $X^u(k) = T_e R_k(u)$ where $R_k\colon K \to K$ is given by $R_k(\ell) = \ell \cdot k$.

[1]This is always the case when the manifold $M$ is a compact.

- The action $\Phi_r \colon K \times K \to K$ is defined by $\Phi_r(g,k) = k \cdot g^{-1}$. For each element $u \in \mathfrak{k}$ of $K$, we shall write $X_u$ the associated (left invariant) vector field. It is defined by $X_u(k) = -T_e L_k(u)$ where $L_k \colon K \to K$ is given by $L_k(\ell) = k \cdot \ell$.
- The action $\Psi \colon K \times K/H \to K/H$ is defined by $\Psi(g, kH) = (g \cdot k)H$. For each element $u \in \mathfrak{k}$, we shall write $Y_u$ the associated vector field. Since the canonical projection $\pi \colon K \to K/H$ is a $K$-equivariant map, then we have $\pi_* X^u = Y_u$ for each $u \in \mathfrak{k}$.
- The action $\Gamma \colon H \times H \to H$ is defined by $\Gamma(g,h) = g \cdot h$. For each element $u$ of the Lie algebra $\mathfrak{h}$ of $H$ we write $Z^u$ the associated (right invariant) vector field.

The associated actions we are going to use are the following.

(a) *The restriction* $\Phi_l \colon G \times K \to K$*, which induces the regular Killing foliation* $\mathcal{K}$.

(b) *The restriction* $\Phi_r \colon G \times K \to K$*, which induces the regular Killing foliation* $\mathcal{K}$.

Since $G \triangleleft K$, the foliation $\mathcal{K}$ is determined by the family of vector fields $\{X^u \ / \ u \in \mathfrak{g}\}$, where $\mathfrak{g}$ is the Lie algebra of $G$, and also by the family $\{X_u \ / \ u \in \mathfrak{g}\}$. The orbits $G(k) = Gk = kG$ have the same dimension $\dim G$.

(c) *The restriction* $\Psi \colon G \times K/H \to K/H$*, which induces the regular Killing foliation* $\mathcal{D}$.

The foliation $\mathcal{D}$ is determined by the family of vector fields $\{Y_u \ / \ u \in \mathfrak{g}\}$. The orbits $G(kH)$ have the same dimension $\dim G - \dim(G \cap H)$.

(d) *The restriction* $\Gamma \colon (G \cap H) \times H \to H$*, which induces the regular Killing foliation* $\mathcal{C}$.

The foliation $\mathcal{C}$ is determined by the family of vector fields $\{Z^u \ / \ u \in \mathfrak{g} \cap \mathfrak{h}\}$. The orbits $(G \cap H)(k)$ have the same dimension $\dim(G \cap H)$.

(e) *The restriction* $\Phi_r \colon GH \times K \to K$*, which induces the regular Killing foliation* $\mathcal{E}$.

Notice that $GH$ is a Lie group since $G$ is normal in $K$. The foliation $\mathcal{E}$ is, in fact, determined by the vector fields $\{X_u \ / \ u \in \mathfrak{g} + \mathfrak{h}\}$. The orbits $(GH)(k)$ have the same dimension $\dim G + \dim H - \dim(G \cap H)$.

**1.3 Twisted product.** In order to prove the finiteness of the basic intersection cohomology we decompose the manifold in a finite number of simpler pieces. These are the twisted products we introduce now.

We fix a trio $(K, G, H)$ and a smooth action $\Theta \colon H \times N \to N$ of $H$ on the manifold $N$. The *twisted product* is the quotient $K \times_H N$ of $K \times N$ by the equivalence relation $(k, z) \sim (k \cdot h^{-1}, \Theta(h,z) = h \cdot z)$. The element of $K \times_H N$ corresponding to $(k,z) \in K \times N$ is denoted by $< k, z >$. This manifold is endowed with the tame action

$$\Phi \colon G \times (K \times_H N) \longrightarrow (K \times_H N),$$

defined by $\Phi(g, < k, z >) = < g \cdot k, z >$. We denote by $\mathcal{W}$ the induced Killing foliation.

We also use the following tame action, namely, the restriction

$$\Theta \colon (G \cap H) \times N \to N$$

whose induced Killing foliation is denoted by $\mathcal{N}$.

The canonical projection $\Pi \colon K \times N \to K \times_H N$ relates the involved foliations as follows:

(a) $\Pi_*(\mathcal{K} \times \mathcal{I}) = \mathcal{W}$, where $\mathcal{I}$ is the pointwise foliation (since the map $\Pi$ is $G$-equivariant).

(b) $\mathsf{S}_{\mathcal{W}} = \{\Pi(K \times S) \ / \ S \in \mathsf{S}_{\mathcal{N}}\} = \Pi\left(\{K\} \times \mathsf{S}_{\mathcal{N}}\right)$ (since $G_{<k,z>} = k(G \cap H)_z k^{-1}$).

## 2 Basic Intersection cohomology

In this section we recall the definition of the basic intersection[2] cohomology and we present the main properties we are going to use in this work. For the rest of this section, we fix a conical foliation $\mathcal{F}$ defied on a manifold $M$. The associated stratification is $\mathsf{S}_{\mathcal{F}}$. The regular stratum of is denoted by $R_{\mathcal{F}}$. We shall write $m = \dim M$, $r = \dim \mathcal{F}$ and $s = m - r = \operatorname{codim}_M \mathcal{F}$.

We are going to deal with differential forms on a product (manifold) $\times [0, 1[^p$, they are restrictions of differential forms defined on (manifold)$\times ]-1, 1[^p$.

**2.1 Perverse forms.** Recall that a *conical chart* is a foliated diffeomorphism $\varphi\colon (\mathbb{R}^{m-n-1} \times c\mathbb{S}^n, \mathcal{H} \times c\mathcal{G}) \to (U, \mathcal{F}_U)$ where $(\mathbb{R}^{m-n-1}, \mathcal{H})$ is a simple foliation and $(\mathbb{S}^n, \mathcal{G})$ is a conical foliation without 0-dimensional leaves. We also shall denote this chart by $(U, \varphi, S)$ where $S$ is the stratum of $\mathsf{S}_{\mathcal{F}}$ verifying $\varphi(\mathbb{R}^{m-n-1} \times \{\vartheta\}) = U \cap S$.

The differential complex $\Pi^*_{\mathcal{F}}(M \times [0, 1[^p)$ of *perverse forms* of $M \times [0, 1[^p$ is introduced by induction on depth $\mathsf{S}_{\mathcal{F}}$. When this depth is 0 then

$$\Pi^*_{\mathcal{F}}(M \times [0, 1[^p) = \Omega^*(M \times [0, 1[^p).$$

Consider now the generic case. A perverse form of $M \times [0, 1[^p$ is first of all a differential form $\omega \in \Omega^*(R_{\mathcal{F}} \times [0, 1[^p)$ such that,

$$\begin{cases} \text{the pull-back} & (\varphi \times \mathbb{I}_{[0,1[^p})^*\omega \in \Omega^*\left(\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times ]0, 1[ \times [0, 1[^p\right) \\ \text{extends to} & \omega_\varphi \in \Pi^*_{\mathcal{H}\times c\mathcal{G}}\left(\mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1[^{p+1}\right) \end{cases}$$

for any conical chart $(U, \varphi)$, where $\mathbb{I}_\bullet$ stands for the identity map. Notice that $\Omega^*(M)$ is included on $\Pi^*_{\mathcal{F}}(M)$[3].

**2.2 Perverse degree.** The amount of transversality of a perverse form $\omega \in \Pi^*_{\mathcal{F}}(M)$ with respect to a singular stratum $S \in \mathsf{S}_{\mathcal{F}}$ is measured by the perverse degree $||\omega||_S$. We recall here the definition of local perverse degree $||\omega||_U \in \{-\infty\} \cup \mathbb{N}$ of $\omega$ relatively to a conical chart $(U, \varphi, S)$:

1. $||\omega||_U = -\infty$ when $\omega_\varphi \equiv 0$ on $\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times \{0\}$,
2. $||\omega||_U \leq p$, with $p \in \mathbb{N}$, when $\omega_\varphi(v_0, \ldots, v_p, -) \equiv 0$ where the vectors $\{v_0, \ldots, v_p\}$ are tangent to the fibers of $P_\varphi : \mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times \{0\} \longrightarrow U \cap S$[4].

This number does not depend on the choice of the conical chart (cf. [11, Proposition 1.3.1]). Finally, we define the *perverse degree* $||\omega||_S$ by

$$||\omega||_S = \sup\left\{||\omega||_U \;/\; (U, \varphi, S) \text{ conical chart}\right\}.$$

The perverse degree of $\omega \in \Omega^*(M)$ verifies $||\omega||_S \leq 0$ for any singular stratum $S \in \mathsf{S}_{\mathcal{F}}$ (cf. 2.1).

[2] We refer the reader to [10],[11] for details.

[3] Through the restriction $\omega \mapsto \omega_{R_{\mathcal{F}}}$.

[4] The map $P_\varphi : \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1[ \longrightarrow U$ is defined by $P_\varphi(x, y, t) = \varphi(x, [y, t])$.

**2.3 Basic cohomology**. The basic cohomology of the foliation $\mathcal{F}$ is an important tool to study its transversal structure and plays the rôle of the cohomology of the orbit space $M/\mathcal{F}$, which can be a wild topological espace. A differential form $\omega \in \Omega^*(M)$ is *basic* if $i_X\omega = i_X d\omega = 0$, for each vector field $X$ on $M$ tangent to the foliation $\mathcal{F}$. For exemple, a function $f$ is basic iff $f$ is constant on the leaves of $\mathcal{F}$. We shall write $\Omega^*(M/\mathcal{F})$ for the complex of basic forms. Its cohomology $H^*(M/\mathcal{F})$ is the *basic cohomology* of $(M,\mathcal{F})$. We also use the *relative basic cohomology* $H^*((M,F)/\mathcal{F})$, that is, the cohomology computed from the complex of basic forms vanishing on the saturated set $F \subset M$. The basic cohomology does not use the stratification $\mathsf{S}_{\mathcal{F}}$.

**2.4 Basic intersection cohomology**. A *perversity* is a map $\overline{p}\colon \mathsf{S}^{\sigma}_{\mathcal{F}} \to \mathbb{Z} \cup \{-\infty,\infty\}$, where $\mathsf{S}^{\sigma}_{\mathcal{F}}$ is the family of singular strata. The *constant perversity* $\overline{\iota}$ is defined by $\overline{\iota}(S) = \iota$, where $\iota \in \mathbb{Z} \cup \{-\infty,\infty\}$.

The basic intersection cohomology appears when one considers basic perverse forms whose perverse degree is controlled by a perversity. We shall put

$$\Omega^*_{\overline{p}}(M/\mathcal{F}) = \left\{\omega \in \Pi^*_{\mathcal{F}}(M) \ / \ \omega \text{ is basic and } \max\left(||\omega||_S, ||d\omega||_S\right) \le \overline{p}(S) \ \ \forall S \in \mathsf{S}^{\sigma}_{\mathcal{F}}\right\}$$

the complex of basic perverse forms whose perverse degree (and that of the their derivative) is bounded by the perversity $\overline{p}$. The cohomology $\boldsymbol{H}^*_{\overline{p}}(M/\mathcal{F})$ of this complex is the *basic intersection cohomology*[5] of $(M,\mathcal{F})$ relatively to the perversity $\overline{p}$.

Consider a twisted product $K\times_H N$. Perversities on $K\times_H N$ and $K \times N$ are determinate by perversities on $N$ by the formula (cf. 1.3(b)):

$$\overline{p}(K \times S) = \overline{p}(\Pi(K \times S)) = \overline{p}(S). \tag{1}$$

**2.5 Mayer-Vietoris**. This is the technique we use in order to decompose the manifold in nicer pieces. An open covering $\{U,V\}$ of $M$ by saturated open subsets is a *basic covering*. It possesses a subordinated partition of the unity made up of basic functions defined on $M$ (see [9]). For a such covering we have the Mayer-Vietoris short sequence

$$0 \to \Omega^*_{\overline{p}}(M/\mathcal{F}) \to \Omega^*_{\overline{p}}(U/\mathcal{F}) \oplus \Omega^*_{\overline{p}}(V/\mathcal{F}) \to \Omega^*_{\overline{p}}((U \cap V)/\mathcal{F}) \to 0,$$

where the map are defined by $\omega \mapsto (\omega,\omega)$ and $(\alpha,\beta) \mapsto \alpha-\beta$. The third map is onto since the elements of the partition of the unity are *controlled functions*, id est, elements of $\Omega^0_{\overline{0}}(-)$ (cf. 2.2). Thus, the sequence is exact. This result is not longer true for more general coverings.

We shall use in this work the two following local calculations (see [11, Proposition 3.5.1 and Proposition 3.5.2] for the proofs).

**Proposition 2.6** *Let $(\mathbb{R}^k,\mathcal{H})$ be a simple foliation. Consider $\overline{p}$ a perversity on $M$ and define the perversity $\overline{p}$ on $\mathbb{R}^k \times M$ by $\overline{p}(\mathbb{R}^k \times S) = \overline{p}(S)$. The canonical projection* $\mathrm{pr} : \mathbb{R}^k \times M \to M$ *induces the isomorphism*

$$\boldsymbol{H}^*_{\overline{p}}(M/\mathcal{F}) \cong \boldsymbol{H}^*_{\overline{p}}\left(\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F}\right).$$

**Proposition 2.7** *Let $\mathcal{G}$ be a conical foliation without 0-dimensional leaves on the sphere $\mathbb{S}^n$. A perversity $\overline{p}$ on $c\mathbb{S}^n$ gives the perversity $\overline{p}$ on $\mathbb{S}^n$ defined by $\overline{p}(S) = \overline{p}(S\times]0,1[)$. The canonical projection* $\mathrm{pr} : \mathbb{S}^n\times]0,1[\to \mathbb{S}^n$ *induces the isomorphism*

$$\boldsymbol{H}^i_{\overline{p}}(c\mathbb{S}^n/c\mathcal{G}) = \begin{cases} \boldsymbol{H}^i_{\overline{p}}(\mathbb{S}^n/\mathcal{G}) & \text{if } i \le \overline{p}(\{\vartheta\}) \\ 0 & \text{if } i > \overline{p}(\{\vartheta\}). \end{cases}$$

[5] BIC for short.

In the next section we shall need the following technical Lemma.

**Lemma 2.8** *Let $\Phi\colon K \times M \to M$ be a smooth action, where $K$ is a compact Lie group, and let $V$ be a fundamental vector field of this action. Consider a normal subgroup $G$ of $K$ and write $\mathcal{F}$ the associated conical foliation on $M$. Then, the interior operator $i_V\colon \Omega^*_{\overline{p}}(M/\mathcal{F}) \longrightarrow \Omega^{*-1}_{\overline{p}}(M/\mathcal{F})$ is well defined, for any perversity $\overline{p}$.*

*Proof.* Since the question is a local one, then it suffices to consider where $M$ is a twisted product $K\times_H N$[6]. Notice that the blow up $\Pi\colon K \times N \to K\times_H N$ is a $K$-equivariant map relatively to the action $\ell \cdot (k, z) = (\ell \cdot k, z)$. This gives $\Pi_*(X^u, 0) = V$ for some $u \in \mathfrak{k}$. From Lemma 3.1 we know that it suffices to prove that the operator

$$i_{(X^u,0)}\colon \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N}) \longrightarrow \Omega^{*-1}_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N})$$

is well defined. Since $G \triangleleft K$ then the vector field $X^u$ preserves the foliation $\mathcal{K}$. So, it suffices to prove that the operator

$$i_{(X^u,0)}\colon \Omega^*_{\overline{p}}(K \times N) \longrightarrow \Omega^{*-1}_{\overline{p}}(K \times N)$$

is well defined. This comes from the fact that $X^u$ acts on the $K$-factor while the perversion conditions are measured on the $N$-factor (cf. (1)). ♣

# 3 The BIC of a twisted product

We compute now the BIC of a twisted product $K\times_H N$ (cf. 1.3) for a perversity $\overline{p}$ (cf. (1)).

**Lemma 3.1** *The natural projection $\Pi\colon K \times N \to K\times_H N$ induces the differential monomorphism*

$$\Pi^*\colon \Omega^*_{\overline{p}}(K\times_H N/\mathcal{W}) \longrightarrow \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N}). \tag{2}$$

*Moreover, given a differential form $\omega$ on $K\times_H R_{\mathcal{W}}$, we have:*

$$\Pi^*\omega \in \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N}) \Longleftrightarrow \omega \in \Omega^*_{\overline{p}}(K\times_H N/\mathcal{W}). \tag{3}$$

*Proof.* Notice that the injectivity of $\Pi^*$ comes from the fact that $\Pi$ is a surjection. For the rest, we proceed in several steps.

(a) *A foliated atlas for $\pi\colon K \to K/H$.*

Since $\pi\colon K \to K/H$ is a $H$-principal bundle then it possesses an atlas $\mathcal{A} = \left\{\varphi\colon \pi^{-1}(U) \longrightarrow U \times H\right\}$ made up with $H$-equivariant charts: $\varphi(k \cdot h^{-1}) = (\pi(k), h \cdot h_0)$ if $\varphi(k) = (\pi(k), h_0)$. We study the foliation $\varphi_*\mathcal{K}$. This equivariance property gives $\varphi_* X_u = (0, Z^u)$ for each $u \in \mathfrak{g} \cap \mathfrak{h}$. Thus, the trace of the foliation $\varphi_*\mathcal{K}$ on the fibers of the canonical projection $\mathrm{pr}: U \times H \to U$ is $C$. On the other hand, since the map $\pi$ is a $G$-equivariant map then $\pi_*\mathcal{K} = \mathcal{D}$, which gives $\mathrm{pr}_*\varphi_*\mathcal{K} = \mathcal{D}$. We conclude that $\varphi_*\mathcal{K} \subset \mathcal{D} \times C$. By dimension reasons we get $\varphi_*\mathcal{K} = \mathcal{D} \times C$. The atlas $\mathcal{A}$ is an $H$-equivariant foliated atlas of $\pi$.

(b) *A foliated atlas for $\Pi\colon K \times N \to K\times_H N$.*

[6] In fact, $N$ is an euclidean space $\mathbb{R}^a$ et $\Theta$ is an orthogonal action.

We claim that $\mathcal{A}_\# = \left\{\overline{\varphi}\colon \pi^{-1}(U)\times_H N \longrightarrow U\times N \ / \ (U,\varphi)\in\mathcal{A}\right\}$ is a foliated atlas of $K\times_H N$ where the map $\overline{\varphi}$ is defined by $\overline{\varphi}(<k,z>) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k),e))^{-1}\cdot k, z)))$. This map is a diffeomorphism whose inverse is $\overline{\varphi}^{-1}(u,z) =< \varphi^{-1}(u,e), z>$. It verifies

$$\overline{\varphi}_*\mathcal{W} \overset{1.3(a)}{=\!=} \overline{\varphi}_*\Pi_*(\mathcal{K}\times\mathcal{I}) = \overline{\varphi}_*\Pi_*(\varphi^{-1}\times\mathbb{I}_N)_*(\mathcal{D}\times\mathcal{C}\times\mathcal{I}).$$

A straightforward calculation shows $\overline{\varphi}\circ\Pi\circ(\varphi^{-1}\times\mathbb{I}_N) = (\mathbb{I}_U\times\Theta)$. Since $\mathcal{C}$ is defined by the action $\Gamma$ then $\Theta_*(\mathcal{C}\times\mathcal{I}) = \mathcal{N}$. Finally we obtain $\overline{\varphi}_*\mathcal{W} = \mathcal{D}\times\mathcal{N}$.

(c) *Last Step*.

Given $\overline{(U,\varphi)}\in\mathcal{A}_\#$, we have the commutative diagram

$$\begin{array}{ccc} U\times H\times N & \xrightarrow{\varphi^{-1}\times\mathbb{I}_N} & K\times N \\ \Big\downarrow{\scriptstyle Q} & & \Big\downarrow{\scriptstyle\Pi} \\ U\times N & \xrightarrow{\overline{\varphi}^{-1}} & K\times_H N \end{array}$$

where $Q(u,h,z) = (u, h^{-1}\cdot z)$, $\Pi^{-1}(\mathrm{Im}\,\overline{\varphi}^{-1}) = \mathrm{Im}\left(\varphi^{-1}\times\mathbb{I}_N\right)$ and the rows are foliated imbeddings. Now, since (2) and (3) are local questions then it suffices to prove that

- $Q^*\colon \Omega^*_{\overline{p}}(U\times N/\mathcal{D}\times\mathcal{N}) \longrightarrow \Omega^*_{\overline{p}}(U\times H\times N/\mathcal{D}\times\mathcal{C}\times\mathcal{N})$ is well-defined, and
- $Q^*\omega\in\Omega^*_{\overline{p}}(U\times H\times N/\mathcal{D}\times\mathcal{C}\times\mathcal{N}) \Longleftrightarrow \omega\in\Omega^*_{\overline{p}}(U\times N/\mathcal{D}\times\mathcal{N})$, for any $\omega\in\Omega^*(U\times R_N)$.

This comes from the fact that the map

$$\nabla\colon (U\times H\times N, \mathcal{D}\times\mathcal{C}\times\mathcal{N}) \longrightarrow (U\times H\times N, \mathcal{D}\times\mathcal{C}\times\mathcal{N}),$$

defined by $\nabla(u,h,z) = (u,h,h^{-1}\cdot z))$, is a foliated diffeomorphism[7] and $Q = \mathrm{pr}_0\circ\nabla$, with $\mathrm{pr}_0\colon U\times H\times N\to U\times N$ canonical projection (cf. Proposition 2.6). ♣

**3.2 The Lie algebra $\mathfrak{k}$.** We suppose in this paragraph that that $G$ is also dense on $K$. Choose $\nu$ a bi-invariant riemannian metric on $K$, which exists by compactness. Consider

$$B = \left\{u_1,\ldots u_a, u_{a+1},\ldots, u_b, u_{b+1},\ldots, u_c, u_{c+1},\ldots, u_f\right\}$$

an orthonormal basis of the Lie algebra $\mathfrak{k}$ of $K$ with $\{u_1,\ldots u_b\}$ basis of the Lie algebra $\mathfrak{g}$ of $G$ and $\{u_{a+1},\ldots u_c\}$ basis of the Lie algebra $\mathfrak{h}$ of $H$. For each indice $1\leq i\leq f$ we shall write $X_i\equiv X_{u_i}$ and $X^i\equiv X^{u_i}$ (cf. 1.2).

Let $\gamma_i\in\Omega^1(K)$ be the dual form of $X_i$, that is, $\gamma_i = i_{X_i}\nu$. Notice that $\delta_{ij} = \gamma_j(X_i)$. These forms are invariant by the left action of $K$. Since the flow of $X^j$ is the multiplication on the left by $\exp(tu_j)$ then $L_{X^j}\gamma_i = 0$ for each $1\leq j\leq f$.

For the differential, we have the formula $d\gamma_l = \displaystyle\sum_{1\leq i<j\leq f} C^l_{ij}\gamma_i\wedge\gamma_j$, where $[X_i,X_j] = \displaystyle\sum_{l=1}^{f} C^l_{ij}X_l$, and $1\leq i,j,l\leq f$. We have several restrictions on these coefficients. Since $G\triangleleft K$ then $\mathfrak{g}$ is an ideal of $\mathfrak{k}$ and therefore we have

$$C^l_{ij} = 0 \text{ for } i\leq b<l.$$

[7] Since $G\cap H\triangleleft H$.

Since $K/G$ is an abelian group (cf. Proposition 1.1.2) then the induced bracket on $\mathfrak{k}/\mathfrak{g}$ is zero and therefore we have

$$C^l_{ij} = 0 \text{ for } b < i, j, l \leq f.$$

These equations imply that

$$d\gamma_l = 0 \text{ for each } b < l. \tag{4}$$

The $\mathcal{E}$-basic differential forms in $\bigwedge^*(\gamma_1, \ldots, \gamma_f)$ are exactly $\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f)$ since they are cycles and the family $\{X_1, \ldots, X_c\}$ generates the foliation $\mathcal{E}$. This gives

$$H^*\left(K\big/\mathcal{E}\right) = \bigwedge{}^*(\gamma_{c+1}, \ldots, \gamma_f). \tag{5}$$

**3.3 Two actions of $H/H_0$.** The Lie group $H$ preserves the foliation $\mathcal{N}$ since the Lie group $G \cap H$ is a normal subgroup of $H$. Put $H_0$ the connected component of $H$ containing the unity element. Since it is a connected compact Lie group then a standard argument shows that

$$\left(\boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})\right)^{H_0} = H^*\left(\left(\Omega^\cdot_{\overline{p}}(N/\mathcal{N})\right)^{H_0}\right) = \boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N}) \tag{6}$$

(cf. [5, Theorem I, Ch. IV, vol. II]). We conclude that the finite group $H/H_0$ acts naturally on $\boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})$.

Since $H_0$ is a connected Lie subgroup of $GH$ then $\left(H^*(K/\mathcal{E})\right)^{H_0} = H^*(K/\mathcal{E})$. We conclude that the finite group $H/H_0$ acts naturally on $H^*(K/\mathcal{E})$.

**Proposition 3.4** *Let $(K, G, H)$ be a trio with $G$ connected and dense in $K$. Then*

$$\boldsymbol{H}^*_{\overline{p}}(K \times_H N/\mathcal{W}) = \left(H^*(K/\mathcal{E}) \otimes \boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0}.$$

*Proof.* Using the blow up $\Pi \colon K \times N \to K \times_H N$, the computation of $\boldsymbol{H}^*_{\overline{p}}(K \times_H N/\mathcal{W})$ can be done by using the complex $\mathrm{Im}\left\{\Pi^* \colon \Omega^*_{\overline{p}}(K \times_H N/\mathcal{F}) \longrightarrow \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N})\right\}$ (cf. Lemma 3.1). We study this complex in several steps. We fix $B = \left\{u_1, \ldots, u_f\right\}$ an orthonormal basis of $\mathfrak{k}$ as in 3.2.

$\langle i \rangle$ <u>Description of</u> $\Omega^*(K \times R_{\mathcal{N}})$.

A differential form $\omega \in \Omega^*(K \times R_{\mathcal{N}})$ is of the form

$$\eta + \sum_{1 \leq i_1 < \cdots < i_\ell \leq f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \ldots, i_\ell}, \tag{7}$$

where the forms $\eta, \eta_{i_1, \ldots, i_\ell} \in \Omega^*(K \times R_{\mathcal{N}})$ verify $i_{X_j}\eta = i_{X_j}\eta_{i_1, \ldots, i_\ell} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_\ell \leq f$.

$\langle ii \rangle$ <u>Description of</u> $\Pi^*_{\mathcal{K} \times \mathcal{N}}(K \times N)$.

Since the foliation $\mathcal{K}$ is regular then we always can choose a conical chart of the form $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ where $(U_1, \varphi_1)$ is a foliated chart of $(K, \mathcal{K})$ and $(U_2, \varphi_2)$ is a conical chart of $(N, \mathcal{N})$. The local blow up of the chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ is constructed from the second factor without modifying the first one. So, the differential forms $\gamma_i$ are always perverse forms and a differential form $\omega \in \Pi^*_{\mathcal{K} \times \mathcal{N}}(K \times N)$ is of the form (7) where $\eta, \eta_{i_1, \ldots, i_\ell} \in \Pi^*_{\mathcal{K} \times \mathcal{N}}(K \times N)$ verify $i_{X_j}\eta = i_{X_j}\eta_{i_1, \ldots, i_\ell} = 0$ for each $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_\ell \leq f$.

⟨*iii*⟩ Description of $\Omega^{^*}(K\times R_{\mathcal N}/\mathcal K\times\mathcal N)$.

Take $\omega\in\Omega^{^*}(K\times R_{\mathcal N}/\mathcal K\times\mathcal N)$. Since $\mathcal K$ is generated by the family $\{X_j\ /\ 1\le j\le b\}$ then $L_{X_j}\omega=0$ for any $1\le j\le b$, or equivalently, $R_g^*\omega=\omega$ for each $g\in G$ since $G$ is connected. By density, $R_k^*\omega=\omega$ for each $k\in K$ and therefore $L_{X_j}\omega=0$ for any $1\le j\le f$ since $K$ is connected. We conclude that $L_{X_j}\eta=L_{X_j}\eta_{i_1,\dots,i_\ell}=0$ for any $1\le j\le f$ and each $1\le i_1<\cdots<i_\ell\le f$. This gives $\omega\in\bigwedge^*(\gamma_1,\dots,\gamma_f)\otimes\Omega^{^*}(R_{\mathcal N})$.

The $\mathcal N$-basic differential forms of $\Omega^{^*}(R_{\mathcal N})$ are exactly $\Omega^{^*}(R_{\mathcal N}/\mathcal N)$. The $\mathcal K$-basic differential forms of $\bigwedge^*(\gamma_1,\dots,\gamma_f)$ are exactly $\bigwedge^*(\gamma_{b+1},\dots,\gamma_f)$ (cf. (4)). From these two facts, we get

$$\Omega^{^*}(K\times R_{\mathcal N}/\mathcal K\times\mathcal N)=\bigwedge{}^{*}(\gamma_{b+1},\dots,\gamma_f)\otimes\Omega^{^*}(R_{\mathcal N}/\mathcal N)$$

as differential graduate commutative algebras.

⟨*iv*⟩ Description of $\Omega^{^*}_{\overline p}(K\times N/\mathcal K\times\mathcal N)$.

From ⟨*ii*⟩ and ⟨*iii*⟩ it suffices to control the perverse degree of the forms

$$\eta+\sum_{b+1\le i_1<\cdots<i_\ell\le f}\gamma_{i_1}\wedge\cdots\wedge\gamma_{i_\ell}\wedge\eta_{i_1,\dots,i_\ell}\in\bigwedge{}^{*}(\gamma_{b+1},\dots,\gamma_f)\otimes\Pi^{^*}_{\mathcal N}(N).$$

Consider $S$ a stratum of $\mathsf S_{\mathcal N}$. From $||\gamma_i||_{K\times S}=0$ and $||\eta||_{K\times S}=||\eta||_S$, we get $||\gamma_{i_1}\wedge\dots\gamma_{i_\ell}\wedge\eta_{i_1,\dots,i_\ell}||_{K\times S}=||\eta_{i_1,\dots,i_\ell}||_S$. We conclude that

$$\Omega^{^*}_{\overline p}(K\times N/\mathcal K\times\mathcal N)\cong\bigwedge{}^{*}(\gamma_{b+1},\dots,\gamma_f)\otimes\Omega^{^*}_{\overline p}(N/\mathcal N)$$

(cf. 1.3(b)).

⟨*v*⟩ Description of $\operatorname{Im}\left\{\Pi^*\colon\Omega^{^*}_{\overline p}(K\times_H N/\mathcal F)\longrightarrow\Omega^{^*}_{\overline p}(K\times N/\mathcal K\times\mathcal N)\right\}$.

We denote by $\{W_{a+1},\dots,W_c\}$ the fundamental vector fields of the action $\Theta\colon H\times N\to N$ associated to the basis $\{u_{a+1},\dots,u_c\}$. Consider now the action $\Upsilon\colon H\times(K\times N)\to(K\times N)$ defined by $\Upsilon(h,(k,z))=(k\cdot h^{-1},\Theta(h,z))$. Its fundamental vector fields associated to the basis $\{u_{a+1},\dots,u_c\}$ are $\{(X_{a+1},W_{a+1}),\dots,(X_c,W_c)\}$. Given $h\in H$, we take $\Upsilon_h\colon K\times N\to K\times N$ the map defined by $\Upsilon_h(k,z)=\Upsilon(h,(k,z))$. Then, we have

$$\operatorname{Im}\Pi^*=\left\{\omega\in\bigwedge{}^{*}(\gamma_{b+1},\dots,\gamma_f)\otimes\Omega^{^*}_{\overline p}(N/\mathcal N)\ \middle/\ \begin{array}{l}\text{(i) } i_{X_i}\omega=-i_{W_i}\omega\text{ if }a<i\le c\}\\ \text{(ii) } L_{X_i}\omega=-L_{W_i}\omega\text{ if }a<i\le c\},\\ \text{(iii) } (\Upsilon_h)^*\omega=\omega\text{ for }h\in H.\end{array}\right\}.$$

Let $H_0$ be the unity connected component of $H$. Recall that the subgroup $H_0$ is normal in $H$ and that the quotient $H/H_0$ is a finite group. Conditions (ii) gives that $\omega$ is $H_0$-invariant. So, condition (iii) can be replaced by: (iv) $(\Upsilon_h)^*\omega=\omega$ for $h\in H/H_0$. Therefore

$$\operatorname{Im}\Pi^*=\left\{\omega\in\bigwedge{}^{*}(\gamma_{b+1},\dots,\gamma_f)\otimes\Omega^{^*}_{\overline p}(N/\mathcal N)\ \middle/\ \begin{array}{l}\text{(i) } i_{X_i}\omega=-i_{W_i}\omega\text{ if }a<i\le c\}\\ \text{(ii) } L_{X_i}\omega=-L_{W_i}\omega\text{ if }a<i\le c\}.\end{array}\right\}^{H/H_0}.$$

Since the group $H/H_0$ is a finite one, we get that the cohomology $H^{^*}(\operatorname{Im}\Pi^*)$ is isomorphic to $\left(H^{^*}(A^{^\cdot})\right)^{H/H_0}$, where $A^*$ is the differential complex

$$\left\{\omega\in\bigwedge{}^{*}(\gamma_{b+1},\dots,\gamma_f)\otimes\Omega^{^*}_{\overline p}(N/\mathcal N)\ \middle/\ \begin{array}{l}\text{(i) } i_{X_i}\omega=-i_{W_i}\omega\text{ if }a<i\le c\}\\ \text{(ii) } L_{X_i}\omega=-L_{W_i}\omega\text{ if }a<i\le c\}\end{array}\right\}.$$

So, it remains to compute $H^{^*}(A^{^\cdot})$. This computation can be simplified by using these three facts:

- $i_{W_i}\omega = L_{W_i}\omega = 0$ for each $a < i \leq b$, since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$.
- $i_{X_i}\gamma_j = \delta_{ij}$ for all $i, j$ (cf. 3.2).
- $d\gamma_j = 0$ for $b < j$ (cf. (4)).

We get that $A^*$ is the differential complex

$$\left\{\omega \in \bigwedge^*(\gamma_{b+1},\ldots,\gamma_f)\otimes\Omega^*_{\overline{p}}(N/\mathcal{N}) \;\middle/\; \begin{array}{l}\text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } b < i \leq c\} \\ \text{(ii) } 0 = L_{W_i}\omega \text{ if } b < i \leq c\}\end{array}\right\} =$$

$$\bigwedge^*(\gamma_{c+1},\ldots,\gamma_f)\otimes\underbrace{\left\{\omega \in \bigwedge^*(\gamma_{b+1},\ldots,\gamma_c)\otimes\Omega^*_{\overline{p}}(N/\mathcal{N}) \;\middle/\; \begin{array}{l}\text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } b < i \leq c\} \\ \text{(ii) } 0 = L_{W_i}\omega \text{ if } b < i \leq c\}\end{array}\right\}}_{B^*}.$$

A straightforward computation gives that the canonical writing of a form $\omega \in \bigwedge^*(\gamma_{b+1},\ldots,\gamma_c)\otimes\Omega^*_{\overline{p}}(N/\mathcal{N})$ verifying (i) is

$$\omega = \omega_0 + \sum_{b<i_1<\cdots<i_\ell\leq c} (-1)^\ell \gamma_{i_1}\wedge\cdots\wedge\gamma_{i_\ell}\wedge(i_{W_{i_\ell}}\cdots i_{W_{i_1}}\omega_0) \tag{8}$$

for some $\omega_0 \in \Omega^*_{\overline{p}}(N/\mathcal{N})$ (cf. Lemma 2.8).

Consider now $b < i, j \leq c$. Since $K/G$ is an abelian group (cf. Proposition 1.1.2) and $H$ is a Lie group then $[W_i, W_j] = \sum_{l=a+1}^{b} C^l_{ij} W_l$. Then, $i_{[W_i,W_j]}\omega_0 = 0$ since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$. So, the canonical writing of a form $\omega \in B^*$ is (8) for some $\omega_0 \in \left\{\eta \in \Omega^*_{\overline{p}}(N/\mathcal{N}) \;/\; L_{W_i}\eta = 0 \text{ if } b < i \leq c\right\} = \left(\Omega^*_{\overline{p}}(N/\mathcal{N})\right)^{H_0}$.

Then, the operator $\Delta\colon B^* \longrightarrow \left(\Omega^*_{\overline{p}}(N/\mathcal{N})\right)^{H_0}$, defined by $\Delta(\omega) = \omega_0$, is a differential isomorphism. We conclude that the differential complex $A^*$ is isomorphic to $\bigwedge^*(\gamma_{c+1},\ldots,\gamma_f)\otimes\left(\Omega^*_{\overline{p}}(N/\mathcal{N})\right)^{H_0}$ and therefore $H^*(A^\cdot) \cong H^*(K/\mathcal{E})\otimes \boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})$ (cf. (5) and (6)). Since the operator $\Delta$ is $(H/H_0)$-equivariant (cf. 3.3) then we get

$$\boldsymbol{H}^*_{\overline{p}}(K\times_H N/\mathcal{W}) \;=\; H^*(\mathrm{Im}\,\Pi^*) = \left(H^*(A^\cdot)\right)^{H/H_0} = \left(H^*(K/\mathcal{E})\otimes\boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0}.$$

This ends the proof. ♣

### 3.5 Remarks.

(a) When the Lie group $G$ is commutative then $K$ is also commutative. Differential forms $\gamma_\bullet$ are $K$-invariants on the left and on the right, so $\left(H^*(K/\mathcal{E})\right)^H = H^*(K/\mathcal{E})$ and therefore

$$\boldsymbol{H}^*_{\overline{p}}(K\times_H N/\mathcal{W}) = H^*(K/\mathcal{E})\otimes\left(\boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0} = H^*(K/\mathcal{E})\otimes\left(\boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})\right)^{H}$$

as it has been proved in [11, Proposition 3.8.4].

(b) Since the foliation $\mathcal{E}$ is a riemannian foliation defined on a compact manifold then we know that the cohomology $H^*(K/\mathcal{E})$ is finite (cf. [4]). So, the finiteness of $\boldsymbol{H}^*_{\overline{p}}(K\times_H N/\mathcal{W})$ depends on the finiteness of $\boldsymbol{H}^*_{\overline{p}}(N/\mathcal{N})$.

# 4 Finiteness of the BIC

We prove in this section that the BIC of a Killing foliation on a compact manifold is finite dimensional. First of all, we present two geometrical tools we shall use in the proof: the isotropy type stratification and the Molino's blow up.

We fix an isometric action $\Phi\colon G\times M \to M$ on the compact manifold $M$. We denote by $\mathcal{F}$ the induced Killing foliation. For the study of $\mathcal{F}$ we can suppose that $G$ is connected (see Lemma 1.1.1). We fix $K$ a tamer group. Notice that the group $G$ is normal in $K$ and the quotient $K/G$ is commutative (cf. Proposition 1.1.2).

**4.1 Isotropy type stratification.** The *isotropy type stratification* $\mathsf{S}_{K,M}$ of $M$ is defined by the equivalence relation[8]:

$$x \sim y \Leftrightarrow K_x \text{ is conjugated to } K_y.$$

When depth $\mathsf{S}_{K,M} > 0$, any closed stratum $S \in \mathsf{S}_{K,M}$ is a $K$-invariant submanifold of $M$ and then it possesses a $K$-invariant tubular neighborhood $(T, \tau, S, \mathbb{R}^m)$ whose structural group is $O(m)$. Recall that there are the following smooth maps associated with this neighborhood:

+ The *radius map* $\rho\colon T \to [0,1[$ defined fiberwise from the assignation $[x,t] \mapsto t$. Each $t \neq 0$ is a regular value of the $\rho$. The pre-image $\rho^{-1}(0)$ is $S$. This map is $K$-invariant, that is, $\rho(k\cdot z) = \rho(z)$.

+ The *contraction* $H\colon T \times [0,1] \to T$ defined fiberwisely from $([x,t],r) \mapsto [x,rt]$. The restriction $H_t\colon T \to T$ is an embedding for each $t \neq 0$ and $H_0 \equiv \tau$. We shall write $H(z,t) = t\cdot z$. This map is $K$-invariant, that is, $t\cdot(k\cdot z) = k\cdot(t\cdot z)$.

The hyper-surface $D = \rho^{-1}(1/2)$ is the *tube* of the tubular neighborhood. It is a $K$-invariant submanifold of $T$. Notice that the map

$$\nabla\colon D\times[0,1[\longrightarrow T,$$

defined by $\nabla(z,t) = (2t)\cdot z$ is a $K$-equivariant smooth map, where $K$ acts trivially on the $[0,1[$-factor. Its restriction $\nabla\colon D\times]0,1[\longrightarrow T\backslash S$ is a $K$-equivariant diffeomorphism.

Denote $S_{min}$ the union of closed (minimal) strata and choose $T_{min}$ a disjoint family of $K$-invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by $D_{min}$. Notice that the induced map $\nabla_{min}\colon D_{min}\times]0,1[\longrightarrow T_{min}\backslash S_{min}$ is a $K$-equivariant diffeomorphism.

**4.2 Molino's blow up**. The Molino' blow up [7] of the foliation $\mathcal{F}$ produces a new foliation $\widehat{\mathcal{F}}$ of the same kind but of smaller depth. We suppose depth $\mathsf{S}_{K,M} > 0$. The *blow up* of $M$ is the compact manifold

$$\widehat{M} = \left\{\left(D_{min}\times]-1,1[\right)\coprod\left((M\backslash S_{min})\times\{-1,1\}\right)\right\}\Big/\sim,$$

where $(z,t) \sim (\nabla_{min}(z,|t|), t/|t|)$, and the map $\mathcal{L}\colon \widehat{M} \longrightarrow M$ defined by

$$\mathcal{L}(v) = \begin{cases} \nabla_{min}(z,|t|) & \text{if } v = (z,t) \in D_{min}\times]-1,1[ \\ z & \text{if } v = (z,j) \in (M\backslash S_{min})\times\{-1,1\}. \end{cases}$$

Notice that $\mathcal{L}$ is a continuous map whose restriction $\mathcal{L}\colon \widehat{M}\backslash\mathcal{L}^{-1}(S_{min}) \to M\backslash S_{min}$ is a $K$-equivariant smooth trivial 2-covering.

[8] For notions related with compact Lie group actions, we refer the reader to [1].

Since the map $\nabla_{min}$ is $K$-equivariant then $\Phi$ induces the action $\widehat{\Phi}\colon K\times\widehat{M}\longrightarrow\widehat{M}$ by saying that the blow-up $\mathcal{L}$ is $K$-equivariant. The open submanifolds $\mathcal{L}^{-1}(T_{min})$ and $\mathcal{L}^{-1}(T_{min}\backslash S_{min})$ are clearly $K$-diffeomorphic to $D_{min}\times]-1,1[$ and $D_{min}\times(]-1,0[\cup]0,1[)$ respectively.

The restriction $\widehat{\Phi}\colon G\times\widehat{M}\longrightarrow\widehat{M}$ is an isometric action with $K$ as a tamer group. The induced Killing foliation is $\widehat{\mathcal{F}}$. Foliations $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are related by $\mathcal{L}$ which is a foliated map. Moreover, if $S$ is a not minimal stratum of $\mathsf{S}_{K,M}$ then there exists an unique stratum $S'\in\mathsf{S}_{K,\widehat{M}}$ such that $\mathcal{L}^{-1}(S)\subset S'$. The family $\{S'\ /\ S\in\mathsf{S}_{K,M}\}$ covers $\widehat{M}$ and verifies the relationship: $S_1\prec S_2\Leftrightarrow S'_1\prec S'_2$. We conclude the important property

$$\text{depth}\ \mathsf{S}_{K,\widehat{M}}<\text{depth}\ \mathsf{S}_{K,M}. \tag{9}$$

**4.3 Finiteness of a tubular neighborhood**. We suppose depth $\mathsf{S}_{K,M}>0$. Consider a closed stratum $S\in\mathsf{S}_{K,M}$. Take $(T,\tau,S,\mathbb{R}^m)$ a $K$-invariant tubular neighborhood. We fix a base point $x\in S$. The isotropy subgroup $K_x$ acts orthogonally on the fiber $\mathbb{R}^m=\tau^{-1}(x)$. So, the induced action $\Lambda_x\colon G_x\times\mathbb{R}^m\to\mathbb{R}^m$ is an isometric action, it gives the Killing foliation $\mathcal{N}$ on $\mathbb{R}^m$.

**Proposition 4.3.1** *If the BIC of $(\mathbb{R}^m,\mathcal{N})$ is finite dimensional then the BIC of $(T,\mathcal{F})$ is also finite dimensional.*

*Proof.* We proceed in two steps.

(a) $K_y=K_x$ *for each* $y\in S$.

The canonical projection $\pi\colon S\to S/K$ is an homogeneous bundle with fiber $K/K_x$. For any open subset $V\subset S/K$ the pull back $\tau^{-1}\pi^{-1}(V)$ is a $K$-invariant subset of $T$, then we can apply the Mayer-Vietoris technics to this kind of subsets (cf. 2.5).

Since the manifold $S/K$ is a compact one then we can find a finite good covering $\{U_i\ /\ i\in I\}$ of it (cf. [2]). An inductive argument on the cardinality of $I$ reduces the proof of the Lemma to the case where $T=\tau^{-1}\pi^{-1}(V)$, where $V$ is a contractible open subset of $S/K$.

Here, the manifold $T$ is $K$-equivalently diffeomorphic to $V\times\left(K\times_{K_x}\mathbb{R}^m\right)$, where $K$ does not act on the first factor. So, the natural retraction of $V$ to a point gives a $K$-equivariant retraction of $T$ to the twisted product $K\times_{K_x}\mathbb{R}^m$. Now the result comes directly from 3.5(b) since $(K,G,K_x)$ is a trio.

(b) General case.

The stratum $S$ is $K$-equivariantly diffeomorphic to the twisted product $K\times_{N(K_x)}F$ where $N(K_x)$ is the normalizer of $K_x$ on $K$ and $F=S^{K_x}$. So, the tubular neighborhood $T$ is $K$-equivariantly diffeomorphic to the twisted product $K\times_{N(H)}N$ where $N$ is the manifold $\tau^{-1}(F)$. The previous case gives that the BIC of $(N,\mathcal{F}_N)$ is finite dimensional. Now the result comes directly from 3.5(b) since $(K,G,N(K_x))$ is a trio. ♣

The main result of this work is the following

**Theorem 4.4** *The BIC of the foliation determined by an isometric action on a compact manifold is finite dimensional.*

*Proof.* Let $\mathcal{F}$ be a Killing foliation defined on a compact manifold $M$ induced by an isometric action $\Phi\colon G\times M\to M$ where $G$ is a Lie group. Without loss of generality we can suppose that the Lie group $G$ is a connected one (cf. Lemma 1.1.1). We fix a tamer group $K$. We know that $G$ is normal in $K$ and the quotient group $K/G$ is commutative (cf. Proposition 1.1.2).

Let us consider the following statement

$$\mathfrak{A}(U,\mathcal{F}) = \text{“The BIC } \boldsymbol{H}^*_{\overline{p}}(U/\mathcal{F}) \text{ is finite dimensional for each perversity } \overline{p}\text{,”}$$

where $U \subset M$ is a $K$-invariant submanifold. We prove $\mathfrak{A}(M,\mathcal{F})$ by induction on $\dim M$. The result is clear when $\dim M = 0$. We suppose $\mathfrak{A}(W,\mathcal{F})$ for any $K$-invariant compact submanifold $W$ of $M$ with $\dim W < \dim M$ and we prove $\mathfrak{A}(M,\mathcal{F})$. We proceed in several steps.

**First step: 0-depth.** Let us suppose depth $\mathsf{S}_{K,M} = 0$. Since $G \triangleleft K$ and $K_x$ is conjugated to $K_y$ then $G_x$ is conjugated to $G_y$, $\forall x, y \in M$. We get that the foliation $\mathcal{F}$ is a (regular) riemannian foliation (cf. [7]). Its BIC is just the basic cohomology (cf. 2.3). Then $\mathfrak{A}(M,\mathcal{F})$ comes from [4].

**Second step: Inside $M$.** Let us suppose depth $\mathsf{S}_{K,M} > 0$. The family $\{M \backslash S_{min}, T_{min}\}$ is a basic covering of $M$ and the we get the exact sequence (cf. 2.5)

$$0 \to \Omega^*_{\overline{p}}(M/\mathcal{F}) \to \Omega^*_{\overline{p}}((M\backslash S_{min})/\mathcal{F}) \oplus \Omega^*_{\overline{p}}(T_{min}/\mathcal{F}) \to \Omega^*_{\overline{p}}((T_{min}\backslash S_{min})/\mathcal{F}) \to 0.$$

The Five Lemma gives

$$\mathfrak{A}(T_{min}\backslash S_{min},\mathcal{F}),\ \ \mathfrak{A}(T_{min},\mathcal{F}) \ \text{ and } \ \mathfrak{A}(M\backslash S_{min},\mathcal{F}) \Longrightarrow \mathfrak{A}(M,\mathcal{F}).$$

Since $T_{min}\backslash S_{min}$ is $K$-diffeomorphic to $D_{min}\times ]0,1[$ (cf. (cf. 4.1)) then $\mathfrak{A}(D_{min},\mathcal{F}) \Longrightarrow \mathfrak{A}(T_{min}\backslash S_{min},\mathcal{F})$. The inequality $\dim D_{min} < \dim M$ gives

$$\mathfrak{A}(T_{min},\mathcal{F}) \ \text{ and } \ \mathfrak{A}(M\backslash S_{min},\mathcal{F}) \Longrightarrow \mathfrak{A}(M,\mathcal{F}).$$

In order to prove $\mathfrak{A}(T_{min},\mathcal{F})$ it suffices to prove $\mathfrak{A}(T,\mathcal{F})$ where $(T,\tau,S,\mathbb{R}^m)$ a $K$-invariant tubular neighborhood of closed stratum $S$ of $\mathsf{S}_{K,M}$. Following Proposition 4.3.1 we have

$$\mathfrak{A}(\mathbb{R}^m,\mathcal{N}) \Longrightarrow \mathfrak{A}(T,\mathcal{F}) \Longrightarrow \mathfrak{A}(T_{min},\mathcal{F}).$$

Consider the orthogonal decomposition $\mathbb{R}^m = \mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$, where $\mathbb{R}^{m_1} = (\mathbb{R}^m)^{G_x}$. The only fixed point of the restriction $\Lambda_x \colon G_x \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ is the origin. So, there exists a Killing foliation[9] $\mathcal{G}$ on the sphere $\mathbb{S}^{m_2-1}$ with $(\mathbb{R}^{m_1}\times\mathbb{R}^{m_2},\mathcal{F}) = (\mathbb{R}^{m_1}\times c\mathbb{S}^{m_2-1},\mathcal{I}\times c\mathcal{G})$. Propositions 2.6 and 2.7 give:

$$\mathfrak{A}(\mathbb{S}^{m_2-1},\mathcal{G}) \Longrightarrow \mathfrak{A}(\mathbb{R}^{m_1}\times c\mathbb{S}^{m_2-1},\mathcal{I}\times c\mathcal{G}) \Longrightarrow \mathfrak{A}(\mathbb{R}^m,\mathcal{N}).$$

Finally, since $\dim \mathbb{S}^{m_2-1} < m \leq \dim T \leq \dim M$ we have

$$\mathfrak{A}(M\backslash S_{min},\mathcal{F}) \Longrightarrow \mathfrak{A}(M,\mathcal{F}). \tag{10}$$

**Third step: Blow-up.** Let us suppose depth $\mathsf{S}_{K,M} > 0$. The family $\left\{\mathcal{L}^{-1}(M\backslash S_{min}),\mathcal{L}^{-1}(T_{min})\right\}$ is a basic covering of $\widehat{M}$ and the we get the exact sequence (cf. 2.5)

$$0 \to \Omega^*_{\overline{p}}\left(\widehat{M}/\widehat{\mathcal{F}}\right) \to \Omega^*_{\overline{p}}\left(\mathcal{L}^{-1}(M\backslash S_{min})/\widehat{\mathcal{F}}\right) \oplus \Omega^*_{\overline{p}}\left(\mathcal{L}^{-1}(T_{min})/\widehat{\mathcal{F}}\right) \to \Omega^*_{\overline{p}}\left(\mathcal{L}^{-1}(T_{min}\backslash S_{min})/\widehat{\mathcal{F}}\right) \to 0.$$

Following 4.2 we have that

- $\mathcal{L}^{-1}(M\backslash S_{min})$ is $K$-diffeomorphic to two copies of $M\backslash S_{min}$,
- $\mathcal{L}^{-1}(T_{min})$ is $K$-diffeomorphic to $D_{min}\times ]-1,1[$,
- $\mathcal{L}^{-1}(T_{min}\backslash S_{min})$ is $K$-diffeomorphic to $D_{min}\times (]-1,0[\cup]0,1[)$.

[9] It is given by the orthogonal action $\Lambda_x \colon G_x \times \mathbb{S}^{m_2-1} \to \mathbb{S}^{m_2-1}$.

Now, the Five Lemma gives

$$\mathfrak{A}(D_{min}, \widehat{\mathcal{F}}) \text{ and } \mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M \backslash S_{min}, \mathcal{F}).$$

But, the inequality $\dim D_{min} < \dim M$ gives

$$\mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M \backslash S_{min}, \mathcal{F}). \tag{11}$$

**Forth step: Final blow-up.** When depth $\mathsf{S}_{K,M} = 0$ we get $\mathfrak{A}(M, \mathcal{F})$ from the First step. Let us suppose depth $\mathsf{S}_{K,M} > 0$. From (10) and (11) we get

$$\mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M, \mathcal{F}).$$

with depth $\mathsf{S}_{K,\widehat{M}} <$ depth $\mathsf{S}_{K,M}$ (cf. (9)). By iterating this procedure we get

$$\mathfrak{A}\left(\widetilde{M}, \widetilde{\mathcal{F}}\right) = \mathfrak{A}\left(\widehat{\widehat{\overset{\cdots}{M}}}, \widehat{\widehat{\overset{\cdots}{\mathcal{F}}}}\right) \Longrightarrow \cdots \Longrightarrow \mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M, \mathcal{F}),$$

with depth $\mathsf{S}_{K,\widetilde{M}} = 0$. We finish the proof by applying again the First Step. ♣

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