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# WEIGHTED COMPOSITION OPERATORS AS DAUGAVET CENTERS

R. Demazeux

## Abstract

We investigate the norm identity  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$  for classes of operators on  $C(S)$ , where  $S$  is a compact Hausdorff space without isolated point, and characterize those weighted composition operators which satisfy this equation for every weakly compact operator  $T : C(S) \rightarrow C(S)$ . We also give a characterization of such weighted composition operator acting on the disk algebra  $A(\mathbb{D})$ .

## 1 Introduction

In 1963, Daugavet proved [3] the norm equality

$$\|I + T\| = 1 + \|T\| \tag{1.1}$$

now known as the Daugavet equation for every compact operator  $T : C([0, 1]) \rightarrow C([0, 1])$ . Over the years, this property was extended to larger classes of operators and various spaces :  $C(S)$  where  $S$  is a compact Hausdorff space without isolated point [5],  $L^1(\mu)$  for measure  $\mu$  without atoms [9], the disk algebra  $A(\mathbb{D})$  or the Hardy space  $H^\infty$  [15]. Actually, if (1.1) holds for every rank-1 operators on  $X$ , then it holds for every weakly compact operators,  $X$  contains a copy of  $\ell_1$ ,  $X$  cannot have an unconditional basis and even cannot embed into a space having an unconditional basis (see the survey [13]).

Recently in [10], the author showed that if we substitute the Identity in (1.1) with an into isometry  $J : L^1[0, 1] \rightarrow L^1[0, 1]$  then equation  $\|J + T\| = 1 + \|T\|$  holds for narrow operators, and particularly for weakly compact operators on  $L^1[0, 1]$ . In arbitrary Banach spaces, this has been investigated by T. Bosenko and V. Kadets in [2]. They introduced the following concept :

**Definition 1.1.** *Let  $X$  be a Banach space. A linear continuous operator  $G : X \rightarrow X$  is said to be a Daugavet center if the norm identity*

$$\|G + T\| = \|G\| + \|T\| \tag{1.2}$$

*holds for every rank-1 operator  $T : X \rightarrow X$ .*

Bosenko and Kadets showed that if  $G : X \rightarrow X$  is a non zero Daugavet center then equation (1.2) holds true for every strong Radon-Nikodým operator on  $X$ , and so for weakly compact operators on  $X$ . Moreover  $G$  fixes a copy of  $\ell_1$ , and  $X$  cannot have an

unconditional basis, merely  $X$  cannot be embedded into a space having an unconditional basis.

In section 2 of this paper, we give a characterization of weighted composition operators on  $C(S)$  which are Daugavet centers ( $S$  compact Hausdorff space without isolated point). We give examples of Daugavet centers which are not weighted composition operators and prove that the set of Daugavet centers in  $C(S)$  is not convex. We also study equation (1.2) for the class of operators whose adjoint has separable range. This encompass the class of operators factorizing through  $c_0$ .

In section 3, we adapt D. Werner's method showing that certain function spaces have the Daugavet property (meaning that the identity operator is a Daugavet center) to characterize weighted composition operators on the disk algebra  $A(\mathbb{D})$  which are Daugavet centers.

## 2 Weighted composition operators as Daugavet center in $C(S)$

Let  $S$  denotes a compact Hausdorff space without isolated point. Considering continuous maps  $\varphi : S \rightarrow S$  and  $u : S \rightarrow \mathbb{C}$ , we study the weighted composition operator  $uC_\varphi : C(S) \rightarrow C(S)$  defined by  $uC_\varphi(f) = u \cdot (f \circ \varphi)$  for all  $f \in C(S)$ . We clearly have  $\|uC_\varphi\| = \|u\|_\infty$ . We investigate the following equation :

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\| \quad (E_{u,\varphi})$$

Note that if we take  $\varphi(s) = s$  (for all  $s$  in  $S$ ) and  $u$  the constant function equal to 1, the previous equation becomes the classical Daugavet equation. We will suppose that  $u$  is not the constant function equal to zero. We want to find conditions on  $\varphi$  and  $u$  implying that every weakly compact operators on  $C(S)$  satisfies equation  $(E_{u,\varphi})$ . A first remark is that  $u$  and  $\varphi$  must be such that the operator  $uC_\varphi$  is not itself compact. By a result of Kamowitz [7],  $uC_\varphi$  is compact if and only if  $\varphi$  is constant on a neighborhood of each connected component of the set where  $u$  is nonzero. So  $\varphi$  should be non constant over at least one nonempty open set in  $S$ .

The main result of this section is the following theorem :

**Theorem 2.1.** *Let  $S$  be a compact Hausdorff space without isolated point,  $u \in C(S)$  and  $\varphi$  be a continuous function from  $S$  to  $S$ .*

*Then every weakly compact operators  $T : C(S) \rightarrow C(S)$  satisfies equation  $(E_{u,\varphi})$  if and only if  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ , for every  $t \in S$  and  $|u|$  is constant on  $S$ .*

The straightforward direction was already proved in [2] for rank-1 operators and for  $u \equiv 1$ . Here we give a direct and simple proof for weakly compact operators, and we check that conditions on  $\varphi$  and  $u$  are necessary.

We first begin with some notations and terminology. The dual space of  $C(S)$  consisting of all regular borel measures on  $S$  of finite variation will be denoted by  $M(S)$ . If  $s \in S$ , we define the corresponding Dirac functional  $\delta_s$  by  $\delta_s(f) = f(s)$  for every  $f \in C(S)$ . Then  $\delta_s \in M(S)$  and  $\|\delta_s\| = 1$ .

Following [12], the key idea is to represent an operator  $T : C(S) \rightarrow C(S)$  by the family of measures  $(\mu_s)_{s \in S}$  defined by  $\mu_s = T^*(\delta_s)$ , such that :

$$(Tf)(s) = \langle Tf, \delta_s \rangle = \langle f, \mu_s \rangle = \int_S f \, d\mu_s.$$

Thus, the (weakly) compact nature of  $T$  is reformulated in terms of continuity of the map  $s \mapsto \mu_s$  in the following (c.f. [4], Th. VI, 7.1) :

**Lemma 2.2.** *Let  $T : C(S) \rightarrow C(S)$  be an operator and  $(\mu_s)_{s \in S}$  be the family of measures associated to  $T$ . Then :*

- i)  $s \mapsto \mu_s$  is continuous from  $S$  to  $M(S) = C(S)^*$  endowed with the weak\*-topology, i.e.  $\sigma(M(S), C(S))$ .*
- ii)  $T$  is weakly compact if and only if  $s \mapsto \mu_s$  is continuous for the weak-topology on  $M(S)$ , i.e. for  $\sigma(M(S), M(S)^*)$ .*
- iii)  $T$  is compact if and only if  $s \mapsto \mu_s$  is continuous for the norm topology on  $M(S)$ .*

Note that  $\|T\| = \sup_{s \in S} \|\mu_s\|$ , and that the operator  $uC_\varphi$  is represented by the family of measures  $(u(s)\delta_{\varphi(s)})_{s \in S}$ . Indeed :

$$(uC_\varphi)^*(\delta_s)(f) = \delta_s(u \cdot f \circ \varphi) = u(s)f(\varphi(s)) = u(s)\delta_{\varphi(s)}(f).$$

The following proposition shows, assuming  $|u|$  is constant, that for every operator  $T$  on  $C(S)$ , there is a  $\lambda \in \mathbb{T}$  such that  $\lambda T$  satisfies equation  $(E_{u,\varphi})$ .

**Proposition 2.3.** *Let  $S$  be a compact Hausdorff space, and  $T : C(S) \rightarrow C(S)$  be an operator. Assume that  $|u|$  is constant. Then*

$$\max_{\lambda \in \mathbb{T}} \|uC_\varphi + \lambda T\| = \|u\|_\infty + \|T\|.$$

**Proof.** Let  $(\mu_s)_{s \in S}$  be the family of measures associated to  $T$ . Then

$$\begin{aligned} \max_{\lambda \in \mathbb{T}} \|uC_\varphi + \lambda T\| &= \max_{\lambda \in \mathbb{T}} \sup_{s \in S} \|u(s)\delta_{\varphi(s)} + \lambda\mu_s\| \\ &= \sup_{s \in S} \max_{\lambda \in \mathbb{T}} \left( |u(s)\delta_{\varphi(s)} + \lambda\mu_s|(\{\varphi(s)\}) + |u(s)\delta_{\varphi(s)} + \lambda\mu_s|(S \setminus \{\varphi(s)\}) \right) \\ &= \sup_{s \in S} \max_{\lambda \in \mathbb{T}} \left( |u(s) + \lambda\mu_s(\{\varphi(s)\})| + |\mu_s|(S \setminus \{\varphi(s)\}) \right) \\ &= \sup_{s \in S} \left( |u(s)| + |\mu_s(\{\varphi(s)\})| + |\mu_s|(S \setminus \{\varphi(s)\}) \right) \\ &= \sup_{s \in S} (\|u\|_\infty + \|\mu_s\|) \quad \text{since } |u(s)| = \|u\|_\infty \\ &= \|u\|_\infty + \|T\|. \end{aligned}$$

■

**Remark 2.4.** i) In the real case, a similar result holds replacing “ $\lambda \in \mathbb{T}$ ” by “ $\lambda \in \{\pm 1\}$ ”.

ii) Without assumption on the modulus of  $u$ , the previous result is not true anymore. For instance, taking  $v \in C(S)$ , we have that  $\max_{\lambda \in \mathbb{T}} \|uC_\varphi + \lambda vC_\psi\| = \| |u| + |v| \|_\infty$  which is not equal to  $\|u\|_\infty + \|v\|_\infty$  in general.

## 2.1 Equation $(E_{u,\varphi})$ for weakly compact operators on $C(S)$

Let  $T : C(S) \rightarrow C(S)$  be an operator and  $(\mu_s)_{s \in S}$  be the family of measures associated to  $T$ . Then

$$\|uC_\varphi + T\| = \sup_{s \in S} \|u(s)\delta_{\varphi(s)} + \mu_s\| = \sup_{s \in S} \left( |u(s) + \mu_s(\{\varphi(s)\})| + |\mu_s|(S \setminus \{\varphi(s)\}) \right)$$

and

$$\|u\|_\infty + \|T\| = \sup_{s \in S} (\|u\|_\infty + \|\mu_s\|) = \sup_{s \in S} \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| + |\mu_s|(S \setminus \{\varphi(s)\}) \right).$$

We have the following lemma which gives a characterization of the operators satisfying equation  $(E_{u,\varphi})$  :

**Lemma 2.5.**

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$$

if and only if

$$\sup_{\{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}} \left( |u(s) + \mu_s(\{\varphi(s)\})| - (\|u\|_\infty + |\mu_s(\{\varphi(s)\})|) \right) = 0 \quad (2.1)$$

for all  $\varepsilon > 0$ .

**Proof.** Sufficient condition : let  $\varepsilon > 0$  and  $U = \{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}$  which is not empty. Then :

$$\begin{aligned} \|uC_\varphi + T\| &\geq \sup_{s \in U} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\ &\geq \sup_{s \in U} \left( |u(s) + \mu_s(\{\varphi(s)\})| + |\mu_s|(S \setminus \{\varphi(s)\}) \right) \\ &\geq \sup_{s \in U} \left( |u(s) + \mu_s(\{\varphi(s)\})| + \|u\|_\infty + \|\mu_s\| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) \right) \\ &\geq \|u\|_\infty + \|T\| - \varepsilon + \sup_{s \in U} \left( |u(s) + \mu_s(\{\varphi(s)\})| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) \right) \\ &\geq \|u\|_\infty + \|T\| - \varepsilon \quad (\text{with (2.1)}). \end{aligned}$$

Necessary condition : let us assume that there exist  $\alpha$  and  $\varepsilon > 0$  such that for all  $s \in S$ ,  $\|\mu_s\| > \|T\| - \varepsilon$  implies

$$|u(s) + \mu_s(\{\varphi(s)\})| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) < -\alpha < 0.$$

Then

$$\begin{aligned} \|uC_\varphi + T\| &= \sup_{s \in S} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\ &= \max \left( \sup_{\{s \mid \|\mu_s\| > \|T\| - \varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\|, \sup_{\{s \mid \|\mu_s\| \leq \|T\| - \varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\| \right). \end{aligned}$$

The second term is lower than  $\|u\|_\infty + \|T\| - \varepsilon$ . For the first term, we write as before

$$\begin{aligned} &\sup_{\{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\ &= \sup_{\{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}} \left( |u(s) + \mu_s(\{\varphi(s)\})| + |\mu_s(S \setminus \{\varphi(s)\})| \right) \\ &= \sup_{\{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}} \left( |u(s) + \mu_s(\{\varphi(s)\})| + \|u\|_\infty + \|\mu_s\| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) \right) \\ &\leq \|u\|_\infty + \|T\| + \sup_{\{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}} \left( |u(s) + \mu_s(\{\varphi(s)\})| - (\|u\|_\infty + |\mu_s(\{\varphi(s)\})|) \right) \\ &\leq \|u\|_\infty + \|T\| - \alpha. \end{aligned}$$

Thus  $\|uC_\varphi + T\| < \|u\|_\infty + \|T\| - \min(\varepsilon, \alpha) < \|u\|_\infty + \|T\|$ , which leads to a contradiction.  $\blacksquare$

As a consequence, we state the following useful corollary :

**Corollary 2.6.** *Assume that the family  $(\mu_s)_{s \in S}$  satisfies the following condition : for every nonempty open set  $U \subset S$ ,*

$$\sup_{s \in U} \left( |u(s) + \mu_s(\{\varphi(s)\})| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) \right) = 0. \quad (2.2)$$

Then

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

**Proof.** Take  $\varepsilon > 0$ , and call  $U = \{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}$ . Thanks to Lemma 2.5, we only need to show that  $U$  is a nonempty open subset of  $S$ . It is clear that  $U$  is nonempty. Take  $s_0 \in U$ . There exists  $f_0 \in C(S)$ ,  $\|f_0\|_\infty \leq 1$  such that  $|\mu_{s_0}(f_0)| > \|T\| - \varepsilon$ . From Lemma 2.2, we know that  $s \mapsto \mu_s$  is continuous for the weak\*-topology on  $M(S)$ , hence  $s \mapsto \mu_s(f_0)$  is continuous. Then  $V = \{s \in S \mid |\mu_s(f_0)| > \|T\| - \varepsilon\}$  is an open neighborhood of  $s_0$  contained in  $U$ . So  $U$  is a nonempty open subset of  $S$ .  $\blacksquare$

Now we can show a first result dealing with weakly compact operators. The following theorem gives sufficient conditions on  $u$  and  $\varphi$  implying that every weakly compact operator on  $C(S)$  satisfies equation  $(E_{u,\varphi})$ .

**Theorem 2.7.** *Let  $S$  be a compact Hausdorff space (without isolated point). Assume that  $|u|$  is constant on  $S$  and  $\varphi(U)$  is infinite for every nonempty open subset  $U$  of  $S$ . Then  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$  for every weakly compact operator  $T : C(S) \rightarrow C(S)$ .*

Note that condition on  $\varphi$  forces  $S$  to have no isolated point.

**Proof.** Assume that  $\varphi(U)$  is infinite for every nonempty open subset  $U$  of  $S$  and that  $|u|$  is constant. If the family of measures  $(\mu_s)_{s \in S}$  representing  $T$  does not satisfy (2.2) of Corollary 2.6, then there exist a nonempty open set  $U \subset S$  and  $\beta > 0$  such that

$$|u(s) + \mu_s(\{\varphi(s)\})| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) < -2\beta \quad \forall s \in U.$$

In particular we have, since  $|u(s)| = \|u\|_\infty$  for all  $s \in S$  :

$$\begin{aligned} |\mu_s(\{\varphi(s)\})| &> 2\beta - \|u\|_\infty + |u(s) + \mu_s(\{\varphi(s)\})| \\ &\geq 2\beta - \|u\|_\infty + |u(s)| - |\mu_s(\{\varphi(s)\})| \\ &= 2\beta - |\mu_s(\{\varphi(s)\})| \end{aligned}$$

which gives

$$|\mu_s(\{\varphi(s)\})| > \beta, \quad \text{for all } s \in U.$$

Take  $t \in S$ . Then  $s \in S \mapsto \mu_s(\{t\}) \in \mathbb{C}$  is continuous. Indeed : from Lemma 2.2,  $s \mapsto \mu_s$  is continuous for the weak-topology on  $M(S)$ . Since  $\mu \mapsto \mu(\{t\})$  belongs to  $M(S)^*$ , it is continuous on  $M(S)$  endowed with the weak-topology.

Let  $s_0 \in U$ , and define

$$U_1 = \{s \in U \mid |\mu_s(\{\varphi(s_0)\})| > \beta\}.$$

From above,  $U_1$  is an open subset of  $U$  (and so of  $S$ ) which contains  $s_0$ . Since  $\varphi(U_1)$  is infinite, one can find  $s_1$  in  $U_1$  satisfying  $\varphi(s_1) \neq \varphi(s_0)$ . Then we have

$$\begin{aligned} |\mu_{s_1}(\{\varphi(s_1)\})| &> \beta, \quad \text{since } s_1 \in U \\ |\mu_{s_1}(\{\varphi(s_0)\})| &> \beta. \end{aligned}$$

Consider now

$$U_2 = \{s \in U_1 \mid |\mu_s(\{\varphi(s_1)\})| > \beta\}.$$

It is an open subset of  $U$  containing  $s_1$ , and it contains an element  $s_2$  such that  $\varphi(s_2) \neq \varphi(s_0)$  and  $\varphi(s_2) \neq \varphi(s_1)$  (since  $\varphi(U_2)$  is infinite). Then we have, since  $s_2 \in U_2 \subset U_1 \subset U$ ,

$$\begin{aligned} |\mu_{s_2}(\{\varphi(s_2)\})| &> \beta \\ |\mu_{s_2}(\{\varphi(s_1)\})| &> \beta \\ |\mu_{s_2}(\{\varphi(s_0)\})| &> \beta. \end{aligned}$$

In such a way we construct a decreasing sequence of open subsets  $U_n \subset U$ , and a sequence of elements  $(s_n)_{n \geq 0}$ ,  $s_n \in U_n$  having the property

$$\begin{aligned} U_{n+1} &= \{s \in U_n \mid |\mu_s(\{\varphi(s_n)\})| > \beta\}, \\ s_{n+1} &\in U_{n+1} \\ \varphi(s_{n+1}) &\notin \{\varphi(s_0), \dots, \varphi(s_n)\}. \end{aligned}$$

So

$$|\mu_{s_n}(\{\varphi(s_j)\})| > \beta, \quad j = 0, \dots, n-1$$

which leads to a contradiction writing that

$$\|T\| \geq \|\mu_{s_n}\| \geq |\mu_{s_n}|(\{\varphi(s_0), \dots, \varphi(s_{n-1})\}) \geq n\beta, \quad \forall n \in \mathbb{N}.$$

■

We now give necessary conditions on  $\varphi$  and  $u$  to ensure that every weakly compact operator on  $C(S)$  satisfies equation  $(E_{u,\varphi})$ . Actually we only need to consider rank-1 operators.

**Theorem 2.8.** *Let  $S$  be a compact Hausdorff space without isolated point. Assume that every rank-1 operator on  $C(S)$  satisfies equation  $(E_{u,\varphi})$ . Then  $|u|$  is constant and  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ , for every  $t \in S$ .*

**Proof :** We first show that  $|u|$  is constant on  $S$ . Arguing by contradiction, assume there exists  $s_0 \in S$  such that  $|u(s_0)| < \|u\|_\infty$ . Then there exists  $\delta > 0$  and an open neighborhood  $U$  of  $s_0$  satisfying

$$\forall s \in U, |u(s)| < \|u\|_\infty - \delta.$$

Choose a continuous function  $v$  such that  $0 \leq v \leq 1$ ,  $v(s_0) = 1$  and  $v(s) < 1$  for all  $s \neq s_0$ . We define the operator  $T = v\delta_\tau$  where  $\tau$  is an element of  $S$ . Then  $\mu_s = T^*(\delta_s) = v(s)\delta_\tau$ ,  $\|\mu_s\| = v(s)$ . Choose  $\varepsilon > 0$  such that we have  $\{s \in S \mid v(s) > 1 - \varepsilon\} \subset U$ . It follows that

$$\begin{aligned} & \sup_{\{s \mid v(s) > 1 - \varepsilon\}} \left( |u(s) + v(s)\delta_\tau(\{\varphi(s)\})| - \left( \|u\|_\infty + |v(s)|\delta_\tau(\{\varphi(s)\}) \right) \right) \\ & \leq \sup_{s \in U} \left( |u(s)| + v(s)\delta_\tau(\{\varphi(s)\}) - \left( \|u\|_\infty + v(s)\delta_\tau(\{\varphi(s)\}) \right) \right) \\ & \leq \sup_{s \in U} |u(s)| - \|u\|_\infty \\ & \leq -\delta < 0. \end{aligned}$$

The family of measures  $(\mu_s)_{s \in S}$  does not satisfy condition (2.1) of Lemma 2.5 and consequently  $T$  does not satisfy equation  $(E_{u,\varphi})$ , which is false since  $T$  is a rank-1 operator. So  $|u|$  is constant.

Now we prove that for every  $t \in S$ ,  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ . Let  $U$  be a nonempty open subset of  $S$ . We want to find  $s \in U$  such that  $\varphi(s) \neq t$ . Consider the rank-1 operator  $T = \delta_t g u$ , where  $g \in C(S)$  such that  $-1 \leq g \leq -\frac{1}{2}$ ,  $g = -\frac{1}{2}$  outside  $U$  and  $\|g\|_\infty = 1$ . Then  $\|T\| = \|u\|_\infty > 0$ , the family of measures associated to  $T$  is given by  $\mu_s = T^*(\delta_s) = u(s)g(s)\delta_t$ , and  $\|\mu_s\| = |u(s)g(s)| = \|u\|_\infty |g(s)|$ .

Take  $\varepsilon = \|u\|_\infty/2$  so that  $V = \{s \in S \mid |u(s)g(s)| > \frac{\|u\|_\infty}{2}\} \subset U$ . Since  $T$  satisfies equation  $(E_{u,\varphi})$ , the family of measures  $(\mu_s)_{s \in S}$  satisfies condition (2.1) of Lemma 2.5 :

$$\begin{aligned} 0 &= \sup_V \left( |u(s) + u(s)g(s)\delta_t(\{\varphi(s)\})| - \left( \|u\|_\infty + |u(s)g(s)|\delta_t(\{\varphi(s)\}) \right) \right) \\ &= \sup_V \left( 2g(s)\|u\|_\infty \delta_t(\{\varphi(s)\}) \right) \end{aligned}$$



which is less than  $\sup_{s \in U} (2g(s)\|u\|_\infty \delta_t(\{\varphi(s)\}))$ . It follows that there exists  $s \in U$  such that  $\delta_t(\{\varphi(s)\}) = 0$ , i.e.  $\varphi(s) \neq t$  and therefore  $U \not\subset \varphi^{-1}(\{t\})$ . ■

**Remark 2.9.** Note that in a topological space  $S$ , and for a continuous map  $\varphi : S \rightarrow S$ , the following conditions are equivalent :

- i) for every  $t \in S$ ,  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$
- ii) for every nonempty open subset  $U$  of  $S$ ,  $\varphi(U)$  is infinite.

Indeed, if there exists a nonempty open subset  $U$  of  $S$  such that  $U \subset \varphi^{-1}(\{t\})$  then  $\varphi$  is constant on  $U$ , so *ii*)  $\Rightarrow$  *i*). Moreover if  $\varphi(U) = \{s_1, \dots, s_n\}$  for an open subset  $U$  of  $S$ ,  $n \geq 1$ , then

$$\begin{aligned} \{s \in U \mid \varphi(s) = s_1\} &= \{s \in U \mid \varphi(s) \neq s_k, 2 \leq k \leq n\} \\ &= \varphi^{-1}(S \setminus \{s_2, \dots, s_n\}) \cap U. \end{aligned}$$

The set  $S \setminus \{s_2, \dots, s_n\}$  is open in  $S$ , so  $\{s \in U \mid \varphi(s) = s_1\}$  is a nonempty open subset of  $U$  (and of  $S$ ) although by *i*),  $\{s \in S \mid \varphi(s) = s_1\}$  must have empty interior.

The previous remark, Theorem 2.7 and Theorem 2.8 give the following :

**Corollary 2.10.** *Let  $S$  be a compact Hausdorff space without isolated point. Then  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$  for every weakly compact operator  $T : C(S) \rightarrow C(S)$  if and only if  $|u|$  is constant on  $S$  and the set  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ , for every  $t \in S$ .*

**Application : a negative answer to a question of Popov [10]**

Note that if  $\varphi$  is onto and  $|u| = 1$  then  $uC_\varphi$  is an isometry on  $C(S)$ . In [10], Popov shows that every into isometry  $J : L^1([0, 1]) \rightarrow L^1([0, 1])$  is a Daugavet center. He raises the question whether this result is true when we substitute  $L^1([0, 1])$  with a Banach space  $X$  having the Daugavet property. Actually, this is not true for  $X = C(S)$ . To see this, consider any composition operator whose symbol  $\varphi$  is onto and constant on a nonempty open subset of  $S$ . Then  $C_\varphi$  is an isometry but there exists rank-1 operators on  $C(S)$  which does not satisfy equation  $(E_{1,\varphi})$ .

After our work was completed, an example was independently produced in [2]. The authors considered a weighted composition operator  $uC_\varphi : C[0, 1] \rightarrow C[0, 1]$  whose symbol  $\varphi$  is constant on  $]1/2, 1]$  and whose weight has not constant modulus on  $[0, 1]$ .

## 2.2 Convex combinations of composition operators

One can wonder if the set of Daugavet centers is a convex set. Actually it is easy to see that this is not true in full generality. Indeed, consider  $u(x) = e^{2i\pi x}$  and  $v(x) = e^{-2i\pi x}$ ,  $x \in [0, 1]$ . Then  $u, v \in C[0, 1]$ ,  $|u| = |v| = 1$  so  $uI$  and  $vI$  are Daugavet centers in  $C([0, 1])$ , but  $(u(x) + v(x))/2 = \cos 2\pi x$  which has not constant modulus on  $[0, 1]$ . Therefore  $(uI + vI)/2$  is not a Daugavet center in  $C[0, 1]$ . Nevertheless it turns out that a convex combination of particular (non zero) Daugavet centers can be a Daugavet center. Let us consider the case of composition operators.

Note that a convex combination of composition operators is not in general a composition operator itself. Indeed, assume that  $C_\varphi = tC_{\psi_1} + (1-t)C_{\psi_2}$  where  $\varphi, \psi_1, \psi_2$  are continuous functions from  $S$  to  $S$  and  $0 < t < 1$ . Assume that  $\varphi \neq \psi_1$  and take  $s_0$  such that  $\varphi(s_0) \neq \psi_1(s_0)$ . Now consider a open subset  $U$  of  $S$  such that  $\varphi(s_0) \in U$  and  $\psi_1(s_0) \notin U$ . Choose  $f \in C(S)$ ,  $\|f\|_\infty = 1$  satisfying  $f(\varphi(s_0)) = 1$  and  $|f| < 1$  out of  $U$ . Then

$$1 = |f(\varphi(s_0))| \leq t|f(\psi_1(s_0))| + (1-t)|f(\psi_2(s_0))| < 1$$

which leads to a contradiction.

Let  $\varphi$  and  $\psi$  be continuous functions from  $S$  to  $S$ . Assume that  $\varphi \neq \psi$ . Define

$$S_1 = \{s \in S \mid \varphi(s) \neq \psi(s)\}.$$

Then  $S_1$  is a nonempty open subset of  $S$  since  $S$  has no isolated point. Consider convex combinations of  $C_\varphi$  and  $C_\psi$ . For  $t \in [0, 1]$ , we define  $T_t = tC_\varphi + (1-t)C_\psi$ . Point out that  $\|T_t\| = 1$ . For convenience, we note

$$\Delta_T(s) = |t + \mu_s(\{\varphi(s)\})| + |1-t + \mu_s(\{\psi(s)\})| - \left(1 + |\mu_s(\{\varphi(s)\})| + |\mu_s(\{\psi(s)\})|\right),$$

and

$$\tilde{\Delta}_T(s) = |1 + \mu_s(\{\varphi(s)\})| - \left(1 + |\mu_s(\{\varphi(s)\})|\right)$$

where  $(\mu_s)_{s \in S}$  is the family of measures representing  $T$  and  $s \in S$ . As for weighted composition operators, we have the following property :

**Proposition 2.11.** *Let  $T$  be an operator on  $C(S)$ . Assume that the family of measures  $(\mu_s)_{s \in S}$  representing  $T$  satisfies the condition : for every nonempty open set  $U \subset S$  :*

-If  $U \cap S_1 \neq \emptyset$ , then

$$\sup_{s \in U \cap S_1} \Delta_T(s) = 0 \tag{2.3}$$

-If  $U \cap S_1 = \emptyset$ , then

$$\sup_{s \in U} \tilde{\Delta}_T(s) = 0. \tag{2.4}$$

Then the following equation holds true :

$$\|T_t + T\| = 1 + \|T\|.$$

**Proof.** One only has to consider open subsets  $U$  of  $S$  of the form  $U = \{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}$  where  $\varepsilon > 0$ . If  $U \cap S_1 = \emptyset$  then  $\varphi = \psi$  on  $U$  so the proof of Lemma 2.5 tells us that  $\|T_t + T\| \geq \sup_{s \in U} \|\delta_{\varphi(s)} + \mu_s\| \geq 1 + \|T\| - \varepsilon$ . Else,  $\|T_t + T\| \geq \sup_{s \in U \cap S_1} \|t\delta_{\varphi(s)} + (1-t)\delta_{\psi(s)} + \mu_s\|$  which is greater than  $1 + \|T\| - \varepsilon$  using the same method as in the proof of Lemma 2.5.  $\blacksquare$

From this we can deduce that any convex combination of composition operators which are Daugavet centers is still a Daugavet center.

**Theorem 2.12.** *Assume that  $C_\varphi$  and  $C_\psi$  are Daugavet center. Then every weakly compact operator  $T$  on  $C(S)$  satisfies the norm equation*

$$\|tC_\varphi + (1-t)C_\psi + T\| = 1 + \|T\|,$$

for all  $t \in [0, 1]$ .

**Proof.** Take  $t \in [0, 1]$ . Argue by contradiction and assume that the family  $(\mu_s)_{s \in S}$  does not satisfy conditions of Proposition 2.11.

*First case :* Let  $U$  be a nonempty open subset of  $S$  such that  $U \cap S_1 \neq \emptyset$  and (2.3) does not hold. Then there exists  $\beta > 0$  such that

$$|t + \mu_s(\{\varphi(s)\})| + |1-t + \mu_s(\{\psi(s)\})| - (1 + |\mu_s(\{\varphi(s)\})| + |\mu_s(\{\psi(s)\})|) < -4\beta$$

for every  $s \in U \cap S_1$ . Then

$$|\mu_s(\{\varphi(s)\})| + |\mu_s(\{\psi(s)\})| > 2\beta, \quad \forall s \in U \cap S_1.$$

Let

$$\begin{aligned} V_1 &= \{s \in U \cap S_1 \mid |\mu_s(\{\varphi(s)\})| > \beta\} \\ V_2 &= \{s \in U \cap S_1 \mid |\mu_s(\{\psi(s)\})| > \beta\}. \end{aligned}$$

Since  $U \cap S_1 \subset V_1 \cup V_2$ , we can assume without loss of generality that  $V_1$  contains a nonempty open set  $V$ . So  $|\mu_s(\{\varphi(s)\})| > \beta$  for every  $s \in V$ . Then we follow the proof of Theorem 2.7 to obtain a contradiction.

*Second case :* If  $U$  is a nonempty open subset of  $S$  such that  $U \subset S \setminus S_1$  and (2.4) does not hold, then the same proof as in Theorem 2.7 leads to a contradiction.  $\blacksquare$

### 2.3 Operators factorizing through an Asplund space

The aim of this section is to extend a result of Ansari in [1] stating that every operator on  $C(S)$  factorizing through  $c_0$  satisfies the Daugavet equation. Let  $T : C(S) \rightarrow C(S)$  be an operator, where  $S$  is a compact Hausdorff space, and  $(\mu_s)_{s \in S}$  the family of measures associated to  $T$ . Note that if  $\mu_s(\{\varphi(s)\}) = 0$  for all  $s \in S$ , and  $|u|$  is constant, then  $T$  trivially satisfies condition (2.2) of corollary (2.6). Actually, it is sufficient that the measures  $(\mu_s)$  almost satisfies this condition :

Define  $S_\varepsilon = \{s \in S \mid |\mu_s(\{\varphi(s)\})| < \varepsilon\}$ , for each  $\varepsilon > 0$ . We have the following :

**Lemma 2.13.** *If  $|u|$  is constant and if the sets  $S_\varepsilon$  are dense in  $S$ , for every  $\varepsilon > 0$ , then*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

**Proof.** Take  $U$  a nonempty open set in  $S$  and  $\varepsilon > 0$ . By density of  $S_\varepsilon$  in  $S$ , there exists  $s_\varepsilon \in U$  satisfying  $|\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})| < \varepsilon$ , and so

$$\begin{aligned} |u(s_\varepsilon) + \mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})| - (\|u\|_\infty + |\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})|) &\geq -2|\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})| \\ &> -2\varepsilon. \end{aligned}$$

Thus for every nonempty open set  $U$  of  $S$ , we have

$$\sup_{s \in U} \left( |u(s) + \mu_s(\{\varphi(s)\})| - \left( \|u\|_\infty + |\mu_s(\{\varphi(s)\})| \right) \right) = 0.$$

We conclude with corollary (2.6). ■

Now we can prove the following result :

**Theorem 2.14.** *Let  $S$  be a compact Hausdorff space without isolated point,  $\varphi : S \rightarrow S$  a continuous map,  $u \in C(S)$  and  $T : C(S) \rightarrow C(S)$  an operator such that  $T^*(M(S))$  is separable. If  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ , for every  $t \in S$ , and if  $|u|$  is constant, then  $T$  satisfies equation  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ .*

**Proof.** Let  $\{\rho_n, n \in \mathbb{N}\}$  be a dense subset of  $T^*(M(S))$ . As previously,  $S_\varepsilon = \{s \in S \mid |\mu_s(\{\varphi(s)\})| < \varepsilon\}$ , where  $\mu_s = T^*(\delta_s)$ , and  $A = \bigcap_{n \geq 0} \{s \in S \mid \rho_n(\{\varphi(s)\}) = 0\}$ . We

want to show that :

- i)  $A$  is dense in  $S$ .
- ii)  $\forall \varepsilon > 0, A \subset S_\varepsilon$ .

Then we conclude with Lemma 2.13.

To prove *i*), we are going to show that  $S \setminus A$  is nowhere dense. Indeed,

$$S \setminus A = \bigcup_{n \geq 0} \{s \in S \mid \rho_n(\{\varphi(s)\}) \neq 0\} = \bigcup_{n \geq 0} \bigcup_{p \geq 1} A_{n,p}$$

where  $A_{n,p} = \{s \in S \mid |\rho_n(\{\varphi(s)\})| > \frac{1}{p}\}$ . Since  $\rho_n$  is a finite measure, this implies that the sets  $\varphi(A_{n,p})$  are finite (hence closed) for every  $n \geq 0, p \geq 1$ . But  $A_{n,p} \subset \varphi^{-1}(\varphi(A_{n,p}))$  which is a finite union of nowhere dense sets (c.f. Remark 2.9). Using Baire's theorem,  $S \setminus A$  is contained in a nowhere dense set, and  $A$  is dense in  $S$ .

Proof of *ii*) : let  $s \in S$  and  $\varepsilon > 0$ . By density of  $(\rho_n)_n$  in  $T^*(M(S))$ , there exists an integer  $n_0 \geq 0$  such that  $\|T^*(\delta_s) - \rho_{n_0}\| < \varepsilon$ . Then  $|\mu_s - \rho_{n_0}|(\{\varphi(s)\}) < \varepsilon$ . Taking  $s \in A$ , it follows that  $|\mu_s(\{\varphi(s)\})| < \varepsilon$ , i.e.  $A \subset S_\varepsilon$ . ■

If  $T : C(S) \rightarrow C(S)$  factorizes through a space  $X$  having a separable dual, then Theorem 2.14 applies. In particular this holds for the class of operators factorizing through  $c_0$ . Actually, regarding operators factorizing through a space  $X$ , one does not need to assume that  $X^*$  is separable in the case where  $S$  is metrizable. We recall the following definition :

**Definition 2.15.** *A Banach space  $X$  is called an Asplund space if its dual space has the Radon-Nikodým property.*

Every dual space which is separable has the Radon-Nikodým property, and so every Banach space with separable dual is Asplund. Asplund spaces are characterized by the fact that every separable subspace has a separable dual.

**Corollary 2.16.** *Let  $S$  be a metric compact space without isolated points,  $\varphi : S \rightarrow S$  a continuous map,  $u \in C(S)$  and  $T : C(S) \rightarrow C(S)$  an operator factorizing through an Asplund space  $X$ . If  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ , for every  $t \in S$ , and if  $|u|$  is constant on  $S$ , then  $T$  satisfies equation  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ .*

**Proof.** Write  $T = T_2T_1$  with  $T_1 : C(S) \rightarrow X$  and  $T_2 : X \rightarrow C(S)$ . Since  $S$  is a metric compact space,  $C(S)$  is separable. So we can assume, by replacing  $X$  by  $\overline{T_1(C(S))}$  that  $X^*$  is separable. Thus  $T^*(M(S))$  is separable, and the result follows from Theorem 2.14.  $\blacksquare$

**Remark 2.17.** Since every compact operator factorizes through a subspace of  $c_0$ , this gives another proof of Theorem 2.7 for compact operators on  $C(S)$ . Moreover every weakly compact operator factorizes through a reflexive space (which is Asplund), giving another proof of Theorem 2.7 for weakly compact operators on  $C(S)$  where  $S$  is a metric compact space without isolated point.

In the case where  $uC_\varphi = I$ , Theorem 2.14 is a particular case of an already known result in Banach spaces with the Daugavet property. If we consider a Banach space  $X$  having the Daugavet property, then every operator  $T : X \rightarrow X$  such that  $T^*(X^*)$  is separable satisfies the Daugavet equation. This can be seen by using a result of Shvidkoy [11] which says that an operator  $T : X \rightarrow X$  not fixing a copy of  $\ell_1$  satisfies the Daugavet equation. Then it is obvious that if  $T$  fixes a copy of  $\ell_1$  then  $T^*$  fixes a copy of  $\ell_\infty$ , hence  $T^*(X^*)$  is not separable.

As another immediate consequence of Theorem 2.14, we have the following for particular weighted composition operators (which can also be viewed directly) :

**Corollary 2.18.** *Let  $S$  be a compact Hausdorff space without isolated points,  $\varphi : S \rightarrow S$  be a continuous map and  $u \in C(S)$ . If for every  $t \in S$ ,  $\varphi^{-1}(\{t\})$  is nowhere dense in  $S$ , and if  $|u|$  is constant, then  $uC_\varphi : C(S) \rightarrow C(S)$  does not factorize through a space having a separable dual space. If  $S$  is metrizable, then  $uC_\varphi$  does not factorize through an Asplund space.*

### 3 Equation $(E_{u,\varphi})$ for classes of operators on $A(\mathbb{D})$

In this section, we want to adapt D. Werner's method in [14] to find new Daugavet centers in subspaces of  $C(S)$ -spaces, and particularly for the disk algebra  $A(\mathbb{D})$ . Actually we will consider weighted composition operators  $uC_\varphi$  on a functional Banach space  $X$  and will formulate conditions on an isometric embedding of  $X$  into  $C(S)$  implying that  $X$  is  $(u, \varphi)$ -nicely embedded. Then we find conditions so that every weakly compact operator on a  $(u, \varphi)$ -nicely embedded space satisfies equation  $\|uC_\varphi + T\| = \|uC_\varphi\| + \|T\|$ .

#### 3.1 General approach

Let  $(X, \|\cdot\|)$  denotes a functional Banach space on  $\Omega$  ( $X \subset \mathcal{F}(\Omega, \mathbb{C})$ ). Consider  $\varphi$  a map such that  $\varphi(\Omega) \subset \Omega$  and  $u \in X$  such that  $0 < \|u\| < \infty$ . Assume that  $uC_\varphi : f \in X \mapsto u \cdot (f \circ \varphi) \in X$  is a weighted composition operator acting continuously on  $X$ . Let  $S$  be a compact Hausdorff space without isolated point. An isometry  $J : X \rightarrow C(S)$  is said to

be a  $(u, \varphi)$ -nice embedding and  $X$  is said to be  $(u, \varphi)$ -nicely embedded into  $C(S)$  if the following conditions are satisfied for every  $s \in S$  :

**(C1)** if  $p_s = (uC_\varphi)^* J^*(\delta_s) \in X^*$ , then  $\|p_s\| = \|u\| > 0$ .

**(C2)**  $\text{Vect}(p_s)$  is an  $L$ -summand in  $X^*$ .

Recall that a closed subspace  $F$  of a Banach space  $E$  is an  $L$ -summand if there exists a projection  $\Pi$  from  $E$  onto  $F$  such that, for every  $x \in E$ ,

$$\|x\| = \|\Pi x\| + \|x - \Pi x\|.$$

We say that  $F$  is an  $M$ -ideal if its annihilator  $F^\perp \subset E^*$  is an  $L$ -summand. Then condition **(C2)** can be reformulated as :  $\ker(p_s)$  is an  $M$ -ideal in  $X$ . Condition **(C1)** forces  $\|uC_\varphi\| = \|u\|$ .

Assume that  $X$  is  $(u, \varphi)$ -nicely embedded in  $C(S)$ . Condition **(C2)** provides us a family of projections  $(\Pi_s)_{s \in S}$  satisfying

$$\|x^*\| = \|\Pi_s x^*\| + \|x^* - \Pi_s x^*\|, \quad \text{for every } x^* \in X^*$$

and a family  $(\pi_s)_{s \in S}$  in  $X^{**}$  such that

$$\Pi_s x^* = \pi_s(x^*) p_s, \quad \text{for every } x^* \in X^*.$$

Note that  $\pi_s(p_s) = 1$ .

Consider the equivalence relation  $\sim$  on  $S$

$$s \sim t \Leftrightarrow \Pi_s = \Pi_t.$$

Note  $E_s$  the class of  $s$  in  $S$ . Then  $E_s$  is closed, and condition **(C1)** tells us that  $E_s = \{t \in S \mid p_t = \lambda p_s, \lambda \in \mathbb{T}\}$ . We will need the following condition :

**(C3)** for all  $s \in S$ , the class  $E_s$  is nowhere dense in  $S$ .

Let  $T : X \rightarrow X$  be an operator, and  $q_s = (JT)^*(\delta_s) \in X^*$ ,  $s \in S$ . Then  $s \mapsto q_s$  is continuous for the weak\*-topology on  $X^*$ , and  $\|T\| = \sup_S \|q_s\|$ .

We can now express some results, whose proofs are similar to those in section 2 and are given in [14] in the particular case where  $\varphi(x) = x$ ,  $x \in \Omega$  and  $u \equiv 1$ .

**Proposition 3.1.** *Suppose  $X$  is  $(u, \varphi)$ -nicely embedded in  $C(S)$ , and  $T$  is an operator acting on  $X$ . Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

*if and only if*

$$\text{for every } \varepsilon > 0, \quad \sup_{\{s \mid \|q_s\| > \|T\| - \varepsilon\}} \left( |1 + \pi_s(q_s)| - (1 + |\pi_s(q_s)|) \right) = 0.$$

**Proposition 3.2.** *Suppose that  $X$  is  $(u, \varphi)$ -nicely embedded in  $C(S)$ , and that condition **(C3)** holds. Let  $T$  be an operator on  $X$ . If we have*

$$\text{for all } t \in S, \quad s \mapsto \pi_t(q_s) \text{ is continuous,}$$

*then  $T$  satisfies equation  $(E_{u, \varphi})$ .*

**Remark 3.3.** Every weakly compact operator  $T$  on  $X$  fulfills conditions of Proposition 3.2, and consequently the equality  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$  holds.

We want to obtain this result for the class of operators whose adjoint has separable range. Let us start with a lemma which will be useful for the proof of the next proposition :

**Lemma 3.4.** ([14], Lemma 2.3) *Suppose  $X$  is  $(u, \varphi)$ -nicely embedded in  $C(S)$ . If  $t_1, \dots, t_k$  are pairwise nonequivalent points (for the relation  $\sim$ ), then*

$$\|x^*\| \geq \sum_{j=1}^k \|\Pi_{t_j}(x^*)\|, \quad \text{for every } x^* \in X^*.$$

**Proposition 3.5.** *Let  $X$  be a  $(u, \varphi)$ -nicely embedded space in  $C(S)$  and satisfying condition **(C3)**, and  $T$  be an operator on  $X$  such that  $T^*(X^*)$  is separable. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

**Proof.** Consider the sets  $S_\varepsilon = \{s \in S \mid |\pi_s(q_s)| < \varepsilon\}$ , where  $q_s = (JT)^*(\delta_s)$  and  $\varepsilon > 0$ . If we show that  $S_\varepsilon$  is dense in  $S$ , for every  $\varepsilon > 0$ , then  $T$  fulfills the condition of Proposition 3.1.

Let  $\{\psi_n, n \in \mathbb{N}\}$  be a dense subset of  $T^*(X^*)$  and define

$$A = \{s \in S \mid \pi_s(\psi_n) = 0, \forall n \in \mathbb{N}\}.$$

As in the proof of Theorem 2.14 we want to show that :

- i)  $A$  is dense in  $S$
- ii)  $\forall \varepsilon > 0, A \subset S_\varepsilon$ .

The proof of *ii)* is similar to the one in Theorem 2.14. For *i)*, we show that  $S \setminus A$  is nowhere dense. Indeed

$$S \setminus A = \bigcup_{n \geq 0} \{s \in S \mid \pi_s(\psi_n) \neq 0\} = \bigcup_{n \geq 0} \bigcup_{p \geq 1} A_{n,p}$$

where  $A_{n,p} = \{s \in S \mid |\pi_s(\psi_n)| > \frac{1}{p}\}$ . By Lemma 3.4 there is a finite number of equivalence classes for  $\sim$  in  $A_{n,p}$  (less than  $p\|\psi_n\|/\|u\|$ ). These equivalence classes are closed and nowhere dense (by condition **C3**). The Baire property yields that  $S \setminus A$  is contained in a nowhere dense set, implying that  $A$  is dense in  $S$ .  $\blacksquare$

In the case where  $X$  is separable, we have the same result as in Corollary 2.16.

**Corollary 3.6.** *Let  $X$  be a  $(u, \varphi)$ -nicely embedded space in  $C(S)$  satisfying condition **(C3)**, and  $T$  be an operator on  $X$  which factorizes through an Asplund space  $E$ . Assume that  $X$  is separable. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

### 3.2 Applications

Obviously one can apply this results to the case  $X = C(S)$  where  $S$  is a compact Hausdorff space without isolated point, with  $J$  the natural inclusion into  $C(S)$ . If  $\varphi : S \rightarrow S$  is continuous and if  $u \in C(S)$  with  $u \neq 0$ , then  $p_s = u(s)\delta_{\varphi(s)}$  so that condition **(C1)** forces  $|u|$  to be constant equal to  $\|u\|_\infty$ . Then  $\Pi_s : \mu \in C(S)^* \mapsto (\mu(\{\varphi(s)\})/u(s))p_s$  is a  $L$ -projection, and condition **(C2)** holds. Finally for  $s$  and  $t$  in  $S$ ,

$$\begin{aligned} s \sim t &\Leftrightarrow \exists \lambda \in \mathbb{T}, p_s = \lambda p_t \\ &\Leftrightarrow \exists \lambda \in \mathbb{T}, |u(s)|\delta_{\varphi(s)} = \lambda |u(t)|\delta_{\varphi(t)} \\ &\Leftrightarrow \varphi(s) = \varphi(t). \end{aligned}$$

Thus  $E_s = \varphi^{-1}(\{\varphi(s)\})$ . So condition **(C3)** is fulfilled if  $\varphi^{-1}(\{t\})$  is nowhere dense, for every  $t \in S$ . Therefore we recover most of the results of section 2.

We now turn to the disk algebra  $A(\mathbb{D})$ . Let  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$  denote the unit disk. The disk algebra  $A(\mathbb{D})$  is the algebra of holomorphic maps on  $\mathbb{D}$  which are continuous on  $\overline{\mathbb{D}}$ , endowed with the supremum norm  $\|f\|_\infty = \sup\{|f(z)| \mid z \in \mathbb{D}\}$ . Considering  $u \neq 0$  and  $\varphi$  in the disk algebra with  $\|\varphi\|_\infty \leq 1$ , one can define the weighted composition operator  $uC_\varphi$  acting on  $A(\mathbb{D})$ , with  $\|uC_\varphi\| = \|u\|_\infty$ . We will assume that  $\varphi$  is not constant (implying  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ), otherwise  $uC_\varphi$  is a rank-1 operator and therefore is not a Daugavet center. On the other hand,  $C_\varphi$  is compact on  $A(\mathbb{D})$  if and only if  $\|\varphi\|_\infty < 1$ . Hence a necessary condition ensuring that every weakly compact operator on  $A(\mathbb{D})$  fulfills equation  $(E_{u,\varphi})$  is that  $\|\varphi\|_\infty = 1$ . Actually, we are going to prove that a strong negation of “ $\|\varphi\|_\infty < 1$ ” is necessary.

Consider the isometry  $J : f \in A(\mathbb{D}) \mapsto J(f) = f|_{\mathbb{T}} \in C(\mathbb{T})$ . It is well known that the image of  $A(\mathbb{D})$  by  $J$  is the closed space  $\{f \in C(\mathbb{T}) \mid \hat{f}(n) = 0, \forall n < 0\}$ . We want conditions on  $\varphi$  and  $u$  implying that  $A(\mathbb{D})$  is  $(u, \varphi)$ -nicely embedded in  $C(\mathbb{T})$ , and moreover that condition **(C3)** is fulfilled.

For  $\omega \in \mathbb{T}$ , let

$$p_\omega := (uC_\varphi)^* J^*(\delta_\omega) = u(\omega)\delta_{\varphi(\omega)|_{A(\mathbb{D})}} \in A(\mathbb{D})^*.$$

Clearly  $\|p_\omega\| = |u(\omega)|$ , so **(C1)** is fulfilled if and only if  $|u|$  is constant on  $\mathbb{T}$ . Assume that  $|u|$  is constant on  $\mathbb{T}$ . To check condition **(C2)**, we have to show that  $\ker(p_\omega)$  is an  $M$ -ideal. But

$$\begin{aligned} \ker(p_\omega) &= \{f \in A(\mathbb{D}) \mid u(\omega)f(\varphi(\omega)) = 0\} \\ &= \{f \in A(\mathbb{D}) \mid f(\varphi(\omega)) = 0\} \end{aligned}$$

since  $u(\omega) \neq 0$ . It is an  $M$ -ideal if and only if  $\varphi(\omega) \in \mathbb{T}$  (see [6] p. 4). It means that **(C2)** is fulfilled if  $\varphi$  is an inner function. Finally, if  $\omega_1, \omega_2 \in \mathbb{T}$ ,

$$\begin{aligned} \omega_1 \sim \omega_2 &\Leftrightarrow \exists \lambda \in \mathbb{T}, p_{\omega_1} = \lambda p_{\omega_2} \\ &\Leftrightarrow \exists \lambda \in \mathbb{T}, |u(\omega_1)|\delta_{\varphi(\omega_1)} = \lambda |u(\omega_2)|\delta_{\varphi(\omega_2)} \text{ on } A(\mathbb{D}) \\ &\Leftrightarrow \varphi(\omega_1) = \varphi(\omega_2). \end{aligned}$$



Thus  $E_\omega = \varphi^{-1}(\{\varphi(\omega)\}) \cap \mathbb{T}$ . If  $\varphi$  is not constant, then condition **(C3)** is fulfilled.

To summarize, we have the following :

**Proposition 3.7.** *Let  $\varphi$  be an inner function and  $u$  be a multiple of an inner function. If  $T : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  is such that  $T^*(A(\mathbb{D})^*)$  is separable, then equation  $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$  holds true.*

Since  $A(\mathbb{D})$  is separable, Remark 2.17 and Corollary 3.6 gives :

**Corollary 3.8.** *Let  $\varphi$  be an inner function and  $u$  be a multiple of an inner function. Then every weakly compact operator  $T : A(\mathbb{D}) \rightarrow A(\mathbb{D})$  satisfies equation*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

This corollary leads to the following remark on (general) essential norms of weighted composition operators on the disk algebra.

**Remark 3.9.** Let  $X$  be a Banach space,  $B(X)$  be the space of bounded operator on  $X$ ,  $\mathcal{K}(X)$  be the closed subspace of  $B(X)$  consisting of compact operators on  $X$  and  $\mathcal{W}(X)$  be the closed subspace of  $B(X)$  consisting of weakly-compact operators on  $X$ . Recall that if  $\mathcal{I}$  is a closed subspace of  $B(X)$ , the essential norm (relatively to  $\mathcal{I}$ ) of  $S \in B(X)$  is the distance from  $S$  to  $\mathcal{I}$  :

$$\|S\|_{e,\mathcal{I}} = \inf\{\|S + T\|; T \in \mathcal{I}\}.$$

This is the canonical norm on the quotient space  $B(X)/\mathcal{I}$ . The classical case corresponds to the case of compact operators  $\mathcal{I} = \mathcal{K}(X)$ . In this case, the above quotient space is the Calkin algebra. General essential norms of weighted composition operators on  $A(\mathbb{D})$  are estimated in [8]. When  $\mathcal{I} \subset \mathcal{W}(A(\mathbb{D}))$ , and in the particular case where  $\varphi$  is an inner function ( $\varphi(\mathbb{D}) \subset \mathbb{D}$ ) and  $u$  is a multiple of an inner function, Corollary 3.8 not only gives us the essential norm relatively to  $\mathcal{I}$  of  $uC_\varphi$ , but how the norm of  $uC_\varphi$  reacts under perturbation by operators in the class  $\mathcal{I}$ .

Although D. Werner's method gives sufficient conditions ensuring that weighted composition operators are Daugavet centers, it turns out that these ones are also necessary.

**Proposition 3.10.** *Suppose that every rank-1 operator on  $A(\mathbb{D})$  satisfies equation  $(E_{u,\varphi})$ . Then  $\varphi$  is inner and  $|u|$  is constant on  $\mathbb{T}$ .*

**Proof.** Assume  $\varphi$  is not inner. Then there exists  $\omega \in \mathbb{T}$  such that  $|\varphi(\omega)| = r < 1$ . As  $u$  is not constant equal to zero, we can assume, taking if necessary a  $\omega' \in \mathbb{T}$  close to  $\omega$ , that  $u(\omega) \neq 0$ . Let  $g \in A(\mathbb{D})$  defined by  $g(z) = (1 + \bar{\omega}z)/2$ , where  $z \in \mathbb{D}$ . We consider the rank-1 operator  $T : f \mapsto Tf = u(\omega)f(\varphi(\omega))g$ , for all  $f \in A(\mathbb{D})$ . We have  $\|T\| = |u(\omega)|$ . For  $0 < \varepsilon < \min(1 - r, |u(\omega)|/3)$ , there exists an arc  $I_\omega \subset \mathbb{T}$  containing  $\omega$  such that for every  $z \in I_\omega$ , we have  $|\varphi(z) - \varphi(\omega)| \leq \varepsilon$ ,  $|1 - g(z)| < 1/2$  and  $|u(z) - u(\omega)| < \varepsilon$ . Let  $f \in A(\mathbb{D})$  with  $\|f\|_\infty = 1$  :

$$\|uC_\varphi(f) - Tf\| = \sup_{|z|=1} |u(z)f(\varphi(z)) - u(\omega)f(\varphi(\omega))g(z)|$$

If  $z \notin I_\omega$ , then  $|u(z)f(\varphi(z)) - u(\omega)f(\varphi(\omega))g(z)| \leq \|u\|_\infty + |u(\omega)| \sup_{z \in \mathbb{T} \setminus I_\omega} |g(z)|$ .

For any  $a, b \in D(0, r + \varepsilon)$ , we have by the Cauchy formula :

$$\begin{aligned} |f(a) - f(b)| &= \left| \frac{1}{2i\pi} \int_{|z|=1} \frac{a-b}{(z-a)(z-b)} f(z) dz \right| \\ &\leq \frac{|a-b|}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta})|}{|e^{i\theta}-a||e^{i\theta}-b|} d\theta \\ &\leq \frac{|a-b|}{(1-(r+\varepsilon))^2}. \end{aligned}$$

Thus for  $z \in I_\omega$ ,  $\varphi(z) \in D(0, r + \varepsilon)$  and we have :

$$\begin{aligned} |u(z)f(\varphi(z)) - u(\omega)f(\varphi(\omega))g(z)| &\leq |u(z)||f(\varphi(z)) - f(\varphi(\omega))| + |u(z) - u(\omega)||f(\varphi(\omega))| \\ &\quad + |u(\omega)f(\varphi(\omega))||1 - g(z)| \\ &\leq \|u\|_\infty \left( \frac{\varepsilon}{(1-(r+\varepsilon))^2} \right) + \varepsilon + \frac{|u(\omega)|}{2} \\ &\leq \|u\|_\infty + \frac{5}{6}|u(\omega)| \end{aligned}$$

for a suitable  $\varepsilon > 0$ . So

$$\|uC_\varphi(f) - Tf\| \leq \max \left( \|u\|_\infty + \frac{5}{6}|u(\omega)|, \|u\|_\infty + |u(\omega)|\delta \right),$$

where  $\delta = \sup_{z \in \mathbb{T} \setminus I_\omega} |g(z)| < 1$ . This gives  $\|C_\varphi - T\| < \|u\|_\infty + |u(\omega)| = \|C_\varphi\| + \|T\|$  which is absurd. So  $\varphi$  is an inner function.

We use a similar argument to show that  $|u|$  is constant on the unit circle. ■

We summarize our results in the following corollary :

**Corollary 3.11.** *Let  $\varphi \in A(\mathbb{D})$ ,  $\varphi(\mathbb{D}) \subset \mathbb{D}$  and  $u \in A(\mathbb{D})$ . Then  $uC_\varphi$  is a Daugavet center in  $A(\mathbb{D})$  if and only if  $\varphi$  is an inner function and  $u$  is a multiple of an inner function.*

**Remark 3.12.** The case of the disk algebra is different from the case of  $C(S)$ . Indeed, we have seen that a function  $\varphi : S \rightarrow S$  could induce a composition operator  $C_\varphi$  on  $C(S)$  which is an isometry but is not a Daugavet center (see section 2). Whereas if  $\varphi \in A(\mathbb{D})$  satisfies  $\varphi(\mathbb{D}) \subset \mathbb{D}$ ,  $\varphi$  is an inner function if and only if  $C_\varphi$  is an isometry, and so  $C_\varphi$  is an isometry if and only if  $C_\varphi$  is a Daugavet center.

Note that in the particular case where  $\varphi$  is a disk automorphism it is easy to see that  $C_\varphi$  is a Daugavet center. Indeed, consider a weakly compact operator  $T$  on  $A(\mathbb{D})$ . Then  $C_\varphi + T = C_\varphi(I + C_\varphi^{-1}T)$ . The fact that  $C_\varphi$  is an isometry implies that  $\|C_\varphi + T\| = \|I + C_\varphi^{-1}T\|$ . Since the disk algebra has the Daugavet property (see [15]) and  $C_\varphi^{-1}$  is itself an isometry, we have

$$\|C_\varphi + T\| = 1 + \|C_\varphi^{-1}T\| = 1 + \|T\|.$$

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