# Weighted composition operators as Daugavet centers

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**Abstract**

We investigate the norm identity $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ for classes of operators on $C(S)$, where $S$ is a compact Hausdorff space without isolated point, and characterize those weighted composition operators which satisfy this equation for every weakly compact operator $T : C(S) \to C(S)$. We also give a characterization of such weighted composition operator acting on the disk algebra $A(\mathbb{D})$.

## 1 Introduction

In 1963, Daugavet proved [3] the norm equality

$$\|I + T\| = 1 + \|T\| \tag{1.1}$$

now known as the Daugavet equation for every compact operator $T : C([0,1]) \to C([0,1])$. Over the years, this property was extended to larger classes of operators and various spaces : $C(S)$ where $S$ is a compact Hausdorff space without isolated point [5], $L^1(\mu)$ for measure $\mu$ without atoms [9], the disk algebra $A(\mathbb{D})$ or the Hardy space $H^\infty$ [15]. Actually, if (1.1) holds for every rank-1 operators on $X$, then it holds for every weakly compact operators, $X$ contains a copy of $\ell_1$, $X$ cannot have an unconditional basis and even cannot embed into a space having an unconditional basis (see the survey [13]).

Recently in [10], the author showed that if we substitute the Identity in (1.1) with an into isometry $J : L^1[0,1] \to L^1[0,1]$ then equation $\|J + T\| = 1 + \|T\|$ holds for narrow operators, and particularly for weakly compact operators on $L^1[0,1]$. In arbitrary Banach spaces, this has been investigated by T. Bosenko and V. Kadets in [2]. They introduced the following concept :

**Definition 1.1.** *Let $X$ be a Banach space. A linear continuous operator $G : X \to X$ is said to be a Daugavet center if the norm identity*

$$\|G + T\| = \|G\| + \|T\| \tag{1.2}$$

*holds for every rank-1 operator $T : X \to X$.*

Bosenko and Kadets showed that if $G : X \to X$ is a non zero Daugavet center then equation (1.2) holds true for every strong Radon-Nikodým operator on $X$, and so for weakly compact operators on $X$. Moreover $G$ fixes a copy of $\ell_1$, and $X$ cannot have an unconditional basis, merely $X$ cannot be embedded into a space having an unconditional basis.

In section 2 of this paper, we give a characterization of weighted composition operators on $C(S)$ which are Daugavet centers ($S$ compact Hausdorff space without isolated point). We give examples of Daugavet centers which are not weighted composition operators and prove that the set of Daugavet centers in $C(S)$ is not convex. We also study equation (1.2) for the class of operators whose adjoint has separable range. This encompass the class of operators factorizing through $c_0$.

In section 3, we adapt D. Werner's method showing that certain function spaces have the Daugavet property (meaning that the identity operator is a Daugavet center) to characterize weighted composition operators on the disk algebra $A(\mathbb{D})$ which are Daugavet centers.

# 2 Weighted composition operators as Daugavet center in $C(S)$

Let $S$ denotes a compact Hausdorff space without isolated point. Considering continuous maps $\varphi : S \to S$ and $u : S \to \mathbb{C}$, we study the weighted composition operator $uC_\varphi : C(S) \to C(S)$ defined by $uC_\varphi(f) = u.(f \circ \varphi)$ for all $f \in C(S)$. We clearly have $\|uC_\varphi\| = \|u\|_\infty$. We investigate the following equation :

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\| \qquad (E_{u,\varphi})$$

Note that if we take $\varphi(s) = s$ (for all $s$ in $S$) and $u$ the constant function equal to 1, the previous equation becomes the classical Daugavet equation. We will suppose that $u$ is not the constant function equal to zero. We want to find conditions on $\varphi$ and $u$ implying that every weakly compact operators on C(S) satisfies equation $(E_{u,\varphi})$. A first remark is that $u$ and $\varphi$ must be such that the operator $uC_\varphi$ is not itself compact. By a result of Kamowitz [7], $uC_\varphi$ is compact if and only if $\varphi$ is constant on a neighborhood of each connected component of the set where $u$ is nonzero. So $\varphi$ should be non constant over at least one nonempty open set in $S$.

The main result of this section is the following theorem :

**Theorem 2.1.** *Let $S$ be a compact Hausdorff space without isolated point, $u \in C(S)$ and $\varphi$ be a continuous function from $S$ to $S$.*
*Then every weakly compact operators $T : C(S) \to C(S)$ satisfies equation $(E_{u,\varphi})$ if and only if $\varphi^{-1}(\{t\})$ is nowhere dense in $S$, for every $t \in S$ and $|u|$ is constant on $S$.*

The straightforward direction was already proved in [2] for rank-1 operators and for $u \equiv 1$. Here we give a direct and simple proof for weakly compact operators, and we check that conditions on $\varphi$ and $u$ are necessary.

We first begin with some notations and terminology. The dual space of $C(S)$ consisting of all regular borel measures on $S$ of finite variation will be denoted by $M(S)$. If $s \in S$, we define the corresponding Dirac functional $\delta_s$ by $\delta_s(f) = f(s)$ for every $f \in C(S)$. Then $\delta_s \in M(S)$ and $\|\delta_s\| = 1$.

Following [12], the key idea is to represent an operator $T : C(S) \to C(S)$ by the family of measures $(\mu_s)_{s\in S}$ defined by $\mu_s = T^*(\delta_s)$, such that :

$$(Tf)(s) = \langle Tf, \delta_s\rangle = \langle f, \mu_s\rangle = \int_S f \; \mathrm{d}\mu_s.$$

Thus, the (weakly) compact nature of $T$ is reformulated in terms of continuity of the map $s \mapsto \mu_s$ in the following (c.f. [4], Th. VI, 7.1) :

**Lemma 2.2.** *Let $T : C(S) \to C(S)$ be an operator and $(\mu_s)_{s\in S}$ be the family of measures associated to $T$. Then :*

*i) $s \mapsto \mu_s$ is continuous from $S$ to $M(S) = C(S)^*$ endowed with the weak$^*$-topology, i.e. $\sigma(M(S), C(S))$.*

*ii) $T$ is weakly compact if and only if $s \mapsto \mu_s$ is continuous for the weak-topology on $M(S)$, i.e. for $\sigma\big(M(S), M(S)^*\big)$.*

*iii) $T$ is compact if and only if $s \mapsto \mu_s$ is continuous for the norm topology on $M(S)$.*

Note that $\|T\| = \sup\limits_{s\in S} \|\mu_s\|$, and that the operator $uC_\varphi$ is represented by the family of measures $(u(s)\delta_{\varphi(s)})_{s\in S}$. Indeed :

$$(uC_\varphi)^*(\delta_s)(f) = \delta_s(u.f \circ \varphi) = u(s)f\big(\varphi(s)\big) = u(s)\delta_{\varphi(s)}(f).$$

The following proposition shows, assuming $|u|$ is constant, that for every operator $T$ on $C(S)$, there is a $\lambda \in \mathbb{T}$ such that $\lambda T$ satisfies equation $(E_{u,\varphi})$.

**Proposition 2.3.** *Let $S$ be a compact Hausdorff space, and $T : C(S) \to C(S)$ be an operator. Assume that $|u|$ is constant. Then*

$$\max_{\lambda\in\mathbb{T}} \|uC_\varphi + \lambda T\| = \|u\|_\infty + \|T\|.$$

**Proof.** Let $(\mu_s)_{s\in S}$ be the family of measures associated to $T$. Then

$$\begin{aligned}
\max_{\lambda\in\mathbb{T}} \|uC_\varphi + \lambda T\| &= \max_{\lambda\in\mathbb{T}} \sup_{s\in S} \|u(s)\delta_{\varphi(s)} + \lambda\mu_s\| \\
&= \sup_{s\in S} \max_{\lambda\in\mathbb{T}} \Big(|u(s)\delta_{\varphi(s)} + \lambda\mu_s|\big(\{\varphi(s)\}\big) + |u(s)\delta_{\varphi(s)} + \lambda\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{s\in S} \max_{\lambda\in\mathbb{T}} \Big(\big|u(s) + \lambda\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{s\in S} \Big(|u(s)| + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{s\in S} (\|u\|_\infty + \|\mu_s\|) \qquad \text{since } |u(s)| = \|u\|_\infty \\
&= \|u\|_\infty + \|T\|.
\end{aligned}$$

■

**Remark 2.4.** i) In the real case, a similar result holds replacing "$\lambda \in \mathbb{T}$" by "$\lambda \in \{\pm 1\}$".

ii) Without assumption on the modulus of $u$, the previous result is not true anymore. For instance, taking $v \in C(S)$, we have that $\max_{\lambda\in\mathbb{T}} \|uC_\varphi + \lambda v C_\psi\| = \||u| + |v|\|_\infty$ which is not equal to $\|u\|_\infty + \|v\|_\infty$ in general.

## 2.1 Equation $(E_{u,\varphi})$ for weakly compact operators on $C(S)$

Let $T : C(S) \to C(S)$ be an operator and $(\mu_s)_{s\in S}$ be the family of measures associated to $T$. Then

$$\|uC_\varphi + T\| = \sup_{s\in S} \|u(s)\delta_{\varphi(s)} + \mu_s\| = \sup_{s\in S} \Big( \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big)$$

and

$$\|u\|_\infty + \|T\| = \sup_{s\in S}(\|u\|_\infty + \|\mu_s\|) = \sup_{s\in S} \Big( \|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big).$$

We have the following lemma which gives a characterization of the operators satisfying equation $(E_{u,\varphi})$ :

**Lemma 2.5.**

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$$

*if and only if*

$$\sup_{\{s\in S|\ \|\mu_s\| > \|T\| - \varepsilon\}} \Big( \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \big(\|u\|_\infty + \big|\mu_s(\{\varphi(s)\})\big|\big)\Big) = 0 \tag{2.1}$$

*for all $\varepsilon > 0$.*

**Proof.** Sufficient condition : let $\varepsilon > 0$ and $U = \{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}$ which is not empty. Then :

$$\begin{aligned}
\|uC_\varphi + T\| &\geq \sup_{s\in U} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\
&\geq \sup_{s\in U} \Big( \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&\geq \sup_{s\in U} \bigg( \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + \|u\|_\infty + \|\mu_s\| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\bigg) \\
&\geq \|u\|_\infty + \|T\| - \varepsilon + \sup_{s\in U} \bigg( \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\bigg) \\
&\geq \|u\|_\infty + \|T\| - \varepsilon \qquad \big(\text{with } (2.1)\big).
\end{aligned}$$

Necessary condition : let us assume that there exist $\alpha$ and $\varepsilon > 0$ such that for all $s \in S$, $\|\mu_s\| > \|T\| - \varepsilon$ implies

$$\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big) < -\alpha < 0.$$

Then

$$\begin{aligned}\|uC_\varphi + T\| &= \sup_{s\in S} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\ &= \max\Big( \sup_{\{s|\ \|\mu_s\|>\|T\|-\varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\|, \sup_{\{s|\ \|\mu_s\|\le\|T\|-\varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\|\Big).\end{aligned}$$

The second term is lower than $\|u\|_\infty + \|T\| - \varepsilon$. For the first term, we write as before

$$\begin{aligned}&\sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\ &= \sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s)+\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\ &= \sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s)+\mu_s\big(\{\varphi(s)\}\big)\big| + \|u\|_\infty + \|\mu_s\| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\Big) \\ &\le \|u\|_\infty + \|T\| + \sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s)+\mu_s\big(\{\varphi(s)\}\big)\big| - \big(\|u\|_\infty + \big|\mu_s(\{\varphi(s)\})\big|\big)\Big) \\ &\le \|u\|_\infty + \|T\| - \alpha.\end{aligned}$$

Thus $\|uC_\varphi + T\| < \|u\|_\infty + \|T\| - \min(\varepsilon,\alpha) < \|u\|_\infty + \|T\|$, which leads to a contradiction. ∎

As a consequence, we state the following useful corollary :

**Corollary 2.6.** *Assume that the family $(\mu_s)_{s\in S}$ satisfies the following condition : for every nonempty open set $U \subset S$,*

$$\sup_{s\in U} \Big(\big|u(s)+\mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\Big) = 0. \tag{2.2}$$

*Then*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

**Proof.** Take $\varepsilon > 0$, and call $U = \{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}$. Thanks to Lemma 2.5, we only need to show that $U$ is a nonempty open subset of $S$. It is clear that $U$ is nonempty. Take $s_0 \in U$. There exists $f_0 \in C(S)$, $\|f_0\|_\infty \le 1$ such that $|\mu_{s_0}(f_0)| > \|T\| - \varepsilon$. From Lemma 2.2, we know that $s \mapsto \mu_s$ is continuous for the weak*-topology on $M(S)$, hence $s \mapsto \mu_s(f_0)$ is continuous. Then $V = \{s \in S \mid |\mu_s(f_0)| > \|T\| - \varepsilon\}$ is an open neighborhood of $s_0$ contained in $U$. So $U$ is a nonempty open subset of $S$. ∎

Now we can show a first result dealing with weakly compact operators. The following theorem gives sufficient conditions on $u$ and $\varphi$ implying that every weakly compact operator on $C(S)$ satisfies equation $(E_{u,\varphi})$.

**Theorem 2.7.** *Let $S$ be a compact Hausdorff space (without isolated point). Assume that $|u|$ is constant on $S$ and $\varphi(U)$ is infinite for every nonempty open subset $U$ of $S$. Then $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ for every weakly compact operator $T : C(S) \to C(S)$.*

Note that condition on $\varphi$ forces $S$ to have no isolated point.

**Proof.** Assume that $\varphi(U)$ is infinite for every nonempty open subset $U$ of $S$ and that $|u|$ is constant. If the family of measures $(\mu_s)_{s\in S}$ representing $T$ does not satisfy (2.2) of Corollary 2.6, then there exist a nonempty open set $U \subset S$ and $\beta > 0$ such that

$$\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big) < -2\beta \quad \forall s \in U.$$

In particular we have, since $|u(s)| = \|u\|_\infty$ for all $s \in S$ :

$$\begin{aligned}\big|\mu_s\big(\{\varphi(s)\}\big)\big| &> 2\beta - \|u\|_\infty + \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| \\ &\geq 2\beta - \|u\|_\infty + |u(s)| - \big|\mu_s\big(\{\varphi(s)\}\big)\big| \\ &= 2\beta - \big|\mu_s\big(\{\varphi(s)\}\big)\big|\end{aligned}$$

which gives

$$\big|\mu_s\big(\{\varphi(s)\}\big)\big| > \beta, \quad \text{for all } s \in U.$$

Take $t \in S$. Then $s \in S \mapsto \mu_s(\{t\}) \in \mathbb{C}$ is continuous. Indeed : from Lemma 2.2, $s \mapsto \mu_s$ is continuous for the weak-topology on $M(S)$. Since $\mu \mapsto \mu(\{t\})$ belongs to $M(S)^*$, it is continuous on $M(S)$ endowed with the weak-topology.

Let $s_0 \in U$, and define

$$U_1 = \big\{s \in U \mid \ \big|\mu_s\big(\{\varphi(s_0)\}\big)\big| > \beta\big\}.$$

From above, $U_1$ is an open subset of $U$ (and so of $S$) which contains $s_0$. Since $\varphi(U_1)$ is infinite, one can find $s_1$ in $U_1$ satifying $\varphi(s_1) \neq \varphi(s_0)$. Then we have

$$\begin{aligned}&\big|\mu_{s_1}\big(\{\varphi(s_1)\}\big)\big| > \beta, \quad \text{since } s_1 \in U \\ &\big|\mu_{s_1}\big(\{\varphi(s_0)\}\big)\big| > \beta.\end{aligned}$$

Consider now

$$U_2 = \big\{s \in U_1 \mid \ \big|\mu_s\big(\{\varphi(s_1)\}\big)\big| > \beta\big\}.$$

It is an open subset of $U$ containing $s_1$, and it contains an element $s_2$ such that $\varphi(s_2) \neq \varphi(s_0)$ and $\varphi(s_2) \neq \varphi(s_1)$ (since $\varphi(U_2)$ is infinite). Then we have, since $s_2 \in U_2 \subset U_1 \subset U$,

$$\begin{aligned}&\big|\mu_{s_2}\big(\{\varphi(s_2)\}\big)\big| > \beta \\ &\big|\mu_{s_2}\big(\{\varphi(s_1)\}\big)\big| > \beta \\ &\big|\mu_{s_2}\big(\{\varphi(s_0)\}\big)\big| > \beta.\end{aligned}$$

In such a way we construct a decreasing sequence of open subsets $U_n \subset U$, and a sequence of elements $(s_n)_{n\geq 0}$, $s_n \in U_n$ having the property

$$\begin{aligned}U_{n+1} &= \big\{s \in U_n \mid \ \big|\mu_s\big(\{\varphi(s_n)\}\big)\big| > \beta\big\}, \\ s_{n+1} &\in U_{n+1} \\ \varphi(s_{n+1}) &\notin \{\varphi(s_0), \ldots, \varphi(s_n)\}.\end{aligned}$$

So

$$\big|\mu_{s_n}\big(\{\varphi(s_j)\}\big)\big| > \beta, \qquad j = 0, \ldots, n-1$$

which leads to a contradiction writing that

$$\|T\| \geq \|\mu_{s_n}\| \geq |\mu_{s_n}|\big(\{\varphi(s_0), \ldots, \varphi(s_{n-1})\}\big) \geq n\beta, \qquad \forall n \in \mathbb{N}.$$

■

We now give necessary conditions on $\varphi$ and $u$ to ensure that every weakly compact operator on $C(S)$ satisfies equation $(E_{u,\varphi})$. Actually we only need to consider rank-1 operators.

**Theorem 2.8.** *Let $S$ be a compact Hausdorff space without isolated point. Assume that every rank-1 operator on $C(S)$ satisfies equation $(E_{u,\varphi})$. Then $|u|$ is constant and $\varphi^{-1}(\{t\})$ is nowhere dense in $S$, for every $t \in S$.*

**Proof :** We first show that $|u|$ is constant on $S$. Arguing by contradiction, assume there exists $s_0 \in S$ such that $|u(s_0)| < \|u\|_\infty$. Then there exists $\delta > 0$ and an open neighborhood $U$ of $s_0$ satisfying

$$\forall s \in U,\ |u(s)| < \|u\|_\infty - \delta.$$

Choose a continuous function $v$ such that : $0 \leq v \leq 1,\ v(s_0) = 1$ and $v(s) < 1$ for all $s \neq s_0$. We define the operator $T = v\delta_\tau$ where $\tau$ is an element of $S$. Then $\mu_s = T^*(\delta_s) = v(s)\delta_\tau,\ \|\mu_s\| = v(s)$. Choose $\varepsilon > 0$ such that we have $\{s \in S \mid\ v(s) > 1-\varepsilon\} \subset U$. It follows that

$$\begin{aligned}
&\sup_{\{s \mid\ v(s) > 1-\varepsilon\}} \Big( \big|u(s) + v(s)\delta_\tau\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + |v(s)|\delta_\tau\big(\{\varphi(s)\}\big)\Big)\Big) \\
&\leq \sup_{s \in U} \Big( |u(s)| + v(s)\delta_\tau\big(\{\varphi(s)\}\big) - \Big(\|u\|_\infty + v(s)\delta_\tau\big(\{\varphi(s)\}\big)\Big)\Big) \\
&\leq \sup_{s \in U} |u(s)| - \|u\|_\infty \\
&\leq -\delta < 0.
\end{aligned}$$

The family of measures $(\mu_s)_{s\in S}$ does not satisfy condition (2.1) of Lemma 2.5 and consequently $T$ does not satisfy equation $(E_{u,\varphi})$, which is false since $T$ is a rank-1 operator. So $|u|$ is constant.

Now we prove that for every $t \in S,\ \varphi^{-1}(\{t\})$ is nowhere dense in $S$. Let $U$ be a nonempty open subset of $S$. We want to find $s \in U$ such that $\varphi(s) \neq t$. Consider the rank-1 operator $T = \delta_t g u$, where $g \in C(S)$ such that $-1 \leq g \leq -\frac{1}{2},\ g = -\frac{1}{2}$ outside $U$ and $\|g\|_\infty = 1$. Then $\|T\| = \|u\|_\infty > 0$, the family of measures associated to $T$ is given by $\mu_s = T^*(\delta_s) = u(s)g(s)\delta_t$, and $\|\mu_s\| = |u(s)g(s)| = \|u\|_\infty |g(s)|$.
Take $\varepsilon = \|u\|_\infty/2$ so that $V = \{s \in S \mid\ |u(s)g(s)| > \frac{\|u\|_\infty}{2}\} \subset U$. Since $T$ satisfies equation $(E_{u,\varphi})$, the family of measures $(\mu_s)_{s\in S}$ satisfies condition (2.1) of Lemma 2.5 :

$$\begin{aligned}
0 &= \sup_V \Big( \big|u(s) + u(s)g(s)\delta_t\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + |u(s)g(s)|\delta_t\big(\{\varphi(s)\}\big)\Big)\Big) \\
&= \sup_V \Big( 2g(s)\|u\|_\infty \delta_t\big(\{\varphi(s)\}\big)\Big)
\end{aligned}$$

which is less than $\sup\limits_{s\in U}\Big(2g(s)\|u\|_\infty\delta_t\big(\{\varphi(s)\}\big)\Big)$. It follows that there exists $s \in U$ such that $\delta_t\big(\{\varphi(s)\}\big) = 0$, *i.e.* $\varphi(s) \neq t$ and therefore $U \not\subset \varphi^{-1}(\{t\})$. ■

**Remark 2.9.** Note that in a topological space $S$, and for a continuous map $\varphi : S \to S$, the following conditions are equivalent :

i) for every $t \in S$, $\varphi^{-1}(\{t\})$ is nowhere dense in $S$

ii) for every nonempty open subset $U$ of $S$, $\varphi(U)$ is infinite.

Indeed, if there exists a nonempty open subset $U$ of $S$ such that $U \subset \varphi^{-1}(\{t\})$ then $\varphi$ is constant on $U$, so *ii)* $\Rightarrow$ *i)*. Moreover if $\varphi(U) = \{s_1, \ldots, s_n\}$ for an open subset $U$ of $S$, $n \geq 1$, then

$$\begin{aligned}\{s \in U \mid \ \varphi(s) = s_1\} &= \{s \in U \mid \varphi(s) \neq s_k,\ 2 \leq k \leq n\}\\ &= \varphi^{-1}\big(S\backslash\{s_2, \ldots, s_n\}\big) \cap U.\end{aligned}$$

The set $S\backslash\{s_2, \ldots, s_n\}$ is open in $S$, so $\{s \in U \mid \ \varphi(s) = s_1\}$ is a nonempty open subset of $U$ (and of $S$) although by *i)*, $\{s \in S \mid \ \varphi(s) = s_1\}$ must have empty interior.

The previous remark, Theorem 2.7 and Theorem 2.8 give the following :

**Corollary 2.10.** *Let $S$ be a compact Hausdorff space without isolated point. Then $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ for every weakly compact operator $T : C(S) \to C(S)$ if and only if $|u|$ is constant on $S$ and the set $\varphi^{-1}(\{t\})$ is nowhere dense in $S$, for every $t \in S$.*

**Application : a negative answer to a question of Popov [10]**

Note that if $\varphi$ is onto and $|u| = 1$ then $uC_\varphi$ is an isometry on $C(S)$. In [10], Popov shows that every into isometry $J : L^1([0,1]) \to L^1([0,1])$ is a Daugavet center. He raises the question whether this result is true when we substitute $L^1([0,1])$ with a Banach space $X$ having the Daugavet property. Actually, this is not true for $X = C(S)$. To see this, consider any composition operator whose symbol $\varphi$ is onto and constant on a nonempty open subset of $S$. Then $C_\varphi$ is an isometry but there exists rank-1 operators on $C(S)$ which does not satisfy equation $(E_{1,\varphi})$.

After our work was completed, an example was independently produced in [2]. The authors considered a weighted composition operator $uC_\varphi : C[0,1] \to C[0,1]$ whose symbol $\varphi$ is constant on $]1/2, 1]$ and whose weight has not constant modulus on $[0,1]$.

## 2.2 Convex combinations of composition operators

One can wonder if the set of Daugavet centers is a convex set. Actually it is easy to see that this is not true in full generality. Indeed, consider $u(x) = e^{2i\pi x}$ and $v(x) = e^{-2i\pi x}$, $x \in [0,1]$. Then $u, v \in C[0,1]$, $|u| = |v| = 1$ so $uI$ and $vI$ are Daugavet centers in $C([0,1])$, but $\big(u(x) + v(x)\big)/2 = \cos 2\pi x$ which has not constant modulus on $[0,1]$. Therefore $(uI + vI)/2$ is not a Daugavet center in $C[0,1]$. Nevertheless it turns out that a convex combination of particular (non zero) Daugavet centers can be a Daugavet center. Let us consider the case of composition operators.

Note that a convex combination of composition operators is not in general a composition operator itself. Indeed, assume that $C_\varphi = tC_{\psi_1} + (1-t)C_{\psi_2}$ where $\varphi, \psi_1, \psi_2$ are continuous functions from $S$ to $S$ and $0 < t < 1$. Assume that $\varphi \neq \psi_1$ and take $s_0$ such that $\varphi(s_0) \neq \psi_1(s_0)$. Now consider a open subset $U$ of $S$ such that $\varphi(s_0) \in U$ and $\psi_1(s_0) \notin U$. Choose $f \in C(S)$, $\|f\|_\infty = 1$ satisfying $f\big(\varphi(s_0)\big) = 1$ and $|f| < 1$ out of $U$. Then

$$1 = |f\big(\varphi(s_0)\big)| \leq t\big|f\big(\psi_1(s_0)\big)\big| + (1-t)\big|f\big(\psi_2(s_0)\big)\big| < 1$$

which leads to a contradiction.

Let $\varphi$ and $\psi$ be continuous functions from $S$ to $S$. Assume that $\varphi \neq \psi$. Define

$$S_1 = \{s \in S \mid \ \varphi(s) \neq \psi(s)\}.$$

Then $S_1$ is a nonempty open subset of $S$ since $S$ has no isolated point. Consider convex combinations of $C_\varphi$ and $C_\psi$. For $t \in [0,1]$,we define $T_t = tC_\varphi + (1-t)C_\psi$. Point out that $\|T_t\| = 1$. For convenience, we note

$$\Delta_T(s) = \big|t + \mu_s\big(\{\varphi(s)\}\big)\big| + \big|1 - t + \mu_s\big(\{\psi(s)\}\big)\big| - \Big(1 + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + \big|\mu_s\big(\{\psi(s)\}\big)\big|\Big),$$

and

$$\tilde{\Delta}_T(s) = \big|1 + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(1 + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)$$

where $(\mu_s)_{s\in S}$ is the family of measures representing $T$ and $s \in S$. As for weighted composition operators, we have the following property :

**Proposition 2.11.** *Let $T$ be an operator on $C(S)$. Assume that the family of measures $(\mu_s)_{s\in S}$ representing $T$ satisfies the condition : for every nonempty open set $U \subset S$ :*

*-If $U \cap S_1 \neq \emptyset$, then*

$$\sup_{s\in U\cap S_1} \Delta_T(s) = 0 \tag{2.3}$$

*-If $U \cap S_1 = \emptyset$, then*

$$\sup_{s\in U} \tilde{\Delta}_T(s) = 0. \tag{2.4}$$

*Then the following equation holds true :*

$$\|T_t + T\| = 1 + \|T\|.$$

**Proof.** One only has to consider open subsets $U$ of $S$ of the form $U = \{s \in S \mid \ \|\mu_s\| > \|T\| - \varepsilon\}$ where $\varepsilon > 0$. If $U \cap S_1 = \emptyset$ then $\varphi = \psi$ on $U$ so the proof of Lemma 2.5 tells us that $\|T_t + T\| \geq \sup_{s\in U} \|\delta_{\varphi(s)} + \mu_s\| \geq 1 + \|T\| - \varepsilon$. Else, $\|T_t + T\| \geq \sup_{s\in U\cap S_1} \|t\delta_{\varphi(s)} + (1-t)\delta_{\psi(s)} + \mu_s\|$ which is greater than $1 + \|T\| - \varepsilon$ using the same method as in the proof of Lemma 2.5. ∎

From this we can deduce that any convex combination of composition operators which are Daugavet centers is still a Daugavet center.

**Theorem 2.12.** *Assume that $C_\varphi$ and $C_\psi$ are Daugavet center. Then every weakly compact operator $T$ on $C(S)$ satisfies the norm equation*

$$\|tC_\varphi + (1-t)C_\psi + T\| = 1 + \|T\|,$$

*for all $t \in [0,1]$.*

**Proof.** Take $t \in [0,1]$. Argue by contradiction and assume that the family $(\mu_s)_{s\in S}$ does not satisfy conditions of Proposition 2.11.
*First case* : Let $U$ be a nonempty open subset of $S$ such that $U \cap S_1 \neq \emptyset$ and (2.3) does not hold. Then there exists $\beta > 0$ such that

$$\big|t + \mu_s\big(\{\varphi(s)\}\big)\big| + \big|1 - t + \mu_s\big(\{\psi(s)\}\big)\big| - \Big(1 + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + \big|\mu_s\big(\{\psi(s)\}\big)\big|\Big) < -4\beta$$

for every $s \in U \cap S_1$. Then

$$\big|\mu_s\big(\{\varphi(s)\}\big)\big| + \big|\mu_s\big(\{\psi(s)\}\big)\big| > 2\beta, \quad \forall s \in U \cap S_1.$$

Let

$$\begin{aligned} V_1 &= \{s \in U \cap S_1 \mid \ \big|\mu_s\big(\{\varphi(s)\}\big)\big| > \beta\} \\ V_2 &= \{s \in U \cap S_1 \mid \ \big|\mu_s\big(\{\psi(s)\}\big)\big| > \beta\}. \end{aligned}$$

Since $U \cap S_1 \subset V_1 \cup V_2$, we can assume without loss of generality that $V_1$ contains a nonempty open set $V$. So $\big|\mu_s\big(\{\varphi(s)\}\big)\big| > \beta$ for every $s \in V$. Then we follow the proof of Theorem 2.7 to obtain a contradiction.
*Second case* : If $U$ is a nonempty open subset of $S$ such that $U \subset S\backslash S_1$ and (2.4) does not hold, then the same proof as in Theorem 2.7 leads to a contradiction. ■

## 2.3 Operators factorizing through an Asplund space

The aim of this section is to extend a result of Ansari in [1] stating that every operator on $C(S)$ factorizing through $c_0$ satisfies the Daugavet equation. Let $T : C(S) \to C(S)$ be an operator, where $S$ is a compact Hausdorff space, and $(\mu_s)_{s\in S}$ the family of measures associated to $T$. Note that if $\mu_s\big(\{\varphi(s)\}\big) = 0$ for all $s \in S$, and $|u|$ is constant, then $T$ trivially satisfies condition (2.2) of corollary (2.6). Actually, it is sufficient that the measures $(\mu_s)$ almost satisfies this condition :

Define $S_\varepsilon = \big\{s \in S \mid \ \big|\mu_s\big(\{\varphi(s)\}\big)\big| < \varepsilon\big\}$, for each $\varepsilon > 0$. We have the following :

**Lemma 2.13.** *If $|u|$ is constant and if the sets $S_\varepsilon$ are dense in $S$, for every $\varepsilon > 0$, then*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

**Proof.** Take $U$ a nonempty open set in $S$ and $\varepsilon > 0$. By density of $S_\varepsilon$ in $S$, there exists $s_\varepsilon \in U$ satisfying $\big|\mu_{s_\varepsilon}\big(\{\varphi(s_\varepsilon)\}\big)\big| < \varepsilon$, and so

$$\begin{aligned} \big|u(s_\varepsilon) + \mu_{s_\varepsilon}\big(\{\varphi(s_\varepsilon)\}\big)\big| - \big(\|u\|_\infty + \big|\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})\big|\big) &\geq -2\big|\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})\big| \\ &> -2\varepsilon. \end{aligned}$$

Thus for every nonempty open set $U$ of $S$, we have

$$\sup_{s\in U}\Big(\big|u(s)+\mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\Big) = 0.$$

We conclude with corollary (2.6). ■

Now we can prove the following result :

**Theorem 2.14.** *Let $S$ be a compact Hausdorff space without isolated point, $\varphi : S \to S$ a continuous map, $u \in C(S)$ and $T : C(S) \to C(S)$ an operator such that $T^*(M(S))$ is separable. If $\varphi^{-1}(\{t\})$ is nowhere dense in $S$, for every $t \in S$, and if $|u|$ is constant, then $T$ satisfies equation $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$.*

**Proof.** Let $\{\rho_n,\ n \in \mathbb{N}\}$ be a dense subset of $T^*\big(M(S)\big)$. As previously, $S_\varepsilon = \big\{s \in S \mid \ \big|\mu_s\big(\{\varphi(s)\}\big)\big| < \varepsilon\big\}$, where $\mu_s = T^*(\delta_s)$, and $A = \bigcap_{n\geq 0}\big\{s \in S \mid \ \rho_n\big(\{\varphi(s)\}\big) = 0\big\}$. We want to show that :

i) $A$ is dense in $S$.

ii) $\forall\varepsilon > 0,\ A \subset S_\varepsilon$.

Then we conclude with Lemma 2.13.
To prove $i)$, we are going to show that $S\backslash A$ is nowhere dense. Indeed,

$$S\backslash A = \bigcup_{n\geq 0}\big\{s \in S \mid \ \rho_n\big(\{\varphi(s)\}\big) \neq 0\big\} = \bigcup_{n\geq 0}\bigcup_{p\geq 1} A_{n,p}$$

where $A_{n,p} = \big\{s \in S \mid \ \big|\rho_n\big(\{\varphi(s)\}\big)\big| > \frac{1}{p}\big\}$. Since $\rho_n$ is a finite measure, this implies that the sets $\varphi(A_{n,p})$ are finite (hence closed) for every $n \geq 0$, $p \geq 1$. But $A_{n,p} \subset \varphi^{-1}\big(\varphi(A_{n,p})\big)$ which a finite union of nowhere dense sets (c.f. Remark 2.9). Using Baire's theorem, $S\backslash A$ is contained in a nowhere dense set, and $A$ is dense in $S$.
Proof of $ii)$ : let $s \in S$ and $\varepsilon > 0$. By density of $(\rho_n)_n$ in $T^*(M(S))$, there exists an integer $n_0 \geq 0$ such that $\|T^*(\delta_s) - \rho_{n_0}\| < \varepsilon$. Then $|\mu_s - \rho_{n_0}|\big(\{\varphi(s)\}\big) < \varepsilon$. Taking $s \in A$, it follows that $\big|\mu_s\big(\{\varphi(s)\}\big)\big| < \varepsilon$, *i.e.* $A \subset S_\varepsilon$. ■

If $T : C(S) \to C(S)$ factorizes through a space $X$ having a separable dual, then Theorem 2.14 applies. In particular this holds for the class of operators factorizing through $c_0$. Actually, regarding operators factorizing through a space $X$, one does not need to assume that $X^*$ is separable in the case where $S$ is metrizable. We recall the following definition :

**Definition 2.15.** *A Banach space $X$ is called an Asplund space if its dual space has the Radon-Nikodým property.*

Every dual space which is separable has the Radon-Nikodým property, and so every Banach space with separable dual is Asplund. Asplund spaces are characterized by the fact that every separable subspace has a separable dual.

**Corollary 2.16.** *Let $S$ be a metric compact space without isolated points, $\varphi : S \to S$ a continuous map, $u \in C(S)$ and $T : C(S) \to C(S)$ an operator factorizing through an Asplund space $X$. If $\varphi^{-1}(\{t\})$ is nowhere dense in $S$, for every $t \in S$, and if $|u|$ is constant on $S$, then $T$ satisfies equation $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$.*

**Proof.** Write $T = T_2T_1$ with $T_1 : C(S) \to X$ and $T_2 : X \to C(S)$. Since $S$ is a metric compact space, $C(S)$ is separable. So we can assume, by replacing $X$ by $\overline{T_1\big(C(S)\big)}$ that $X^*$ is separable. Thus $T^*(M(S))$ is separable, and the result follows from Theorem 2.14. ∎

**Remark 2.17.** Since every compact operator factorizes through a subspace of $c_0$, this gives another proof of Theorem 2.7 for compact operators on $C(S)$. Moreover every weakly compact operator factorizes through a reflexive space (which is Asplund), giving another proof of Theorem 2.7 for weakly compact operators on $C(S)$ where $S$ is a metric compact space without isolated point.

In the case where $uC_\varphi = I$, Theorem 2.14 is a particular case of an already known result in Banach spaces with the Daugavet property. If we consider a Banach space $X$ having the Daugavet property, then every operator $T : X \to X$ such that $T^*(X^*)$ is separable satisfies the Daugavet equation. This can be seen by using a result of Shvidkoy [11] which says that an operator $T : X \to X$ not fixing a copy of $\ell_1$ satisfies the Daugavet equation. Then it is obvious that if $T$ fixes a copy of $\ell_1$ then $T^*$ fixes a copy of $\ell_\infty$, hence $T^*(X^*)$ is not separable.

As another immediate consequence of Theorem 2.14, we have the following for particular weighted composition operators (which can also be viewed directly) :

**Corollary 2.18.** *Let $S$ be a compact Hausdorff space without isolated points, $\varphi : S \to S$ be a continuous map and $u \in C(S)$. If for every $t \in S$, $\varphi^{-1}(\{t\})$ is nowhere dense in $S$, and if $|u|$ is constant, then $uC_\varphi : C(S) \to C(S)$ does not factorize through a space having a separable dual space. If $S$ is metrizable, then $uC_\varphi$ does not factorize through an Asplund space.*

# 3 Equation $(E_{u,\varphi})$ for classes of operators on $A(\mathbb{D})$

In this section, we want to adapt D. Werner's method in [14] to find new Daugavet centers in subspaces of $C(S)$-spaces, and particularly for the disk algebra $A(\mathbb{D})$. Actually we will consider weighted composition operators $uC_\varphi$ on a functionnal Banach space $X$ and will formulate conditions on an isometric embedding of $X$ into $C(S)$ implying that $X$ is $(u,\varphi)$-nicely embedded. Then we find conditions so that every weakly compact operator on a $(u,\varphi)$-nicely embedded space satisfies equation $\|uC_\varphi + T\| = \|uC_\varphi\| + \|T\|$.

## 3.1 General approach

Let $(X, \|.\|)$ denotes a functional Banach space on $\Omega$ $\big(X \subset \mathcal{F}(\Omega,\mathbb{C})\big)$. Consider $\varphi$ a map such that $\varphi(\Omega) \subset \Omega$ and $u \in X$ such that $0 < \|u\| < \infty$. Assume that $uC_\varphi : f \in X \mapsto u.(f \circ \varphi) \in X$ is a weighted composition operator acting continuously on $X$. Let $S$ be a compact Hausdorff space without isolated point. An isometry $J : X \to C(S)$ is said to be a $(u,\varphi)$-*nice embedding* and $X$ is said to be $(u,\varphi)$-*nicely embedded* into $C(S)$ if the following conditions are satisfied for every $s \in S$ :

**(C1)** if $p_s = (uC_\varphi)^* J^*(\delta_s) \in X^*$, then $\|p_s\| = \|u\| > 0$.

**(C2)** $\mathrm{Vect}(p_s)$ is an $L$-summand in $X^*$.

Recall that a closed subspace $F$ of a Banach space $E$ is an $L$-summand if there exists a projection $\Pi$ from $E$ onto $F$ such that, for every $x \in E$,

$$\|x\| = \|\Pi x\| + \|x - \Pi x\|.$$

We say that $F$ is an $M$-ideal if its annihilator $F^\perp \subset E^*$ is an $L$-summand. Then condition **(C2)** can be reformulated as : $\ker(p_s)$ is an $M$-ideal in $X$. Condition **(C1)** forces $\|uC_\varphi\| = \|u\|$.

Assume that $X$ is $(u,\varphi)$-nicely embedded in $C(S)$. Condition **(C2)** provides us a family of projections $(\Pi_s)_{s\in S}$ satisfying

$$\|x^*\| = \|\Pi_s x^*\| + \|x^* - \Pi_s x^*\|, \quad \text{for every } x^* \in X^*$$

and a family $(\pi_s)_{s\in S}$ in $X^{**}$ such that

$$\Pi_s x^* = \pi_s(x^*) p_s, \quad \text{for every } x^* \in X^*.$$

Note that $\pi_s(p_s) = 1$.

Consider the equivalence relation $\sim$ on $S$

$$s \sim t \Leftrightarrow \Pi_s = \Pi_t.$$

Note $E_s$ the class of $s$ in $S$. Then $E_s$ is closed, and condition **(C1)** tells us that $E_s = \{t \in S \mid \ p_t = \lambda p_s,\ \lambda \in \mathbb{T}\}$. We will need the following condition :

**(C3)** for all $s \in S$, the class $E_s$ is nowhere dense in $S$.

Let $T : X \to X$ be an operator, and $q_s = (JT)^*(\delta_s) \in X^*,\ s \in S$. Then $s \mapsto q_s$ is continuous for the weak*-topology on $X^*$, and $\|T\| = \sup_S \|q_s\|$.
We can now express some results, whose proofs are similar to those in section 2 and are given in [14] in the particular case where $\varphi(x) = x,\ x \in \Omega$ and $u \equiv 1$.

**Proposition 3.1.** *Suppose $X$ is $(u,\varphi)$-nicely embedded in $C(S)$, and $T$ is an operator acting on $X$. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

*if and only if*

$$for\ every\ \varepsilon > 0, \qquad \sup_{\{s \mid \ \|q_s\| > \|T\| - \varepsilon\}} \Big( |1 + \pi_s(q_s)| - \big(1 + |\pi_s(q_s)|\big) \Big) = 0.$$

**Proposition 3.2.** *Suppose that $X$ is $(u,\varphi)$-nicely embedded in $C(S)$, and that condition **(C3)** holds. Let $T$ be an operator on $X$. If we have*

$$\text{for all } t \in S,\ s \mapsto \pi_t(q_s) \text{ is continuous,}$$

*then $T$ satisfies equation $(E_{u,\varphi})$.*

**Remark 3.3.** Every weakly compact operator $T$ on $X$ fulfills conditions of Proposition 3.2, and consequently the equality $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ holds.

We want to obtain this result for the class of operators whose adjoint has separable range. Let us start with a lemma which will be useful for the proof of the next proposition :

**Lemma 3.4.** *([14], Lemma 2.3) Suppose $X$ is $(u,\varphi)$-nicely embedded in $C(S)$. If $t_1,\ldots,t_k$ are pairwise nonequivalent points (for the relation $\sim$), then*

$$\|x^*\| \geq \sum_{j=1}^{k} \|\Pi_{t_j}(x^*)\|, \quad \textit{for every } x^* \in X^*.$$

**Proposition 3.5.** *Let $X$ be a $(u,\varphi)$-nicely embedded space in $C(S)$ and satisfying condition **(C3)**, and $T$ be an operator on $X$ such that $T^*(X^*)$ is separable. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

**Proof.** Consider the sets $S_\varepsilon = \{s \in S \mid \; |\pi_s(q_s)| < \varepsilon\}$, where $q_s = (JT)^*(\delta_s)$ and $\varepsilon > 0$. If we show that $S_\varepsilon$ is dense in $S$, for every $\varepsilon > 0$, then $T$ fulfills the condition of Proposition 3.1.

Let $\{\psi_n, \; n \in \mathbb{N}\}$ be a dense subset of $T^*(X^*)$ and define

$$A = \big\{s \in S \mid \; \pi_s(\psi_n) = 0, \; \forall n \in \mathbb{N}\big\}.$$

As in the proof of Theorem 2.14 we want to show that :

i) $A$ is dense in $S$

ii) $\forall \varepsilon > 0, \; A \subset S_\varepsilon$.

The proof of *ii)* is similar to the one in Theorem 2.14. For *i)*, we show that $S \backslash A$ is nowhere dense. Indeed

$$S\backslash A = \bigcup_{n\geq 0} \big\{s \in S \mid \; \pi_s(\psi_n) \neq 0\big\} = \bigcup_{n\geq 0}\bigcup_{p\geq 1} A_{n,p}$$

where $A_{n,p} = \big\{s \in S \mid \; \big|\pi_s(\psi_n)\big| > \frac{1}{p}\big\}$. By Lemma 3.4 there is a finite number of equivalence classes for $\sim$ in $A_{n,p}$ (less than $p\|\psi_n\|/\|u\|$). These equivalence classes are closed and nowhere dense (by condition **C3**). The Baire property yields that $S\backslash A$ is contained in a nowhere dense set, implying that $A$ is dense in $S$. ∎

In the case where $X$ is separable, we have the same result as in Corollary 2.16.

**Corollary 3.6.** *Let $X$ be a $(u,\varphi)$-nicely embedded space in $C(S)$ satisfying condition **(C3)**, and $T$ be an operator on $X$ which factorizes through an Asplund space $E$. Assume that $X$ is separable. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

## 3.2 Applications

Obviously one can apply this results to the case $X = C(S)$ where $S$ is a compact Hausdorff space without isolated point, with $J$ the natural inclusion into $C(S)$. If $\varphi : S \to S$ is continuous and if $u \in C(S)$ with $u \neq 0$, then $p_s = u(s)\delta_{\varphi(s)}$ so that condition **(C1)** forces $|u|$ to be constant equal to $\|u\|_\infty$. Then $\Pi_s : \mu \in C(S)^* \mapsto \big(\mu\big(\{\varphi(s)\}\big)/u(s)\big)p_s$ is a $L$-projection, and condition **(C2)** holds. Finally for $s$ and $t$ in $S$,

$$\begin{aligned} s \sim t &\Leftrightarrow \exists \lambda \in \mathbb{T},\ p_s = \lambda p_t \\ &\Leftrightarrow \exists \lambda \in \mathbb{T},\ |u(s)|\delta_{\varphi(s)} = \lambda |u(t)|\delta_{\varphi(t)} \\ &\Leftrightarrow \varphi(s) = \varphi(t). \end{aligned}$$

Thus $E_s = \varphi^{-1}\big(\{\varphi(s)\}\big)$. So condition **(C3)** is fulfilled if $\varphi^{-1}(\{t\})$ is nowhere dense, for every $t \in S$. Therefore we recover most of the results of section 2.

We now turn to the disk algebra $A(\mathbb{D})$. Let $\mathbb{D} = \{z \in \mathbb{C} \mid \ |z| < 1\}$ denote the unit disk. The disk algebra $A(\mathbb{D})$ is the algebra of holomorphic maps on $\mathbb{D}$ which are continuous on $\overline{\mathbb{D}}$, endowed with the supremum norm $\|f\|_\infty = \sup\{|f(z)| \mid \ z \in \mathbb{D}\}$. Considering $u \neq 0$ and $\varphi$ in the disk algebra with $\|\varphi\|_\infty \leq 1$, one can define the weighted composition operator $uC_\varphi$ acting on $A(\mathbb{D})$, with $\|uC_\varphi\| = \|u\|_\infty$. We will assume that $\varphi$ is not constant (implying $\varphi(\mathbb{D}) \subset \mathbb{D}$), otherwise $uC_\varphi$ is a rank-1 operator and therefore is not a Daugavet center. On the other hand, $C_\varphi$ is compact on $A(\mathbb{D})$ if and only if $\|\varphi\|_\infty < 1$. Hence a necessary condition ensuring that every weakly compact operator on $A(\mathbb{D})$ fulfills equation $(E_{u,\varphi})$ is that $\|\varphi\|_\infty = 1$. Actually, we are going to prove that a strong negation of "$\|\varphi\|_\infty < 1$" is necessary.

Consider the isometry $J : f \in A(\mathbb{D}) \mapsto J(f) = f_{|\mathbb{T}} \in C(\mathbb{T})$. It is well known that the image of $A(\mathbb{D})$ by $J$ is the closed space $\{f \in C(\mathbb{T}) \mid \ \hat{f}(n) = 0,\ \forall n < 0\}$. We want conditions on $\varphi$ and $u$ implying that $A(\mathbb{D})$ is $(u, \varphi)$-nicely embedded in $C(\mathbb{T})$, and moreover that condition **(C3)** is fulfilled.

For $\omega \in \mathbb{T}$, let

$$p_\omega := (uC_\varphi)^* J^*(\delta_\omega) = u(\omega)\delta_{\varphi(\omega)_{|A(\mathbb{D})}} \in A(\mathbb{D})^*.$$

Clearly $\|p_\omega\| = |u(\omega)|$, so **(C1)** is fulfilled if and only if $|u|$ is constant on $\mathbb{T}$. Assume that $|u|$ is constant on $\mathbb{T}$. To check condition **(C2)**, we have to show that $\ker(p_\omega)$ is an $M$-ideal. But

$$\begin{aligned} \ker(p_\omega) &= \{f \in A(\mathbb{D}) \mid \ u(\omega)f\big(\varphi(\omega)\big) = 0\} \\ &= \{f \in A(\mathbb{D}) \mid \ f\big(\varphi(\omega)\big) = 0\} \end{aligned}$$

since $u(\omega) \neq 0$. It is an $M$-ideal if and only if $\varphi(\omega) \in \mathbb{T}$ (see [6] p. 4). It means that **(C2)** is fulfilled if $\varphi$ is an inner function. Finally, if $\omega_1, \omega_2 \in \mathbb{T}$,

$$\begin{aligned} \omega_1 \sim \omega_2 &\Leftrightarrow \exists \lambda \in \mathbb{T},\ p_{\omega_1} = \lambda p_{\omega_2} \\ &\Leftrightarrow \exists \lambda \in \mathbb{T},\ |u(\omega_1)|\delta_{\varphi(\omega_1)} = \lambda |u(\omega_2)|\delta_{\varphi(\omega_2)} \text{ on } A(\mathbb{D}) \\ &\Leftrightarrow \varphi(\omega_1) = \varphi(\omega_2). \end{aligned}$$

Thus $E_\omega = \varphi^{-1}\big(\{\varphi(\omega)\}\big) \cap \mathbb{T}$. If $\varphi$ is not constant, then condition **(C3)** is fulfilled.

To summarize, we have the following :

**Proposition 3.7.** *Let $\varphi$ be an inner function and $u$ be a multiple of an inner function. If $T : A(\mathbb{D}) \to A(\mathbb{D})$ is such that $T^*\big(A(\mathbb{D})^*\big)$ is separable, then equation $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ holds true.*

Since $A(\mathbb{D})$ is separable, Remark 2.17 and Corollary 3.6 gives :

**Corollary 3.8.** *Let $\varphi$ be an inner function and $u$ be a multiple of an inner function. Then every weakly compact operator $T : A(\mathbb{D}) \to A(\mathbb{D})$ satifies equation*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

This corollary leads to the following remark on (general) essential norms of weighted composition operators on the disk algebra.

**Remark 3.9.** Let $X$ be a Banach space, $B(X)$ be the space of bounded operator on $X$, $\mathcal{K}(X)$ be the closed subspace of $B(X)$ consisting of compact operators on $X$ and $\mathcal{W}(X)$ be the closed subspace of $B(X)$ consisting of weakly-compact operators on $X$. Recall that if $\mathcal{I}$ is a closed subspace of $B(X)$, the essential norm (relatively to $\mathcal{I}$) of $S \in B(X)$ is the distance from $S$ to $\mathcal{I}$ :

$$\|S\|_{e,\mathcal{I}} = \inf\{\|S + T\|;\ T \in \mathcal{I}\}.$$

This is the canonical norm on the quotient space $B(X)/\mathcal{I}$. The classical case corresponds to the case of compact operators $\mathcal{I} = \mathcal{K}(X)$. In this case, the above quotient space is the Calkin algebra. General essential norms of weighted composition operators on $A(\mathbb{D})$ are estimated in [8]. When $\mathcal{I} \subset \mathcal{W}\big(A(\mathbb{D})\big)$, and in the particular case where $\varphi$ is an inner function ($\varphi(\mathbb{D}) \subset \mathbb{D}$) and $u$ is a multiple of an inner function, Corollary 3.8 not only gives us the essential norm relatively to $\mathcal{I}$ of $uC_\varphi$, but how the norm of $uC_\varphi$ reacts under perturbation by operators in the class $\mathcal{I}$.

Although D. Werner's method gives sufficient conditions ensuring that weighted composition operators are Daugavet centers, it turns out that these ones are also necessary.

**Proposition 3.10.** *Suppose that every rank-1 operator on $A(\mathbb{D})$ satisfies equation $(E_{u,\varphi})$. Then $\varphi$ is inner and $|u|$ is constant on $\mathbb{T}$.*

**Proof.** Assume $\varphi$ is not inner. Then there exists $\omega \in \mathbb{T}$ such that $|\varphi(\omega)| = r < 1$. As $u$ is not constant equal to zero, we can assume, taking if necessary a $\omega' \in \mathbb{T}$ close to $\omega$, that $u(\omega) \neq 0$. Let $g \in A(\mathbb{D})$ defined by $g(z) = (1 + \bar{\omega}z)/2$, where $z \in \mathbb{D}$. We consider the rank-1 operator $T : f \mapsto Tf = u(\omega)f\big(\varphi(\omega)\big)g$, for all $f \in A(\mathbb{D})$. We have $\|T\| = |u(\omega)|$. For $0 < \varepsilon < \min(1 - r, |u(\omega)|/3)$, there exists an arc $I_\omega \subset \mathbb{T}$ containing $\omega$ such that for every $z \in I_\omega$, we have $|\varphi(z) - \varphi(\omega)| \leq \varepsilon$, $|1 - g(z)| < 1/2$ and $|u(z) - u(\omega)| < \varepsilon$. Let $f \in A(\mathbb{D})$ with $\|f\|_\infty = 1$ :

$$\|uC_\varphi(f) - Tf\| = \sup_{|z|=1} \big|u(z)f\big(\varphi(z)\big) - u(\omega)f\big(\varphi(\omega)\big)g(z)\big|$$

If $z \notin I_\omega$, then $\left|u(z)f\big(\varphi(z)\big) - u(\omega)f\big(\varphi(\omega)\big)g(z)\right| \le \|u\|_\infty + |u(\omega)| \sup_{z\in\mathbb{T}\setminus I_\omega} |g(z)|$.

For any $a, b \in D(0, r+\varepsilon)$, we have by the Cauchy formula :

$$\begin{aligned}|f(a) - f(b)| &= \left|\frac{1}{2i\pi}\int_{|z|=1} \frac{a-b}{(z-a)(z-b)} f(z)\mathrm{d}z\right| \\ &\le \frac{|a-b|}{2\pi}\int_0^{2\pi} \frac{\left|f\big(e^{i\theta}\big)\right|}{\left|e^{i\theta}-a\right|\left|e^{i\theta}-b\right|}\mathrm{d}\theta \\ &\le \frac{|a-b|}{\big(1-(r+\varepsilon)\big)^2}.\end{aligned}$$

Thus for $z \in I_\omega,\ \varphi(z) \in D(0, r+\varepsilon)$ and we have :

$$\begin{aligned}\left|u(z)f\big(\varphi(z)\big) - u(\omega)f\big(\varphi(\omega)\big)g(z)\right| &\le |u(z)|\left|f\big(\varphi(z)\big) - f\big(\varphi(\omega)\big)\right| + |u(z)-u(\omega)|\left|f\big(\varphi(\omega)\big)\right| \\ &\quad + \left|u(\omega)f\big(\varphi(\omega)\big)\right||1-g(z)| \\ &\le \|u\|_\infty\left(\frac{\varepsilon}{\big(1-(r+\varepsilon)\big)^2}\right) + \varepsilon + \frac{|u(\omega)|}{2} \\ &\le \|u\|_\infty + \frac{5}{6}|u(\omega)|\end{aligned}$$

for a suitable $\varepsilon > 0$. So

$$\|uC_\varphi(f) - Tf\| \le \max\left(\|u\|_\infty + \frac{5}{6}|u(\omega)|, \|u\|_\infty + |u(\omega)|\delta\right),$$

where $\delta = \sup_{z\in\mathbb{T}\setminus I_\omega} |g(z)| < 1$. This gives $\|C_\varphi - T\| < \|u\|_\infty + |u(\omega)| = \|C_\varphi\| + \|T\|$ which is absurd. So $\varphi$ is an inner function.

We use a similar argument to show that $|u|$ is constant on the unit circle. ■

We summarize our results in the following corollary :

**Corollary 3.11.** *Let $\varphi \in A(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $u \in A(\mathbb{D})$. Then $uC_\varphi$ is a Daugavet center in $A(\mathbb{D})$ if and only if $\varphi$ is an inner function and $u$ is a multiple of an inner function.*

**Remark 3.12.** The case of the disk algebra is different from the case of $C(S)$. Indeed, we have seen that a function $\varphi : S \to S$ could induce a composition operator $C_\varphi$ on $C(S)$ which is an isometry but is not a Daugavet center (see section 2). Whereas if $\varphi \in A(\mathbb{D})$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi$ is an inner function if and only if $C_\varphi$ is an isometry, and so $C_\varphi$ is an isometry if and only if $C_\varphi$ is a Daugavet center.

Note that in the particular case where $\varphi$ is a disk automorphism it is easy to see that $C_\varphi$ is a Daugavet center. Indeed, consider a weakly compact operator $T$ on $A(\mathbb{D})$. Then $C_\varphi + T = C_\varphi(I + C_\varphi^{-1}T)$. The fact that $C_\varphi$ is an isometry implies that $\|C_\varphi + T\| = \|I + C_\varphi^{-1}T\|$. Since the disk algebra has the Daugavet property (see [15]) and $C_\varphi^{-1}$ is itself an isometry, we have

$$\|C_\varphi + T\| = 1 + \|C_\varphi^{-1}T\| = 1 + \|T\|.$$

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