Weighted composition operators as Daugavet centers

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Abstract

We investigate the norm identity $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ for classes of operators on $C(S)$, where S is a compact Hausdorff space without isolated point, and characterize those weighted composition operators which satisfy this equation for every weakly compact operator $T : C(S) \to C(S)$. We also give a characterization of such weighted composition operator acting on the disk algebra $A(\mathbb{D})$.

1 Introduction

In 1963, Daugavet proved [3] the norm equality

$$\|I + T\| = 1 + \|T\| \tag{1.1}$$

now known as the Daugavet equation for every compact operator $T : C([0,1]) \to C([0,1])$. Over the years, this property was extended to larger classes of operators and various spaces : $C(S)$ where S is a compact Hausdorff space without isolated point [5], $L^1(\mu)$ for measure μ without atoms [9], the disk algebra $A(\mathbb{D})$ or the Hardy space H^∞ [15]. Actually, if (1.1) holds for every rank-1 operators on X, then it holds for every weakly compact operators, X contains a copy of ℓ_1, X cannot have an unconditional basis and even cannot embed into a space having an unconditional basis (see the survey [13]).

Recently in [10], the author showed that if we substitute the Identity in (1.1) with an into isometry $J : L^1[0,1] \to L^1[0,1]$ then equation $\|J + T\| = 1 + \|T\|$ holds for narrow operators, and particularly for weakly compact operators on $L^1[0,1]$. In arbitrary Banach spaces, this has been investigated by T. Bosenko and V. Kadets in [2]. They introduced the following concept :

Definition 1.1. *Let X be a Banach space. A linear continuous operator $G : X \to X$ is said to be a Daugavet center if the norm identity*

$$\|G + T\| = \|G\| + \|T\| \tag{1.2}$$

holds for every rank-1 operator $T : X \to X$.

Bosenko and Kadets showed that if $G : X \to X$ is a non zero Daugavet center then equation (1.2) holds true for every strong Radon-Nikodým operator on X, and so for weakly compact operators on X. Moreover G fixes a copy of ℓ_1, and X cannot have an unconditional basis, merely X cannot be embedded into a space having an unconditional basis.

In section 2 of this paper, we give a characterization of weighted composition operators on $C(S)$ which are Daugavet centers (S compact Hausdorff space without isolated point). We give examples of Daugavet centers which are not weighted composition operators and prove that the set of Daugavet centers in $C(S)$ is not convex. We also study equation (1.2) for the class of operators whose adjoint has separable range. This encompass the class of operators factorizing through c_0.

In section 3, we adapt D. Werner's method showing that certain function spaces have the Daugavet property (meaning that the identity operator is a Daugavet center) to characterize weighted composition operators on the disk algebra $A(\mathbb{D})$ which are Daugavet centers.

2 Weighted composition operators as Daugavet center in $C(S)$

Let S denotes a compact Hausdorff space without isolated point. Considering continuous maps $\varphi : S \to S$ and $u : S \to \mathbb{C}$, we study the weighted composition operator $uC_\varphi : C(S) \to C(S)$ defined by $uC_\varphi(f) = u.(f \circ \varphi)$ for all $f \in C(S)$. We clearly have $\|uC_\varphi\| = \|u\|_\infty$. We investigate the following equation :

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\| \qquad (E_{u,\varphi})$$

Note that if we take $\varphi(s) = s$ (for all s in S) and u the constant function equal to 1, the previous equation becomes the classical Daugavet equation. We will suppose that u is not the constant function equal to zero. We want to find conditions on φ and u implying that every weakly compact operators on C(S) satisfies equation $(E_{u,\varphi})$. A first remark is that u and φ must be such that the operator uC_φ is not itself compact. By a result of Kamowitz [7], uC_φ is compact if and only if φ is constant on a neighborhood of each connected component of the set where u is nonzero. So φ should be non constant over at least one nonempty open set in S.

The main result of this section is the following theorem :

Theorem 2.1. *Let S be a compact Hausdorff space without isolated point, $u \in C(S)$ and φ be a continuous function from S to S.*
Then every weakly compact operators $T : C(S) \to C(S)$ satisfies equation $(E_{u,\varphi})$ if and only if $\varphi^{-1}(\{t\})$ is nowhere dense in S, for every $t \in S$ and $|u|$ is constant on S.

The straightforward direction was already proved in [2] for rank-1 operators and for $u \equiv 1$. Here we give a direct and simple proof for weakly compact operators, and we check that conditions on φ and u are necessary.

We first begin with some notations and terminology. The dual space of $C(S)$ consisting of all regular borel measures on S of finite variation will be denoted by $M(S)$. If $s \in S$, we define the corresponding Dirac functional δ_s by $\delta_s(f) = f(s)$ for every $f \in C(S)$. Then $\delta_s \in M(S)$ and $\|\delta_s\| = 1$.

Following [12], the key idea is to represent an operator $T : C(S) \to C(S)$ by the family of measures $(\mu_s)_{s\in S}$ defined by $\mu_s = T^*(\delta_s)$, such that :

$$(Tf)(s) = \langle Tf, \delta_s\rangle = \langle f, \mu_s\rangle = \int_S f \; \mathrm{d}\mu_s.$$

Thus, the (weakly) compact nature of T is reformulated in terms of continuity of the map $s \mapsto \mu_s$ in the following (c.f. [4], Th. VI, 7.1) :

Lemma 2.2. *Let $T : C(S) \to C(S)$ be an operator and $(\mu_s)_{s\in S}$ be the family of measures associated to T. Then :*

i) $s \mapsto \mu_s$ is continuous from S to $M(S) = C(S)^*$ endowed with the weak*-topology, i.e. $\sigma(M(S), C(S))$.

ii) T is weakly compact if and only if $s \mapsto \mu_s$ is continuous for the weak-topology on $M(S)$, i.e. for $\sigma\big(M(S), M(S)^*\big)$.

iii) T is compact if and only if $s \mapsto \mu_s$ is continuous for the norm topology on $M(S)$.

Note that $\|T\| = \sup_{s\in S} \|\mu_s\|$, and that the operator uC_φ is represented by the family of measures $(u(s)\delta_{\varphi(s)})_{s\in S}$. Indeed :

$$(uC_\varphi)^*(\delta_s)(f) = \delta_s(u.f \circ \varphi) = u(s)f\big(\varphi(s)\big) = u(s)\delta_{\varphi(s)}(f).$$

The following proposition shows, assuming $|u|$ is constant, that for every operator T on $C(S)$, there is a $\lambda \in \mathbb{T}$ such that λT satisfies equation $(E_{u,\varphi})$.

Proposition 2.3. *Let S be a compact Hausdorff space, and $T : C(S) \to C(S)$ be an operator. Assume that $|u|$ is constant. Then*

$$\max_{\lambda\in\mathbb{T}} \|uC_\varphi + \lambda T\| = \|u\|_\infty + \|T\|.$$

Proof. Let $(\mu_s)_{s\in S}$ be the family of measures associated to T. Then

$$\begin{aligned}
\max_{\lambda\in\mathbb{T}} \|uC_\varphi + \lambda T\| &= \max_{\lambda\in\mathbb{T}} \sup_{s\in S} \|u(s)\delta_{\varphi(s)} + \lambda\mu_s\| \\
&= \sup_{s\in S} \max_{\lambda\in\mathbb{T}} \Big(|u(s)\delta_{\varphi(s)} + \lambda\mu_s|\big(\{\varphi(s)\}\big) + |u(s)\delta_{\varphi(s)} + \lambda\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{s\in S} \max_{\lambda\in\mathbb{T}} \Big(\big|u(s) + \lambda\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{s\in S} \Big(|u(s)| + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{s\in S} (\|u\|_\infty + \|\mu_s\|) \qquad \text{since } |u(s)| = \|u\|_\infty \\
&= \|u\|_\infty + \|T\|.
\end{aligned}$$

■

Remark 2.4. i) In the real case, a similar result holds replacing "$\lambda \in \mathbb{T}$" by "$\lambda \in \{\pm 1\}$".

ii) Without assumption on the modulus of u, the previous result is not true anymore. For instance, taking $v \in C(S)$, we have that $\max_{\lambda\in\mathbb{T}} \|uC_\varphi + \lambda vC_\psi\| = \||u| + |v|\|_\infty$ which is not equal to $\|u\|_\infty + \|v\|_\infty$ in general.

2.1 Equation $(E_{u,\varphi})$ for weakly compact operators on $C(S)$

Let $T : C(S) \to C(S)$ be an operator and $(\mu_s)_{s\in S}$ be the family of measures associated to T. Then

$$\|uC_\varphi + T\| = \sup_{s\in S} \|u(s)\delta_{\varphi(s)} + \mu_s\| = \sup_{s\in S} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big)$$

and

$$\|u\|_\infty + \|T\| = \sup_{s\in S}(\|u\|_\infty + \|\mu_s\|) = \sup_{s\in S} \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big).$$

We have the following lemma which gives a characterization of the operators satisfying equation $(E_{u,\varphi})$:

Lemma 2.5.

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$$

if and only if

$$\sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \big(\|u\|_\infty + \big|\mu_s(\{\varphi(s)\})\big|\big)\Big) = 0 \tag{2.1}$$

for all $\varepsilon > 0$.

Proof. Sufficient condition : let $\varepsilon > 0$ and $U = \{s \in S \mid \|\mu_s\| > \|T\| - \varepsilon\}$ which is not empty. Then :

$$\begin{aligned}
\|uC_\varphi + T\| &\geq \sup_{s\in U} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\
&\geq \sup_{s\in U} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&\geq \sup_{s\in U} \bigg(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + \|u\|_\infty + \|\mu_s\| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\bigg) \\
&\geq \|u\|_\infty + \|T\| - \varepsilon + \sup_{s\in U} \bigg(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\bigg) \\
&\geq \|u\|_\infty + \|T\| - \varepsilon \qquad \big(\text{with (2.1)}\big).
\end{aligned}$$

Necessary condition : let us assume that there exist α and $\varepsilon > 0$ such that for all $s \in S$, $\|\mu_s\| > \|T\| - \varepsilon$ implies

$$\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big) < -\alpha < 0.$$

Then

$$
\begin{aligned}
\|uC_\varphi + T\| &= \sup_{s\in S} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\
&= \max\Big(\sup_{\{s|\ \|\mu_s\|>\|T\|-\varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\|, \sup_{\{s|\ \|\mu_s\|\le\|T\|-\varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\|\Big).
\end{aligned}
$$

The second term is lower than $\|u\|_\infty + \|T\| - \varepsilon$. For the first term, we write as before

$$
\begin{aligned}
&\sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \|u(s)\delta_{\varphi(s)} + \mu_s\| \\
&= \sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + |\mu_s|\big(S\backslash\{\varphi(s)\}\big)\Big) \\
&= \sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| + \|u\|_\infty + \|\mu_s\| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\Big) \\
&\le \|u\|_\infty + \|T\| + \sup_{\{s\in S|\ \|\mu_s\|>\|T\|-\varepsilon\}} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \big(\|u\|_\infty + \big|\mu_s(\{\varphi(s)\})\big|\big)\Big) \\
&\le \|u\|_\infty + \|T\| - \alpha.
\end{aligned}
$$

Thus $\|uC_\varphi + T\| < \|u\|_\infty + \|T\| - \min(\varepsilon, \alpha) < \|u\|_\infty + \|T\|$, which leads to a contradiction. ∎

As a consequence, we state the following useful corollary :

Corollary 2.6. *Assume that the family $(\mu_s)_{s\in S}$ satisfies the following condition : for every nonempty open set $U \subset S$,*

$$
\sup_{s\in U} \Big(\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)\Big) = 0. \tag{2.2}
$$

Then

$$
\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.
$$

Proof. Take $\varepsilon > 0$, and call $U = \{s \in S \mid \ \|\mu_s\| > \|T\| - \varepsilon\}$. Thanks to Lemma 2.5, we only need to show that U is a nonempty open subset of S. It is clear that U is nonempty. Take $s_0 \in U$. There exists $f_0 \in C(S)$, $\|f_0\|_\infty \le 1$ such that $|\mu_{s_0}(f_0)| > \|T\| - \varepsilon$. From Lemma 2.2, we know that $s \mapsto \mu_s$ is continuous for the weak*-topology on $M(S)$, hence $s \mapsto \mu_s(f_0)$ is continuous. Then $V = \{s \in S \mid \ |\mu_s(f_0)| > \|T\| - \varepsilon\}$ is an open neighborhood of s_0 contained in U. So U is a nonempty open subset of S. ∎

Now we can show a first result dealing with weakly compact operators. The following theorem gives sufficient conditions on u and φ implying that every weakly compact operator on $C(S)$ satisfies equation $(E_{u,\varphi})$.

Theorem 2.7. *Let S be a compact Hausdorff space (without isolated point). Assume that $|u|$ is constant on S and $\varphi(U)$ is infinite for every nonempty open subset U of S. Then $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ for every weakly compact operator $T : C(S) \to C(S)$.*

Note that condition on φ forces S to have no isolated point.

Proof. Assume that $\varphi(U)$ is infinite for every nonempty open subset U of S and that $|u|$ is constant. If the family of measures $(\mu_s)_{s\in S}$ representing T does not satisfy (2.2) of Corollary 2.6, then there exist a nonempty open set $U \subset S$ and $\beta > 0$ such that

$$\big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big) < -2\beta \quad \forall s \in U.$$

In particular we have, since $|u(s)| = \|u\|_\infty$ for all $s \in S$:

$$\begin{aligned}\big|\mu_s\big(\{\varphi(s)\}\big)\big| &> 2\beta - \|u\|_\infty + \big|u(s) + \mu_s\big(\{\varphi(s)\}\big)\big| \\ &\geq 2\beta - \|u\|_\infty + |u(s)| - \big|\mu_s\big(\{\varphi(s)\}\big)\big| \\ &= 2\beta - \big|\mu_s\big(\{\varphi(s)\}\big)\big|\end{aligned}$$

which gives

$$\big|\mu_s\big(\{\varphi(s)\}\big)\big| > \beta, \quad \text{for all } s \in U.$$

Take $t \in S$. Then $s \in S \mapsto \mu_s(\{t\}) \in \mathbb{C}$ is continuous. Indeed : from Lemma 2.2, $s \mapsto \mu_s$ is continuous for the weak-topology on $M(S)$. Since $\mu \mapsto \mu(\{t\})$ belongs to $M(S)^*$, it is continuous on $M(S)$ endowed with the weak-topology.

Let $s_0 \in U$, and define

$$U_1 = \big\{s \in U \mid \ \big|\mu_s\big(\{\varphi(s_0)\}\big)\big| > \beta\big\}.$$

From above, U_1 is an open subset of U (and so of S) which contains s_0. Since $\varphi(U_1)$ is infinite, one can find s_1 in U_1 satifying $\varphi(s_1) \neq \varphi(s_0)$. Then we have

$$\begin{aligned}&\big|\mu_{s_1}\big(\{\varphi(s_1)\}\big)\big| > \beta, \quad \text{since } s_1 \in U \\ &\big|\mu_{s_1}\big(\{\varphi(s_0)\}\big)\big| > \beta.\end{aligned}$$

Consider now

$$U_2 = \big\{s \in U_1 \mid \ \big|\mu_s\big(\{\varphi(s_1)\}\big)\big| > \beta\big\}.$$

It is an open subset of U containing s_1, and it contains an element s_2 such that $\varphi(s_2) \neq \varphi(s_0)$ and $\varphi(s_2) \neq \varphi(s_1)$ (since $\varphi(U_2)$ is infinite). Then we have, since $s_2 \in U_2 \subset U_1 \subset U$,

$$\begin{aligned}&\big|\mu_{s_2}\big(\{\varphi(s_2)\}\big)\big| > \beta \\ &\big|\mu_{s_2}\big(\{\varphi(s_1)\}\big)\big| > \beta \\ &\big|\mu_{s_2}\big(\{\varphi(s_0)\}\big)\big| > \beta.\end{aligned}$$

In such a way we construct a decreasing sequence of open subsets $U_n \subset U$, and a sequence of elements $(s_n)_{n\geq 0}$, $s_n \in U_n$ having the property

$$\begin{aligned}U_{n+1} &= \big\{s \in U_n \mid \ \big|\mu_s\big(\{\varphi(s_n)\}\big)\big| > \beta\big\}, \\ s_{n+1} &\in U_{n+1} \\ \varphi(s_{n+1}) &\notin \{\varphi(s_0), \ldots, \varphi(s_n)\}.\end{aligned}$$

So

$$\big|\mu_{s_n}\big(\{\varphi(s_j)\}\big)\big| > \beta, \qquad j = 0, \ldots, n-1$$

which leads to a contradiction writing that

$$\|T\| \geq \|\mu_{s_n}\| \geq |\mu_{s_n}|\big(\{\varphi(s_0), \dots, \varphi(s_{n-1})\}\big) \geq n\beta, \qquad \forall n \in \mathbb{N}.$$

■

We now give necessary conditions on φ and u to ensure that every weakly compact operator on $C(S)$ satisfies equation $(E_{u,\varphi})$. Actually we only need to consider rank-1 operators.

Theorem 2.8. *Let S be a compact Hausdorff space without isolated point. Assume that every rank-1 operator on $C(S)$ satisfies equation $(E_{u,\varphi})$. Then $|u|$ is constant and $\varphi^{-1}(\{t\})$ is nowhere dense in S, for every $t \in S$.*

Proof : We first show that $|u|$ is constant on S. Arguing by contradiction, assume there exists $s_0 \in S$ such that $|u(s_0)| < \|u\|_\infty$. Then there exists $\delta > 0$ and an open neighborhood U of s_0 satisfying

$$\forall s \in U,\ |u(s)| < \|u\|_\infty - \delta.$$

Choose a continuous function v such that : $0 \leq v \leq 1,\ v(s_0) = 1$ and $v(s) < 1$ for all $s \neq s_0$. We define the operator $T = v\delta_\tau$ where τ is an element of S. Then $\mu_s = T^*(\delta_s) = v(s)\delta_\tau,\ \|\mu_s\| = v(s)$. Choose $\varepsilon > 0$ such that we have $\{s \in S \mid\ v(s) > 1 - \varepsilon\} \subset U$. It follows that

$$\begin{aligned}
&\sup_{\{s \mid\ v(s) > 1-\varepsilon\}} \left(\big|u(s) + v(s)\delta_\tau\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + |v(s)|\delta_\tau\big(\{\varphi(s)\}\big)\Big) \right) \\
&\leq \sup_{s \in U} \left(|u(s)| + v(s)\delta_\tau\big(\{\varphi(s)\}\big) - \Big(\|u\|_\infty + v(s)\delta_\tau\big(\{\varphi(s)\}\big)\Big) \right) \\
&\leq \sup_{s \in U} |u(s)| - \|u\|_\infty \\
&\leq -\delta < 0.
\end{aligned}$$

The family of measures $(\mu_s)_{s \in S}$ does not satisfy condition (2.1) of Lemma 2.5 and consequently T does not satisfy equation $(E_{u,\varphi})$, which is false since T is a rank-1 operator. So $|u|$ is constant.

Now we prove that for every $t \in S,\ \varphi^{-1}(\{t\})$ is nowhere dense in S. Let U be a nonempty open subset of S. We want to find $s \in U$ such that $\varphi(s) \neq t$. Consider the rank-1 operator $T = \delta_t g u$, where $g \in C(S)$ such that $-1 \leq g \leq -\frac{1}{2},\ g = -\frac{1}{2}$ outside U and $\|g\|_\infty = 1$. Then $\|T\| = \|u\|_\infty > 0$, the family of measures associated to T is given by $\mu_s = T^*(\delta_s) = u(s)g(s)\delta_t$, and $\|\mu_s\| = |u(s)g(s)| = \|u\|_\infty |g(s)|$.
Take $\varepsilon = \|u\|_\infty/2$ so that $V = \{s \in S \mid\ |u(s)g(s)| > \frac{\|u\|_\infty}{2}\} \subset U$. Since T satisfies equation $(E_{u,\varphi})$, the family of measures $(\mu_s)_{s \in S}$ satisfies condition (2.1) of Lemma 2.5 :

$$\begin{aligned}
0 &= \sup_V \left(\big|u(s) + u(s)g(s)\delta_t\big(\{\varphi(s)\}\big)\big| - \Big(\|u\|_\infty + |u(s)g(s)|\delta_t\big(\{\varphi(s)\}\big)\Big) \right) \\
&= \sup_V \Big(2g(s)\|u\|_\infty \delta_t\big(\{\varphi(s)\}\big) \Big)
\end{aligned}$$

which is less than $\sup_{s\in U}\Big(2g(s)\|u\|_\infty\delta_t\big(\{\varphi(s)\}\big)\Big)$. It follows that there exists $s\in U$ such that $\delta_t\big(\{\varphi(s)\}\big)=0$, *i.e.* $\varphi(s)\neq t$ and therefore $U\not\subset\varphi^{-1}(\{t\})$. ■

Remark 2.9. Note that in a topological space S, and for a continuous map $\varphi:S\to S$, the following conditions are equivalent :

i) for every $t\in S$, $\varphi^{-1}(\{t\})$ is nowhere dense in S

ii) for every nonempty open subset U of S, $\varphi(U)$ is infinite.

Indeed, if there exists a nonempty open subset U of S such that $U\subset\varphi^{-1}(\{t\})$ then φ is constant on U, so $ii)\Rightarrow i)$. Moreover if $\varphi(U)=\{s_1,\dots,s_n\}$ for an open subset U of S, $n\geq 1$, then

$$\begin{aligned}\{s\in U\mid\ \varphi(s)=s_1\}&=\{s\in U\mid\varphi(s)\neq s_k,\ 2\leq k\leq n\}\\&=\varphi^{-1}\big(S\backslash\{s_2,\dots,s_n\}\big)\cap U.\end{aligned}$$

The set $S\backslash\{s_2,\dots,s_n\}$ is open in S, so $\{s\in U\mid\ \varphi(s)=s_1\}$ is a nonempty open subset of U (and of S) although by $i)$, $\{s\in S\mid\ \varphi(s)=s_1\}$ must have empty interior.

The previous remark, Theorem 2.7 and Theorem 2.8 give the following :

Corollary 2.10. *Let S be a compact Hausdorff space without isolated point. Then $\|uC_\varphi+T\|=\|u\|_\infty+\|T\|$ for every weakly compact operator $T:C(S)\to C(S)$ if and only if $|u|$ is constant on S and the set $\varphi^{-1}(\{t\})$ is nowhere dense in S, for every $t\in S$.*

Application : a negative answer to a question of Popov [10]

Note that if φ is onto and $|u|=1$ then uC_φ is an isometry on $C(S)$. In [10], Popov shows that every into isometry $J:L^1([0,1])\to L^1([0,1])$ is a Daugavet center. He raises the question whether this result is true when we substitute $L^1([0,1])$ with a Banach space X having the Daugavet property. Actually, this is not true for $X=C(S)$. To see this, consider any composition operator whose symbol φ is onto and constant on a nonempty open subset of S. Then C_φ is an isometry but there exists rank-1 operators on $C(S)$ which does not satisfy equation $(E_{1,\varphi})$.

After our work was completed, an example was independently produced in [2]. The authors considered a weighted composition operator $uC_\varphi:C[0,1]\to C[0,1]$ whose symbol φ is constant on $]1/2,1]$ and whose weight has not constant modulus on $[0,1]$.

2.2 Convex combinations of composition operators

One can wonder if the set of Daugavet centers is a convex set. Actually it is easy to see that this is not true in full generality. Indeed, consider $u(x)=e^{2i\pi x}$ and $v(x)=e^{-2i\pi x}$, $x\in[0,1]$. Then $u,v\in C[0,1]$, $|u|=|v|=1$ so uI and vI are Daugavet centers in $C([0,1])$, but $\big(u(x)+v(x)\big)/2=\cos 2\pi x$ which has not constant modulus on $[0,1]$. Therefore $(uI+vI)/2$ is not a Daugavet center in $C[0,1]$. Nevertheless it turns out that a convex combination of particular (non zero) Daugavet centers can be a Daugavet center. Let us consider the case of composition operators.

Note that a convex combination of composition operators is not in general a composition operator itself. Indeed, assume that $C_\varphi = tC_{\psi_1} + (1-t)C_{\psi_2}$ where φ, ψ_1, ψ_2 are continuous functions from S to S and $0 < t < 1$. Assume that $\varphi \neq \psi_1$ and take s_0 such that $\varphi(s_0) \neq \psi_1(s_0)$. Now consider a open subset U of S such that $\varphi(s_0) \in U$ and $\psi_1(s_0) \notin U$. Choose $f \in C(S)$, $\|f\|_\infty = 1$ satisfying $f\big(\varphi(s_0)\big) = 1$ and $|f| < 1$ out of U. Then

$$1 = |f\big(\varphi(s_0)\big)| \leq t\big|f\big(\psi_1(s_0)\big)\big| + (1-t)\big|f\big(\psi_2(s_0)\big)\big| < 1$$

which leads to a contradiction.

Let φ and ψ be continuous functions from S to S. Assume that $\varphi \neq \psi$. Define

$$S_1 = \{s \in S \mid \ \varphi(s) \neq \psi(s)\}.$$

Then S_1 is a nonempty open subset of S since S has no isolated point. Consider convex combinations of C_φ and C_ψ. For $t \in [0,1]$,we define $T_t = tC_\varphi + (1-t)C_\psi$. Point out that $\|T_t\| = 1$. For convenience, we note

$$\Delta_T(s) = \big|t + \mu_s\big(\{\varphi(s)\}\big)\big| + \big|1 - t + \mu_s\big(\{\psi(s)\}\big)\big| - \Big(1 + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + \big|\mu_s\big(\{\psi(s)\}\big)\big|\Big),$$

and

$$\tilde{\Delta}_T(s) = \big|1 + \mu_s\big(\{\varphi(s)\}\big)\big| - \Big(1 + \big|\mu_s\big(\{\varphi(s)\}\big)\big|\Big)$$

where $(\mu_s)_{s\in S}$ is the family of measures representing T and $s \in S$. As for weighted composition operators, we have the following property :

Proposition 2.11. *Let T be an operator on $C(S)$. Assume that the family of measures $(\mu_s)_{s\in S}$ representing T satisfies the condition : for every nonempty open set $U \subset S$:*

-If $U \cap S_1 \neq \emptyset$, then

$$\sup_{s\in U\cap S_1} \Delta_T(s) = 0 \tag{2.3}$$

-If $U \cap S_1 = \emptyset$, then

$$\sup_{s\in U} \tilde{\Delta}_T(s) = 0. \tag{2.4}$$

Then the following equation holds true :

$$\|T_t + T\| = 1 + \|T\|.$$

Proof. One only has to consider open subsets U of S of the form $U = \{s \in S \mid \ \|\mu_s\| > \|T\| - \varepsilon\}$ where $\varepsilon > 0$. If $U \cap S_1 = \emptyset$ then $\varphi = \psi$ on U so the proof of Lemma 2.5 tells us that $\|T_t + T\| \geq \sup_{s\in U} \|\delta_{\varphi(s)} + \mu_s\| \geq 1 + \|T\| - \varepsilon$. Else, $\|T_t + T\| \geq \sup_{s\in U\cap S_1} \|t\delta_{\varphi(s)} + (1-t)\delta_{\psi(s)} + \mu_s\|$ which is greater than $1 + \|T\| - \varepsilon$ using the same method as in the proof of Lemma 2.5. ■

From this we can deduce that any convex combination of composition operators which are Daugavet centers is still a Daugavet center.

Theorem 2.12. *Assume that C_φ and C_ψ are Daugavet center. Then every weakly compact operator T on $C(S)$ satisfies the norm equation*

$$\|tC_\varphi + (1-t)C_\psi + T\| = 1 + \|T\|,$$

for all $t \in [0,1]$.

Proof. Take $t \in [0,1]$. Argue by contradiction and assume that the family $(\mu_s)_{s\in S}$ does not satisfy conditions of Proposition 2.11.
First case : Let U be a nonempty open subset of S such that $U \cap S_1 \neq \emptyset$ and (2.3) does not hold. Then there exists $\beta > 0$ such that

$$\big|t + \mu_s\big(\{\varphi(s)\}\big)\big| + \big|1 - t + \mu_s\big(\{\psi(s)\}\big)\big| - \Big(1 + \big|\mu_s\big(\{\varphi(s)\}\big)\big| + \big|\mu_s\big(\{\psi(s)\}\big)\big|\Big) < -4\beta$$

for every $s \in U \cap S_1$. Then

$$\big|\mu_s\big(\{\varphi(s)\}\big)\big| + \big|\mu_s\big(\{\psi(s)\}\big)\big| > 2\beta, \quad \forall s \in U \cap S_1.$$

Let

$$\begin{aligned} V_1 &= \{s \in U \cap S_1 \mid \ \big|\mu_s\big(\{\varphi(s)\}\big)\big| > \beta\} \\ V_2 &= \{s \in U \cap S_1 \mid \ \big|\mu_s\big(\{\psi(s)\}\big)\big| > \beta\}. \end{aligned}$$

Since $U \cap S_1 \subset V_1 \cup V_2$, we can assume without loss of generality that V_1 contains a nonempty open set V. So $\big|\mu_s\big(\{\varphi(s)\}\big)\big| > \beta$ for every $s \in V$. Then we follow the proof of Theorem 2.7 to obtain a contradiction.
Second case : If U is a nonempty open subset of S such that $U \subset S \backslash S_1$ and (2.4) does not hold, then the same proof as in Theorem 2.7 leads to a contradiction. ■

2.3 Operators factorizing through an Asplund space

The aim of this section is to extend a result of Ansari in [1] stating that every operator on $C(S)$ factorizing through c_0 satisfies the Daugavet equation. Let $T : C(S) \to C(S)$ be an operator, where S is a compact Hausdorff space, and $(\mu_s)_{s\in S}$ the family of measures associated to T. Note that if $\mu_s\big(\{\varphi(s)\}\big) = 0$ for all $s \in S$, and $|u|$ is constant, then T trivially satisfies condition (2.2) of corollary (2.6). Actually, it is sufficient that the measures (μ_s) almost satisfies this condition :

Define $S_\varepsilon = \big\{s \in S \mid \ \big|\mu_s\big(\{\varphi(s)\}\big)\big| < \varepsilon\big\}$, for each $\varepsilon > 0$. We have the following :

Lemma 2.13. *If $|u|$ is constant and if the sets S_ε are dense in S, for every $\varepsilon > 0$, then*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

Proof. Take U a nonempty open set in S and $\varepsilon > 0$. By density of S_ε in S, there exists $s_\varepsilon \in U$ satisfying $\big|\mu_{s_\varepsilon}\big(\{\varphi(s_\varepsilon)\}\big)\big| < \varepsilon$, and so

$$\begin{aligned} \big|u(s_\varepsilon) + \mu_{s_\varepsilon}\big(\{\varphi(s_\varepsilon)\}\big)\big| - \big(\|u\|_\infty + \big|\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})\big|\big) &\geq -2\big|\mu_{s_\varepsilon}(\{\varphi(s_\varepsilon)\})\big| \\ &> -2\varepsilon. \end{aligned}$$

Thus for every nonempty open set U of S, we have

$$\sup_{s\in U}\left(\left|u(s)+\mu_s\big(\{\varphi(s)\}\big)\right|-\left(\|u\|_\infty+\left|\mu_s\big(\{\varphi(s)\}\big)\right|\right)\right)=0.$$

We conclude with corollary (2.6). ∎

Now we can prove the following result :

Theorem 2.14. *Let S be a compact Hausdorff space without isolated point, $\varphi : S \to S$ a continuous map, $u \in C(S)$ and $T : C(S) \to C(S)$ an operator such that $T^*(M(S))$ is separable. If $\varphi^{-1}(\{t\})$ is nowhere dense in S, for every $t \in S$, and if $|u|$ is constant, then T satisfies equation $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$.*

Proof. Let $\{\rho_n,\ n \in \mathbb{N}\}$ be a dense subset of $T^*\big(M(S)\big)$. As previously, $S_\varepsilon = \big\{s \in S \mid \ \left|\mu_s\big(\{\varphi(s)\}\big)\right| < \varepsilon\big\}$, where $\mu_s = T^*(\delta_s)$, and $A = \bigcap_{n\geq 0}\big\{s \in S \mid \ \rho_n\big(\{\varphi(s)\}\big) = 0\big\}$. We want to show that :

i) A is dense in S.

ii) $\forall\varepsilon > 0,\ A \subset S_\varepsilon$.

Then we conclude with Lemma 2.13.
To prove $i)$, we are going to show that $S\backslash A$ is nowhere dense. Indeed,

$$S\backslash A = \bigcup_{n\geq 0}\big\{s \in S \mid \ \rho_n\big(\{\varphi(s)\}\big) \neq 0\big\} = \bigcup_{n\geq 0}\bigcup_{p\geq 1} A_{n,p}$$

where $A_{n,p} = \big\{s \in S \mid \ \left|\rho_n\big(\{\varphi(s)\}\big)\right| > \frac{1}{p}\big\}$. Since ρ_n is a finite measure, this implies that the sets $\varphi(A_{n,p})$ are finite (hence closed) for every $n \geq 0$, $p \geq 1$. But $A_{n,p} \subset \varphi^{-1}\big(\varphi(A_{n,p})\big)$ which a finite union of nowhere dense sets (c.f. Remark 2.9). Using Baire's theorem, $S\backslash A$ is contained in a nowhere dense set, and A is dense in S.
Proof of $ii)$: let $s \in S$ and $\varepsilon > 0$. By density of $(\rho_n)_n$ in $T^*(M(S))$, there exists an integer $n_0 \geq 0$ such that $\|T^*(\delta_s) - \rho_{n_0}\| < \varepsilon$. Then $|\mu_s - \rho_{n_0}|\big(\{\varphi(s)\}\big) < \varepsilon$. Taking $s \in A$, it follows that $\left|\mu_s\big(\{\varphi(s)\}\big)\right| < \varepsilon$, *i.e.* $A \subset S_\varepsilon$. ∎

If $T : C(S) \to C(S)$ factorizes through a space X having a separable dual, then Theorem 2.14 applies. In particular this holds for the class of operators factorizing through c_0. Actually, regarding operators factorizing through a space X, one does not need to assume that X^* is separable in the case where S is metrizable. We recall the following definition :

Definition 2.15. *A Banach space X is called an Asplund space if its dual space has the Radon-Nikodým property.*

Every dual space which is separable has the Radon-Nikodým property, and so every Banach space with separable dual is Asplund. Asplund spaces are characterized by the fact that every separable subspace has a separable dual.

Corollary 2.16. *Let S be a metric compact space without isolated points, $\varphi : S \to S$ a continuous map, $u \in C(S)$ and $T : C(S) \to C(S)$ an operator factorizing through an Asplund space X. If $\varphi^{-1}(\{t\})$ is nowhere dense in S, for every $t \in S$, and if $|u|$ is constant on S, then T satisfies equation $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$.*

Proof. Write $T = T_2T_1$ with $T_1 : C(S) \to X$ and $T_2 : X \to C(S)$. Since S is a metric compact space, $C(S)$ is separable. So we can assume, by replacing X by $\overline{T_1\big(C(S)\big)}$ that X^* is separable. Thus $T^*(M(S))$ is separable, and the result follows from Theorem 2.14. ■

Remark 2.17. Since every compact operator factorizes through a subspace of c_0, this gives another proof of Theorem 2.7 for compact operators on $C(S)$. Moreover every weakly compact operator factorizes through a reflexive space (which is Asplund), giving another proof of Theorem 2.7 for weakly compact operators on $C(S)$ where S is a metric compact space without isolated point.

In the case where $uC_\varphi = I$, Theorem 2.14 is a particular case of an already known result in Banach spaces with the Daugavet property. If we consider a Banach space X having the Daugavet property, then every operator $T : X \to X$ such that $T^*(X^*)$ is separable satisfies the Daugavet equation. This can be seen by using a result of Shvidkoy [11] which says that an operator $T : X \to X$ not fixing a copy of ℓ_1 satisfies the Daugavet equation. Then it is obvious that if T fixes a copy of ℓ_1 then T^* fixes a copy of ℓ_∞, hence $T^*(X^*)$ is not separable.

As another immediate consequence of Theorem 2.14, we have the following for particular weighted composition operators (which can also be viewed directly) :

Corollary 2.18. *Let S be a compact Hausdorff space without isolated points, $\varphi : S \to S$ be a continuous map and $u \in C(S)$. If for every $t \in S$, $\varphi^{-1}(\{t\})$ is nowhere dense in S, and if $|u|$ is constant, then $uC_\varphi : C(S) \to C(S)$ does not factorize through a space having a separable dual space. If S is metrizable, then uC_φ does not factorize through an Asplund space.*

3 Equation $(E_{u,\varphi})$ for classes of operators on $A(\mathbb{D})$

In this section, we want to adapt D. Werner's method in [14] to find new Daugavet centers in subspaces of $C(S)$-spaces, and particularly for the disk algebra $A(\mathbb{D})$. Actually we will consider weighted composition operators uC_φ on a functionnal Banach space X and will formulate conditions on an isometric embedding of X into $C(S)$ implying that X is (u, φ)-nicely embedded. Then we find conditions so that every weakly compact operator on a (u, φ)-nicely embedded space satisfies equation $\|uC_\varphi + T\| = \|uC_\varphi\| + \|T\|$.

3.1 General approach

Let $(X, \|.\|)$ denotes a functional Banach space on Ω $\big(X \subset \mathcal{F}(\Omega, \mathbb{C})\big)$. Consider φ a map such that $\varphi(\Omega) \subset \Omega$ and $u \in X$ such that $0 < \|u\| < \infty$. Assume that $uC_\varphi : f \in X \mapsto u.(f \circ \varphi) \in X$ is a weighted composition operator acting continuously on X. Let S be a compact Hausdorff space without isolated point. An isometry $J : X \to C(S)$ is said to be a (u,φ)-*nice embedding* and X is said to be (u,φ)-*nicely embedded* into $C(S)$ if the following conditions are satisfied for every $s \in S$:

(C1) if $p_s = (uC_\varphi)^* J^*(\delta_s) \in X^*$, then $\|p_s\| = \|u\| > 0$.

(C2) $\mathrm{Vect}(p_s)$ is an L-summand in X^*.

Recall that a closed subspace F of a Banach space E is an L-summand if there exists a projection Π from E onto F such that, for every $x \in E$,

$$\|x\| = \|\Pi x\| + \|x - \Pi x\|.$$

We say that F is an M-ideal if its annihilator $F^\perp \subset E^*$ is an L-summand. Then condition **(C2)** can be reformulated as : $\ker(p_s)$ is an M-ideal in X. Condition **(C1)** forces $\|uC_\varphi\| = \|u\|$.

Assume that X is (u,φ)-nicely embedded in $C(S)$. Condition **(C2)** provides us a family of projections $(\Pi_s)_{s\in S}$ satisfying

$$\|x^*\| = \|\Pi_s x^*\| + \|x^* - \Pi_s x^*\|, \quad \text{for every } x^* \in X^*$$

and a family $(\pi_s)_{s\in S}$ in X^{**} such that

$$\Pi_s x^* = \pi_s(x^*) p_s, \quad \text{for every } x^* \in X^*.$$

Note that $\pi_s(p_s) = 1$.

Consider the equivalence relation $\sim$ on S

$$s \sim t \Leftrightarrow \Pi_s = \Pi_t.$$

Note E_s the class of s in S. Then E_s is closed, and condition **(C1)** tells us that $E_s = \{t \in S \mid \ p_t = \lambda p_s,\ \lambda \in \mathbb{T}\}$. We will need the following condition :

(C3) for all $s \in S$, the class E_s is nowhere dense in S.

Let $T : X \to X$ be an operator, and $q_s = (JT)^*(\delta_s) \in X^*,\ s \in S$. Then $s \mapsto q_s$ is continuous for the weak*-topology on X^*, and $\|T\| = \sup_S \|q_s\|$.
We can now express some results, whose proofs are similar to those in section 2 and are given in [14] in the particular case where $\varphi(x) = x,\ x \in \Omega$ and $u \equiv 1$.

Proposition 3.1. *Suppose X is (u,φ)-nicely embedded in $C(S)$, and T is an operator acting on X. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

if and only if

$$\textit{for every } \varepsilon > 0, \quad \sup_{\{s \mid \ \|q_s\| > \|T\| - \varepsilon\}} \Big(|1 + \pi_s(q_s)| - \big(1 + |\pi_s(q_s)|\big) \Big) = 0.$$

Proposition 3.2. *Suppose that X is (u,φ)-nicely embedded in $C(S)$, and that condition **(C3)** holds. Let T be an operator on X. If we have*

$$\textit{for all } t \in S,\ s \mapsto \pi_t(q_s) \textit{ is continuous,}$$

then T satisfies equation $(E_{u,\varphi})$.

Remark 3.3. Every weakly compact operator T on X fulfills conditions of Proposition 3.2, and consequently the equality $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ holds.

We want to obtain this result for the class of operators whose adjoint has separable range. Let us start with a lemma which will be useful for the proof of the next proposition :

Lemma 3.4. *([14], Lemma 2.3) Suppose X is (u,φ)-nicely embedded in $C(S)$. If $t_1,\dots,t_k$ are pairwise nonequivalent points (for the relation $\sim$), then*

$$\|x^*\| \geq \sum_{j=1}^{k} \|\Pi_{t_j}(x^*)\|, \quad \textit{for every } x^* \in X^*.$$

Proposition 3.5. *Let X be a (u,φ)-nicely embedded space in $C(S)$ and satisfying condition **(C3)**, and T be an operator on X such that $T^*(X^*)$ is separable. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

Proof. Consider the sets $S_\varepsilon = \{s \in S \mid \ |\pi_s(q_s)| < \varepsilon\}$, where $q_s = (JT)^*(\delta_s)$ and $\varepsilon > 0$. If we show that S_ε is dense in S, for every $\varepsilon > 0$, then T fulfills the condition of Proposition 3.1.

Let $\{\psi_n,\ n \in \mathbb{N}\}$ be a dense subset of $T^*(X^*)$ and define

$$A = \big\{s \in S \mid \ \pi_s(\psi_n) = 0,\ \forall n \in \mathbb{N}\big\}.$$

As in the proof of Theorem 2.14 we want to show that :

i) A is dense in S

ii) $\forall \varepsilon > 0,\ A \subset S_\varepsilon$.

The proof of *ii)* is similar to the one in Theorem 2.14. For *i)*, we show that $S\backslash A$ is nowhere dense. Indeed

$$S\backslash A = \bigcup_{n\geq 0} \big\{s \in S \mid \ \pi_s(\psi_n) \neq 0\big\} = \bigcup_{n\geq 0} \bigcup_{p\geq 1} A_{n,p}$$

where $A_{n,p} = \big\{s \in S \mid \ \big|\pi_s(\psi_n)\big| > \frac{1}{p}\big\}$. By Lemma 3.4 there is a finite number of equivalence classes for $\sim$ in $A_{n,p}$ (less than $p\|\psi_n\|/\|u\|$). These equivalence classes are closed and nowhere dense (by condition **C3**). The Baire property yields that $S\backslash A$ is contained in a nowhere dense set, implying that A is dense in S. ■

In the case where X is separable, we have the same result as in Corollary 2.16.

Corollary 3.6. *Let X be a (u,φ)-nicely embedded space in $C(S)$ satisfying condition **(C3)**, and T be an operator on X which factorizes through an Asplund space E. Assume that X is separable. Then*

$$\|uC_\varphi + T\| = \|u\| + \|T\|$$

3.2 Applications

Obviously one can apply this results to the case $X = C(S)$ where S is a compact Hausdorff space without isolated point, with J the natural inclusion into $C(S)$. If $\varphi : S \to S$ is continuous and if $u \in C(S)$ with $u \neq 0$, then $p_s = u(s)\delta_{\varphi(s)}$ so that condition **(C1)** forces $|u|$ to be constant equal to $\|u\|_\infty$. Then $\Pi_s : \mu \in C(S)^* \mapsto \big(\mu\big(\{\varphi(s)\}\big)/u(s)\big)p_s$ is a L-projection, and condition **(C2)** holds. Finally for s and t in S,

$$\begin{aligned} s \sim t &\Leftrightarrow \exists \lambda \in \mathbb{T},\ p_s = \lambda p_t \\ &\Leftrightarrow \exists \lambda \in \mathbb{T},\ |u(s)|\delta_{\varphi(s)} = \lambda |u(t)|\delta_{\varphi(t)} \\ &\Leftrightarrow \varphi(s) = \varphi(t). \end{aligned}$$

Thus $E_s = \varphi^{-1}\big(\{\varphi(s)\}\big)$. So condition **(C3)** is fulfilled if $\varphi^{-1}(\{t\})$ is nowhere dense, for every $t \in S$. Therefore we recover most of the results of section 2.

We now turn to the disk algebra $A(\mathbb{D})$. Let $\mathbb{D} = \{z \in \mathbb{C} \mid \ |z| < 1\}$ denote the unit disk. The disk algebra $A(\mathbb{D})$ is the algebra of holomorphic maps on $\mathbb{D}$ which are continuous on $\overline{\mathbb{D}}$, endowed with the supremum norm $\|f\|_\infty = \sup\{|f(z)| \mid \ z \in \mathbb{D}\}$. Considering $u \neq 0$ and φ in the disk algebra with $\|\varphi\|_\infty \leq 1$, one can define the weighted composition operator uC_φ acting on $A(\mathbb{D})$, with $\|uC_\varphi\| = \|u\|_\infty$. We will assume that φ is not constant (implying $\varphi(\mathbb{D}) \subset \mathbb{D}$), otherwise uC_φ is a rank-1 operator and therefore is not a Daugavet center. On the other hand, C_φ is compact on $A(\mathbb{D})$ if and only if $\|\varphi\|_\infty < 1$. Hence a necessary condition ensuring that every weakly compact operator on $A(\mathbb{D})$ fulfills equation $(E_{u,\varphi})$ is that $\|\varphi\|_\infty = 1$. Actually, we are going to prove that a strong negation of "$\|\varphi\|_\infty < 1$" is necessary.

Consider the isometry $J : f \in A(\mathbb{D}) \mapsto J(f) = f_{|\mathbb{T}} \in C(\mathbb{T})$. It is well known that the image of $A(\mathbb{D})$ by J is the closed space $\{f \in C(\mathbb{T}) \mid \ \hat{f}(n) = 0,\ \forall n < 0\}$. We want conditions on φ and u implying that $A(\mathbb{D})$ is (u, φ)-nicely embedded in $C(\mathbb{T})$, and moreover that condition **(C3)** is fulfilled.

For $\omega \in \mathbb{T}$, let

$$p_\omega := (uC_\varphi)^* J^*(\delta_\omega) = u(\omega)\delta_{\varphi(\omega)}{}_{|_{A(\mathbb{D})}} \in A(\mathbb{D})^*.$$

Clearly $\|p_\omega\| = |u(\omega)|$, so **(C1)** is fulfilled if and only if $|u|$ is constant on $\mathbb{T}$. Assume that $|u|$ is constant on $\mathbb{T}$. To check condition **(C2)**, we have to show that $\ker(p_\omega)$ is an M-ideal. But

$$\begin{aligned} \ker(p_\omega) &= \{f \in A(\mathbb{D}) \mid \ u(\omega) f\big(\varphi(\omega)\big) = 0\} \\ &= \{f \in A(\mathbb{D}) \mid \ f\big(\varphi(\omega)\big) = 0\} \end{aligned}$$

since $u(\omega) \neq 0$. It is an M-ideal if and only if $\varphi(\omega) \in \mathbb{T}$ (see [6] p. 4). It means that **(C2)** is fulfilled if φ is an inner function. Finally, if $\omega_1, \omega_2 \in \mathbb{T}$,

$$\begin{aligned} \omega_1 \sim \omega_2 &\Leftrightarrow \exists \lambda \in \mathbb{T},\ p_{\omega_1} = \lambda p_{\omega_2} \\ &\Leftrightarrow \exists \lambda \in \mathbb{T},\ |u(\omega_1)|\delta_{\varphi(\omega_1)} = \lambda |u(\omega_2)|\delta_{\varphi(\omega_2)} \text{ on } A(\mathbb{D}) \\ &\Leftrightarrow \varphi(\omega_1) = \varphi(\omega_2). \end{aligned}$$

Thus $E_\omega = \varphi^{-1}\big(\{\varphi(\omega)\}\big) \cap \mathbb{T}$. If φ is not constant, then condition **(C3)** is fulfilled.

To summarize, we have the following :

Proposition 3.7. *Let φ be an inner function and u be a multiple of an inner function. If $T : A(\mathbb{D}) \to A(\mathbb{D})$ is such that $T^*\big(A(\mathbb{D})^*\big)$ is separable, then equation $\|uC_\varphi + T\| = \|u\|_\infty + \|T\|$ holds true.*

Since $A(\mathbb{D})$ is separable, Remark 2.17 and Corollary 3.6 gives :

Corollary 3.8. *Let φ be an inner function and u be a multiple of an inner function. Then every weakly compact operator $T : A(\mathbb{D}) \to A(\mathbb{D})$ satifies equation*

$$\|uC_\varphi + T\| = \|u\|_\infty + \|T\|.$$

This corollary leads to the following remark on (general) essential norms of weighted composition operators on the disk algebra.

Remark 3.9. Let X be a Banach space, $B(X)$ be the space of bounded operator on X, $\mathcal{K}(X)$ be the closed subspace of $B(X)$ consisting of compact operators on X and $\mathcal{W}(X)$ be the closed subspace of $B(X)$ consisting of weakly-compact operators on X. Recall that if $\mathcal{I}$ is a closed subspace of $B(X)$, the essential norm (relatively to $\mathcal{I}$) of $S \in B(X)$ is the distance from S to $\mathcal{I}$:

$$\|S\|_{e,\mathcal{I}} = \inf\{\|S+T\|;\ T \in \mathcal{I}\}.$$

This is the canonical norm on the quotient space $B(X)/\mathcal{I}$. The classical case corresponds to the case of compact operators $\mathcal{I} = \mathcal{K}(X)$. In this case, the above quotient space is the Calkin algebra. General essential norms of weighted composition operators on $A(\mathbb{D})$ are estimated in [8]. When $\mathcal{I} \subset \mathcal{W}\big(A(\mathbb{D})\big)$, and in the particular case where φ is an inner function ($\varphi(\mathbb{D}) \subset \mathbb{D}$) and u is a multiple of an inner function, Corollary 3.8 not only gives us the essential norm relatively to $\mathcal{I}$ of uC_φ, but how the norm of uC_φ reacts under perturbation by operators in the class $\mathcal{I}$.

Although D. Werner's method gives sufficient conditions ensuring that weighted composition operators are Daugavet centers, it turns out that these ones are also necessary.

Proposition 3.10. *Suppose that every rank-1 operator on $A(\mathbb{D})$ satisfies equation $(E_{u,\varphi})$. Then φ is inner and $|u|$ is constant on $\mathbb{T}$.*

Proof. Assume φ is not inner. Then there exists $\omega \in \mathbb{T}$ such that $|\varphi(\omega)| = r < 1$. As u is not constant equal to zero, we can assume, taking if necessary a $\omega' \in \mathbb{T}$ close to ω, that $u(\omega) \neq 0$. Let $g \in A(\mathbb{D})$ defined by $g(z) = (1 + \bar{\omega} z)/2$, where $z \in \mathbb{D}$. We consider the rank-1 operator $T : f \mapsto Tf = u(\omega) f\big(\varphi(\omega)\big) g$, for all $f \in A(\mathbb{D})$. We have $\|T\| = |u(\omega)|$. For $0 < \varepsilon < \min(1 - r, |u(\omega)|/3)$, there exists an arc $I_\omega \subset \mathbb{T}$ containing ω such that for every $z \in I_\omega$, we have $|\varphi(z) - \varphi(\omega)| \leq \varepsilon$, $|1 - g(z)| < 1/2$ and $|u(z) - u(\omega)| < \varepsilon$. Let $f \in A(\mathbb{D})$ with $\|f\|_\infty = 1$:

$$\|uC_\varphi(f) - Tf\| = \sup_{|z|=1} \big|u(z) f\big(\varphi(z)\big) - u(\omega) f\big(\varphi(\omega)\big) g(z)\big|$$

If $z \notin I_\omega$, then $\left|u(z)f\big(\varphi(z)\big) - u(\omega)f\big(\varphi(\omega)\big)g(z)\right| \leq \|u\|_\infty + |u(\omega)| \sup\limits_{z \in \mathbb{T}\setminus I_\omega} |g(z)|$.

For any $a, b \in D(0, r+\varepsilon)$, we have by the Cauchy formula :

$$\begin{aligned} |f(a) - f(b)| &= \left| \frac{1}{2i\pi} \int_{|z|=1} \frac{a-b}{(z-a)(z-b)} f(z)\mathrm{d}z \right| \\ &\leq \frac{|a-b|}{2\pi} \int_0^{2\pi} \frac{\left|f\big(e^{i\theta}\big)\right|}{\left|e^{i\theta}-a\right|\left|e^{i\theta}-b\right|}\mathrm{d}\theta \\ &\leq \frac{|a-b|}{\big(1-(r+\varepsilon)\big)^2}. \end{aligned}$$

Thus for $z \in I_\omega,\ \varphi(z) \in D(0, r+\varepsilon)$ and we have :

$$\begin{aligned} \left|u(z)f\big(\varphi(z)\big) - u(\omega)f\big(\varphi(\omega)\big)g(z)\right| &\leq |u(z)|\left|f\big(\varphi(z)\big) - f\big(\varphi(\omega)\big)\right| + |u(z)-u(\omega)|\left|f\big(\varphi(\omega)\big)\right| \\ &\quad + \left|u(\omega)f\big(\varphi(\omega)\big)\right||1-g(z)| \\ &\leq \|u\|_\infty \left(\frac{\varepsilon}{\big(1-(r+\varepsilon)\big)^2} \right) + \varepsilon + \frac{|u(\omega)|}{2} \\ &\leq \|u\|_\infty + \frac{5}{6}|u(\omega)| \end{aligned}$$

for a suitable $\varepsilon > 0$. So

$$\|uC_\varphi(f) - Tf\| \leq \max\left(\|u\|_\infty + \frac{5}{6}|u(\omega)|, \|u\|_\infty + |u(\omega)|\delta \right),$$

where $\delta = \sup\limits_{z \in \mathbb{T}\setminus I_\omega} |g(z)| < 1$. This gives $\|C_\varphi - T\| < \|u\|_\infty + |u(\omega)| = \|C_\varphi\| + \|T\|$ which is absurd. So φ is an inner function.

We use a similar argument to show that $|u|$ is constant on the unit circle. ■

We summarize our results in the following corollary :

Corollary 3.11. *Let $\varphi \in A(\mathbb{D})$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $u \in A(\mathbb{D})$. Then uC_φ is a Daugavet center in $A(\mathbb{D})$ if and only if φ is an inner function and u is a multiple of an inner function.*

Remark 3.12. The case of the disk algebra is different from the case of $C(S)$. Indeed, we have seen that a function $\varphi : S \to S$ could induce a composition operator C_φ on $C(S)$ which is an isometry but is not a Daugavet center (see section 2). Whereas if $\varphi \in A(\mathbb{D})$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$, φ is an inner function if and only if C_φ is an isometry, and so C_φ is an isometry if and only if C_φ is a Daugavet center.

Note that in the particular case where φ is a disk automorphism it is easy to see that C_φ is a Daugavet center. Indeed, consider a weakly compact operator T on $A(\mathbb{D})$. Then $C_\varphi + T = C_\varphi(I + C_\varphi^{-1}T)$. The fact that C_φ is an isometry implies that $\|C_\varphi + T\| = \|I + C_\varphi^{-1}T\|$. Since the disk algebra has the Daugavet property (see [15]) and C_φ^{-1} is itself an isometry, we have

$$\|C_\varphi + T\| = 1 + \|C_\varphi^{-1}T\| = 1 + \|T\|.$$

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