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► **To cite this version:**

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza. Some translation-invariant Banach function spaces which contain  $c_0$ . *Studia Mathematica*, INSTYTUT MATEMATYCZNY \* POLSKA AKADEMIA NAUK, 2004, 163 (2), pp.137 - 155. hal-00441506

HAL Id: hal-00441506

<https://hal-univ-artois.archives-ouvertes.fr/hal-00441506>

Submitted on 16 Dec 2009

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# Some translation-invariant Banach function spaces which contain $c_0$

*Pascal Lefèvre, Daniel Li,  
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**Abstract.** *We produce several situations where some natural subspaces of classical Banach spaces of functions over a compact abelian group contain the space  $c_0$ .*

**Mathematics Subject Classification.** Primary: 43A46, 46B20; Secondary: 42A55, 42B35, 43A07, 46E30

**Key-words.** cotype; Hilbert set; invariant mean; Orlicz space;  $q$ -Rider set; Riemann-integrable function; Rosenthal set; set of uniform convergence; Sidon set;  $p$ -Sidon set.

## 1 Introduction

Let  $G$  be a compact abelian group and  $\Gamma = \widehat{G}$  its dual group. It is a familiar theme in Harmonic Analysis to compare the “thinness” properties of a subset  $\Lambda \subseteq \Gamma$  with the Banach space properties of the space  $X_\Lambda$ , where  $X$  is a Banach space of Haar-integrable functions on  $G$  and  $X_\Lambda$  is the subspace of  $X$  consisting of the  $f \in X$  whose spectrum lies in  $\Lambda$ :  $\widehat{f}(\gamma) = 0$  if  $\gamma \notin \Lambda$ . We refer to Kwapien-Pelczyński’s classical paper [17] for such investigations.

It is known that, denoting by  $\Psi_2$  the Orlicz function  $e^{x^2} - 1$ :

- (1) If  $L_\Lambda^{\Psi_2} = L_\Lambda^2$ , then  $\Lambda$  is a Sidon set (Pisier [35], Théorème 6.2);
- (2) If  $\mathcal{C}_\Lambda$  has a finite cotype, then  $\Lambda$  is a Sidon set (Bourgain-Milman [3]).

Recall that  $\Lambda$  is a *Sidon set* if every continuous function on  $G$  with spectrum in  $\Lambda$  has an absolutely convergent Fourier series.

In a previous paper, we proved, among other facts, the following extension of (1) ([19], Theorem 2.3):

- (1') If  $L_\Lambda^{\Psi_2}$  has cotype 2, then  $\Lambda$  is a Sidon set;

We also showed the following variant of (2) ([19], Theorem 1.2):

- (2') If  $U_\Lambda$  has a finite cotype, then  $\Lambda$  is a Sidon set,

where  $U = U(\mathbb{T})$  is the space of the continuous functions on the circle group  $\mathbb{T}$  whose Fourier series converges uniformly on  $\mathbb{T}$ .

In this work, we study what are the implications on  $\Lambda$  of the fact that some Banach space  $X_\Lambda$  contains, or not, the space  $c_0$ . In particular, we shall extend (1') and (2').

The paper is organized as follows. In Section 2, we show that if  $\psi$  is an Orlicz function which violates the  $\Delta_2$ -condition, in a strong sense:  $\lim_{x \rightarrow +\infty} \frac{\psi(2x)}{\psi(x)} = +\infty$  (which is the case of  $\Psi_2$ ), and if  $X_0$  is a linear subspace of  $L^\infty$  on which the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\psi$  are not equivalent, then the closure  $X$  of  $X_0$  in  $L^\psi$  contains  $c_0$ . It follows that if  $\Lambda$  is not a Sidon set, then  $L_\Lambda^{\Psi_2}$  contains  $c_0$ , and *a fortiori* that if  $L_\Lambda^{\Psi_2}$  has a finite cotype, then  $\Lambda$  is a Sidon set, which generalizes (1').

In Section 3, we extend (2') by showing that: If  $\Lambda$  is not a set of uniform convergence (*i.e.* if  $U_\Lambda \neq \mathcal{C}_\Lambda$ ), then  $U_\Lambda$  does contain  $c_0$ . In particular, if  $U_\Lambda$  has a finite cotype, then  $U_\Lambda = \mathcal{C}_\Lambda$ , so  $\mathcal{C}_\Lambda$  has a finite cotype and therefore, in view of (2),  $\Lambda$  is a Sidon set. This explains why the proof of (2') in [19] mimicked Bourgain and Milman's.

In Section 4, we use the notion of invariant mean in  $L^\infty(G)$ . We say that  $\Lambda$  is a *Lust-Piquard set* if, for every function  $f \in L_\Lambda^\infty$ , the product  $\gamma f$  of  $f$  with every character  $\gamma \in \Gamma$  has a unique invariant mean. Of course, if every  $f \in L_\Lambda^\infty$  is continuous (*i.e.*  $\Lambda$  is a Rosenthal set), then  $\Lambda$  is a Lust-Piquard set. F. Lust-Piquard ([27]) showed that there are Lust-Piquard sets which are not Rosenthal sets, and, more precisely, that  $\Lambda = \mathbb{P} \cap (5\mathbb{Z} + 2)$ , where  $\mathbb{P}$  is the set of the prime numbers, is a Lust-Piquard set such that  $\mathcal{C}_\Lambda$  contains  $c_0$  (if  $\Lambda$  is a Rosenthal set,  $\mathcal{C}_\Lambda$  cannot contain  $c_0$ ). We construct here another kind of "big" Lust-Piquard set  $\Lambda$ , namely a Hilbert set. Then  $\mathcal{C}_\Lambda$  contains  $c_0$  by a result of the second-named author ([22], Theorem 2).

In Section 5, we investigate under which conditions the space  $\mathcal{C}_\Lambda$  is complemented in  $L_\Lambda^\infty$ . We conjecture that this happens only if  $\mathcal{C}_\Lambda = L_\Lambda^\infty$ , *i.e.*  $\Lambda$  is a Rosenthal set. We are only able to show that, under that condition of complementation,  $\mathcal{C}_\Lambda$  does not contain  $c_0$ , and, moreover, every  $f \in L_\Lambda^\infty$  which is Riemann-integrable is actually in  $\mathcal{C}_\Lambda$ .

**Notation.** Throughout this paper,  $G$  is a compact abelian group, and  $\Gamma = \widehat{G}$  is its (discrete) dual group. The Haar measure of  $G$  is denoted by  $m$ , and integration with respect to  $m$  by  $dt$  or  $dx$ . We shall write the group structure of  $\Gamma$  additively, so that, for  $\gamma \in \Gamma$ , the character  $(-\gamma) \in \Gamma$  is the function  $\overline{\gamma} \in \mathcal{C}(G)$ . When  $G$  is the circle group  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ , we identify, as usual, the character  $e_n: t \mapsto e^{int}$  with the integer  $n \in \mathbb{Z}$ , and so the dual group  $\Gamma$  to  $\mathbb{Z}$ ; the Haar measure is then  $dt/2\pi$ .

For  $f \in L^1(G)$ , the Fourier coefficient of  $f$  at  $\gamma \in \Gamma$  is  $\widehat{f}(\gamma) = \int_G f(t) \overline{\gamma(t)} dt$ . If  $X$  is a linear function subspace of  $L^1(G)$ , we denote by  $X_\Lambda$  the subspace of those  $f \in X$  for which the Fourier coefficients vanish outside of  $\Lambda$ .

When we say that a Banach space  $X$  contains a Banach space  $Y$ , we mean that  $X$  contains a (closed) subspace isomorphic to  $Y$ .

**Acknowledgements.** This work was partly supported by a Picasso project (EGIDE-MCYT) between the french and spanish governments.

We thank the referee for a very careful reading of this paper and for many suggestions to improve the writing.

## 2 Subspaces of Orlicz spaces

Let  $\psi$  be an Orlicz function, that is, an increasing convex function  $\psi: [0, +\infty[ \rightarrow [0, +\infty[$  such that  $\psi(0) = 0$  and  $\psi(+\infty) = +\infty$ . We shall assume that  $\psi$  violates the  $\Delta_2$ -condition, in the following strong sense:

$$\lim_{x \rightarrow +\infty} \frac{\psi(2x)}{\psi(x)} = +\infty. \quad (*)$$

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. The Orlicz space  $L^\psi(\Omega)$  is the space of all the (equivalence classes of) measurable functions  $f: \Omega \rightarrow \mathbb{C}$  for which there is a constant  $C \geq 0$  such that

$$\int_{\Omega} \psi\left(\frac{|f(t)|}{C}\right) d\mathbb{P}(t) \leq 1$$

and then  $\|f\|_\psi$  is the least possible constant  $C$ .

Observe that (\*) implies that there exists  $a > 0$  such that  $\psi(2t) \geq 4\psi(t)$  for every  $t \geq a$ . Hence, for all  $n \geq 0$ , one has  $\psi(2^n a) \geq 4^n \psi(a)$ . It follows that, for  $2^n a \leq x < 2^{n+1} a$ , we have

$$\psi(x) \geq \psi(2^n a) \geq 4^n \psi(a) \geq \left(\frac{x}{2a}\right)^2 \psi(a) = C x^2.$$

Hence  $\psi(x) \geq C x^2$  for every  $x \geq a$ , and so the norm  $\|\cdot\|_\psi$  is stronger than the norm of  $L^2$ .

**Theorem 2.1** *Suppose that  $\psi$  is an Orlicz function as above. Let  $X_0$  be a linear subspace of  $L^\infty(\Omega)$  on which the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\psi$  are not equivalent. Then there exists in  $X_0$  a sequence which is equivalent, in the closure  $X$  of  $X_0$  for the norm  $\|\cdot\|_\psi$ , to the canonical basis of  $c_0$ .*

**Proof.** We first remark that, thanks to (\*), we can choose, for each  $n \geq 1$ , a positive number  $x_n$  such that

$$\psi\left(\frac{x}{2}\right) \leq \frac{1}{2^n} \psi(x), \quad \forall x \geq x_n.$$

Since  $\psi$  increases, we have for every  $x \geq 0$ :

$$\psi\left(\frac{x}{2}\right) \leq \frac{1}{2^n} \psi(x) + \psi(x_n).$$

Next,  $\psi$  is continuous since it is convex. Hence there exists  $a > 0$  such that  $\psi(a) = 1$ . Then, since  $\psi$  is increasing, we have, for every  $f \in L^\infty(\Omega)$ :

$$\int_{\Omega} \psi\left(a \frac{|f|}{\|f\|_\infty}\right) d\mathbb{P} \leq 1,$$

and so  $\|f\|_\psi \leq (1/a) \|f\|_\infty$ .

Now, let  $\alpha_n, n \geq 1$ , be positive numbers less than  $a/2$  such that  $\sum_{n \geq 1} \alpha_n < a$ . We shall construct inductively a sequence of functions  $f_n \in X_0$ , with  $\|f_n\|_\psi = 1$ , and a sequence of positive numbers  $\beta_n \leq 1/2^n$  such that:

- (i)  $\mathbb{P}(\{|f_n| > \alpha_n\}) \leq \beta_n$ , for every  $n \geq 1$ ;
- (ii) if we set  $M_1 = 1$  and, for  $n \geq 2$ :

$$M_n = \psi\left(\frac{\|f_1\|_\infty + \cdots + \|f_{n-1}\|_\infty}{2}\right),$$

then  $(M_n + \psi(x_n))\beta_n \leq 1/2^n$ ;

- (iii) for every  $n \geq 1$ ,  $\|g_n\|_\psi \geq 1/2$ , with  $g_n = f_n \mathbb{I}_{\{|f_n| > \alpha_n\}}$ .

For this, we start with  $\beta_1$  such that  $(1 + \psi(x_1))\beta_1 = 1/2$ . Since the norms  $\|\cdot\|_\psi$  and  $\|\cdot\|_2$  are not equivalent on  $X_0$ , there is an  $f_1 \in X_0$  with  $\|f_1\|_\psi = 1$  and  $\mathbb{P}(\{|f_1| > \alpha_1\}) \leq \beta_1$ . Suppose now that  $f_1, \dots, f_{n-1}$  and  $\beta_1, \dots, \beta_{n-1}$  have been constructed. We choose then  $\beta_n \leq 1/2^n$  in order that  $(M_n + \psi(x_n))\beta_n \leq 1/2^n$ . Since the norms  $\|\cdot\|_\psi$  and  $\|\cdot\|_2$  are not equivalent on  $X_0$ , we can find  $f_n \in X_0$  such that  $\|f_n\|_\psi = 1$  and  $\|f_n\|_2$  is so small that

$$\mathbb{P}(\{|f_n| > \alpha_n\}) \leq \beta_n.$$

Since  $\|f_n - g_n\|_\psi \leq (1/a) \|f_n - g_n\|_\infty \leq \alpha_n/a$ , we have  $\|g_n\|_\psi \geq \|f_n\|_\psi - \alpha_n/a \geq 1/2$ , and that finishes the construction.

Now, consider

$$g = \sum_{n=1}^{+\infty} |g_n|.$$

Set  $A_n = \{|f_n| > \alpha_n\}$  and, for  $n \geq 1$ :

$$B_n = A_n \setminus \bigcup_{j>n} A_j.$$

We have  $\mathbb{P}(\limsup A_n) = 0$ , because  $\sum_{n \geq 1} \mathbb{P}(A_n) \leq \sum_{n \geq 1} \beta_n < +\infty$ . Now  $g$  vanishes out of  $\bigcup_{n \geq 1} B_n \cup (\limsup A_n)$  and  $\int_{B_n} \psi(|g_n|) d\mathbb{P} \leq \int_\Omega \psi(|f_n|) d\mathbb{P} \leq 1$ . Therefore

$$\begin{aligned}
\int_{\Omega} \psi\left(\frac{|g|}{4}\right) d\mathbb{P} &= \sum_{n=1}^{+\infty} \int_{B_n} \psi\left(\frac{|g|}{4}\right) d\mathbb{P} \\
&\leq \sum_{n=1}^{+\infty} \int_{B_n} \frac{1}{2} \left[ \psi\left(\frac{\|f_1\|_{\infty} + \dots + \|f_{n-1}\|_{\infty}}{2}\right) + \psi\left(\frac{|g_n|}{2}\right) \right] d\mathbb{P} \\
&\quad \text{by convexity of } \psi \text{ and because } g_j = 0 \text{ on } B_n \text{ for } j > n \\
&\leq \frac{1}{2} \sum_{n=1}^{+\infty} M_n \mathbb{P}(A_n) + \frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{2^n} \int_{B_n} \psi(|g_n|) d\mathbb{P} + \frac{1}{2} \sum_{n=1}^{+\infty} \psi(x_n) \mathbb{P}(A_n) \\
&\leq \frac{1}{2} \sum_{n=1}^{+\infty} (M_n + \psi(x_n)) \beta_n + \frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{2^n} \leq 1.
\end{aligned}$$

Hence  $g \in L^{\psi}(\Omega)$ .

It follows that the series  $\sum_{n \geq 1} g_n$  is weakly unconditionally Cauchy in  $X$ . Since  $\|g_n\|_{\psi} \geq 1/2$ , it has, by the Bessaga-Pelczyński's theorem, a subsequence which is equivalent to the canonical basis of  $c_0$ . The same is true for  $(f_n)_{n \geq 1}$  since

$$\sum_{n=1}^{+\infty} \|f_n - g_n\|_{\psi} \leq \frac{1}{a} \sum_{n=1}^{+\infty} \|f_n - g_n\|_{\infty} \leq \frac{1}{a} \sum_{n=1}^{+\infty} \alpha_n < 1.$$

That ends the proof.  $\square$

Of course, the proof shows that the assumption that the norm  $\|\cdot\|_{\psi}$  is not equivalent to  $\|\cdot\|_2$  can be replaced by the non-equivalence of  $\|\cdot\|_{\psi}$  with many other norms. We only used the fact that the topology of convergence in measure is not equivalent on  $X_0$  to the topology defined by  $\|\cdot\|_{\psi}$ .

When we apply this result to the probability space  $(G, m)$ , we get (see [19], Theorem 2.3):

**Theorem 2.2** *Let  $\psi$  be as in Theorem 2.1 and let  $G$  be a compact abelian group. Then, for  $\Lambda \subseteq \Gamma = \widehat{G}$ , either  $L_{\Lambda}^{\psi}$  has cotype 2, or it contains  $c_0$ .*

*In particular, either  $\Lambda$  is a Sidon set and  $L_{\Lambda}^{\Psi_2} = L_{\Lambda}^2$ , or  $L_{\Lambda}^{\Psi_2}$  contains  $c_0$  (and so it has no finite cotype).*

**Proof.** Observe that when  $L_{\Lambda}^{\psi} \neq L_{\Lambda}^2$ , the norms  $\|\cdot\|_{\psi}$  and  $\|\cdot\|_2$  are not equivalent on  $X_0 = \mathcal{P}_{\Lambda}$ , the subspace of the trigonometric polynomials whose spectrum is contained in  $\Lambda$ . So the first part follows directly from Theorem 2.1. The second one follows from Pisier's characterization of Sidon sets ([35], Théorème 6.2):  $\Lambda$  is a Sidon set if and only if  $L_{\Lambda}^{\Psi_2} = L_{\Lambda}^2$ .  $\square$

**Remark.** It is proved in [19], Theorem 2.3, that  $\Lambda$  is a  $\Lambda(\psi)$ -set (i.e.  $L_{\Lambda}^{\psi} = L_{\Lambda}^2$ ) when  $L_{\Lambda}^{\Psi_2} \subseteq L_{\Lambda}^{\psi} \subseteq L_{\Lambda}^2$  and  $L_{\Lambda}^{\psi}$  has cotype 2.

### 3 Uniform convergence

A function  $f \in \mathcal{C}(\mathbb{T})$  is said to have a *uniformly convergent Fourier series* if  $\|S_k(f) - f\|_\infty \xrightarrow{k \rightarrow +\infty} 0$ , where

$$S_k(f) = \sum_{j=-k}^k \widehat{f}(j) e_j.$$

The space  $U(\mathbb{T})$  of *uniformly convergent Fourier series* is the space of all such  $f \in \mathcal{C}(\mathbb{T})$ . With the norm

$$\|f\|_U = \sup_{k \geq 1} \|S_k(f)\|_\infty,$$

$U(\mathbb{T})$  becomes a Banach space.

A set  $\Lambda \subseteq \mathbb{Z}$  is said to be a *set of uniform convergence (UC-set)* if  $U_\Lambda = \mathcal{C}_\Lambda$  as linear spaces. They are then isomorphic as Banach spaces. There exist sets  $\Lambda$  which are not UC-sets, but for which  $\mathcal{C}_\Lambda$  does not contain  $c_0$  (for instance, a Rosenthal set which contains arbitrarily long arithmetical progressions [38]). For  $U_\Lambda$  the situation is different; we have:

**Theorem 3.1** *If  $\Lambda$  is not a UC-set, then  $U_\Lambda$  contains  $c_0$ .*

**Corollary 3.2** *If  $U_\Lambda$  has a finite cotype, then  $\Lambda$  is a Sidon set.*

**Proof.** If  $U_\Lambda$  has a finite cotype, it cannot contain  $c_0$ . Hence  $U_\Lambda$  is isomorphic to  $\mathcal{C}_\Lambda$ . It follows that  $\mathcal{C}_\Lambda$  has a finite cotype, and so  $\Lambda$  is a Sidon set, by Bourgain-Milman's theorem [3].  $\square$

**Remark.** This result was proved in [19], Theorem 1.2, by adapting the proof of Bourgain and Milman. Now it becomes clear why this proof happened to mimic the original one.

**Proof of Theorem 3.1.** Since  $\Lambda$  is not a UC-set, there exists a trigonometric polynomial  $P_1 \in \mathcal{C}_\Lambda$  such that  $\|P_1\|_U = 1$  and  $\|P_1\|_\infty \leq 1/2$ . Let  $N_1 \geq 2$  such that  $\widehat{P_1}(n) = 0$  for  $|n| \geq N_1$ . The spaces  $U_{\Lambda \setminus \Lambda \cap \{-N_1+1, \dots, 0, \dots, N_1-1\}}$  and  $\mathcal{C}_{\Lambda \setminus \Lambda \cap \{-N_1+1, \dots, 0, \dots, N_1-1\}}$  remain non-isomorphic, and so we can find a trigonometric polynomial  $P_2$  such that  $\widehat{P_2}(n) = 0$  for  $|n| \leq N_1 - 1$  with  $\|P_2\|_U = 1$  and  $\|P_2\|_\infty \leq 1/4$ . Carrying on this construction, we get a sequence of integers  $2 \leq N_1 < N_2 < \dots$  and a sequence of trigonometric polynomials  $P_l \in \mathcal{C}_\Lambda$  such that  $\|P_l\|_U = 1$ ,  $\|P_l\|_\infty \leq 1/2^l$  and  $\widehat{P_l}(n) = 0$  for  $n \notin \{\pm N_{l-1}, \dots, \pm(N_l - 1)\}$ .

Now, fix an integer  $L \geq 1$  and a sequence  $a_1, \dots, a_L$  of complex numbers.

For each  $k \geq 1$ , let  $l_k$  such that  $N_{l_k} \leq k < N_{l_k+1}$ . We have, when  $L \geq l_k + 1$ :

$$\begin{aligned} \left\| S_k \left( \sum_{l=1}^L a_l P_l \right) \right\|_{\infty} &\leq \left\| \sum_{l=1}^{l_k} a_l P_l \right\|_{\infty} + \|a_{l_k+1} S_k(P_{l_k+1})\|_{\infty} \\ &\leq \max_{1 \leq j \leq l_k} |a_j| \sum_{l=1}^{l_k} \|P_l\|_{\infty} + |a_{l_k+1}| \|P_{l_k+1}\|_{\infty} \\ &\leq 2 \max\{|a_1|, \dots, |a_{l_k}|, |a_{l_k+1}|, \dots, |a_L|\}. \end{aligned}$$

The inequality  $\|S_k(\sum_{l=1}^L a_l P_l)\|_{\infty} \leq 2 \max\{|a_1|, \dots, |a_{l_k}|, |a_{l_k+1}|, \dots, |a_L|\}$  remains trivially true for  $L \leq l_k$ , because in this case  $S_k(\sum_{l=1}^L a_l P_l) = \sum_{l=1}^L a_l P_l$ . Therefore we get

$$\left\| \sum_{l=1}^L a_l P_l \right\|_U \leq 2 \max\{|a_1|, \dots, |a_L|\}.$$

It follows that the series  $\sum_{l \geq 1} P_l$  is weakly unconditionally Cauchy. Since it is obviously not convergent,  $U_{\Lambda}$  contains a subspace isomorphic to  $c_0$  by Bessaga-Pełczyński's theorem (see [6], pages 44–45, Theorem 6 and Theorem 8).  $\square$

**Remark 1.** There is the stronger notion of *CUC*-set.  $\Lambda \subseteq \mathbb{Z}$  is a *CUC*-set if  $\left\| \sum_{j=k_1}^{k_2} \widehat{f}(j) e_j - f \right\|_{\infty} \xrightarrow[k_2 \rightarrow +\infty]{k_1 \rightarrow -\infty} 0$  for every  $f \in \mathcal{C}_{\Lambda}$ . Obviously, for subsets of  $\mathbb{N}$ ,

the two notions coincide. Theorem 3.1 is not valid for *CUC*-sets: let  $H$  be a Hadamard lacunary sequence. Then  $\Lambda = H - H$  is not a *CUC*-set (Fournier [8]), but it is *UC* and Rosenthal, so that  $U_{\Lambda} = \mathcal{C}_{\Lambda}$  does not contain  $c_0$ .

However, it is not known whether  $\mathcal{C}_{\Lambda_1 \cup \Lambda_2}$  lacks  $c_0$  whenever this is true for  $\mathcal{C}_{\Lambda_1}$  and  $\mathcal{C}_{\Lambda_2}$ . If we replace the space  $\mathcal{C}(G)$  by  $U(\mathbb{T})$ , the answer is in the negative. Indeed, J. Fournier shows ([8]), completing Soardi and Travaglini's work [43], that there exist two *UC*-sets  $\Lambda_1, \Lambda_2 \subseteq \mathbb{Z}$ , which are Rosenthal sets, but  $\Lambda_1 \cup \Lambda_2 = H + H - H$  is not *UC*. Therefore  $U_{\Lambda_1} = \mathcal{C}_{\Lambda_1}$  and  $U_{\Lambda_2} = \mathcal{C}_{\Lambda_2}$  do not contain  $c_0$ , though  $U_{\Lambda_1 \cup \Lambda_2}$  contains  $c_0$ .

**Remark 2.** *UC*-sets  $\Lambda$  for which  $\mathcal{C}_{\Lambda}$  contains  $c_0$  are constructed in [24].

**Remark 3.** We stated Theorem 3.1 for uniform convergence because it is the classical case. Actually, J. Fournier ([8], page 72) and S. Hartman ([13], page 107) introduced the space  $L^1 - UC$  as the set of all  $f \in L^1(\mathbb{T})$  for which  $\|S_k(f) - f\|_1 \xrightarrow[k \rightarrow +\infty]{} 0$ . It is normed by  $\|f\|_{UL^1} = \sup_{k \geq 1} \|S_k(f)\|_1$ .  $\Lambda$  is said to be an  $L^1 - UC$ -set if  $(L^1 - UC)_{\Lambda} = L^1_{\Lambda}$ . The same proof as above shows that if  $(L^1 - UC)_{\Lambda} \neq L^1_{\Lambda}$ , then  $(L^1 - UC)_{\Lambda}$  contains  $c_0$ . More generally, let  $\Lambda \subseteq \mathbb{Z}$  and let  $X$  be a Banach space contained, as a linear subspace, in  $L^1(\mathbb{T})$  such that the linear space generated by  $X \cap \Lambda$  is dense in  $X$ . We can define  $X - UC$  in an obvious way, and we have: if  $X - UC$  is not isomorphic to  $X$ , then it contains  $c_0$ .



We give another consequence of Theorem 3.1. Recall (see [30]) that  $\Lambda \subseteq \Gamma$  is a *Riesz set* if every measure with spectrum in  $\Lambda$  is absolutely continuous, with respect to the Haar measure (in short,  $\mathcal{M}_\Lambda = L_\Lambda^1$ ).

**Corollary 3.3** *If  $U_\Lambda$  does not contain  $c_0$ , then  $\Lambda$  is a Riesz set.*

**Proof.** If  $U_\Lambda \not\supseteq c_0$ , then  $U_\Lambda = \mathcal{C}_\Lambda$ , by Theorem 3.1, and so  $\mathcal{C}_\Lambda \not\supseteq c_0$ . It follows then that  $\Lambda$  is a Riesz set (F. Lust-Piquard [25], her first Théorème 3.1). Let us recall why. For  $\mu \in \mathcal{M}_\Lambda$ , the convolution operator  $C_\mu: f \in \mathcal{C}(G) \mapsto f * \mu \in \mathcal{C}_\Lambda \subseteq \mathcal{C}(G)$  is weakly compact, because  $\mathcal{C}(G)$  has Pełczyński's property (V) and  $\mathcal{C}_\Lambda \not\supseteq c_0$ . Its adjoint operator  $\nu \in \mathcal{M}(G) \mapsto \nu * \mu \in \mathcal{M}_\Lambda$  is also weakly compact. Hence, if  $(K_j)_j$  is an approximate unit for the convolution, there is a sequence  $(j_n)_n$  such that  $K_{j_n} * \mu$  is weakly convergent. Since  $K_j * \mu$  converges weak-star to  $\mu$ , it follows that  $\mu \in L_\Lambda^1$ .  $\square$

**Remark.** Another proof can be given, without using Theorem 3.1, but using that  $U(\mathbb{T})$  has Pełczyński's property (V) (Saccone [42], Theorem 2.2; for  $U_{\mathbb{N}}(\mathbb{T})$ , see Bourgain [1], Lemme 2 and Lemme 3, and Saccone [41], Theorem 4.1). Then, as before,  $K_{j_n} * \mu$  is weakly convergent, in  $U(\mathbb{T})^*$  this time. So there are convex combinations which converge in the norm of  $U(\mathbb{T})^*$ . But then they converge in the norm of  $U_{\mathbb{N}}(\mathbb{T})^*$ , and so  $u \in L^1(G)$  (see D. Oberlin [33], page 310). Note that Oberlin's argument (as well as Bourgain's one) depends on Carleson's Theorem (via [47]).

## 4 Invariant means and Hilbert sets

An *invariant mean*  $M$  on  $L^\infty(G)$  is a continuous linear functional on  $L^\infty(G)$  such that  $M(\mathbf{1}) = \|M\| = 1$  and  $M(f_x) = M(f)$  for every  $f \in L^\infty(G)$ . The Haar measure  $m$  defines an invariant mean, and W. Rudin ([40]) showed that, for infinite compact abelian groups  $G$ , there always exist other invariant means on  $L^\infty(G)$ . A function  $f \in L^\infty(G)$  has a unique invariant mean if  $M(f) = \widehat{f}(0)$  for every invariant mean  $M$  on  $L^\infty(G)$ . Every continuous function (or, even, every Riemann-integrable function: [39], page 38, or [44]) has a unique invariant mean.

**Definition 4.1** *A subset  $\Lambda$  of  $\Gamma = \widehat{G}$  is called a Lust-Piquard set if  $\gamma f$  has a unique invariant mean for every  $f \in L_\Lambda^\infty$  and every  $\gamma \in \Gamma$ .*

In other words,  $\Lambda$  is a Lust-Piquard set if for every invariant mean  $M$  on  $L^\infty(G)$  and every  $f \in L_\Lambda^\infty$ , one has:

$$M(\gamma f) = \widehat{f}(-\gamma).$$

In [26] (and then in [21]; see also [28]), F. Lust-Piquard called them *totally ergodic sets*. We use a different name because J. Bourgain ([2], 2.I, page 206), used the terminology “ergodic set” for another property (see also [24]).

Note that it is required that the invariant means agree on  $\bigcup_{\gamma \in \Gamma} L_{\Lambda - \gamma}^\infty$ , and not only on  $L_\Lambda^\infty$ , because the invariant means may coincide on  $L_\Lambda^\infty$  for trivial reasons; for instance, all the invariant means are equal to 0 on  $L_{2\mathbb{Z}+1}^\infty$  (since  $f(x + 1/2) = -f(x)$  for  $f \in L_{2\mathbb{Z}+1}^\infty$ ). It is clear that if  $\Lambda$  is a Lust-Piquard set, then  $\Lambda - \gamma$  is also a Lust-Piquard set for every  $\gamma \in \Gamma$ .

It is obvious that every Rosenthal set is a Lust-Piquard set (since every continuous function has a unique invariant mean), and it is shown in [21] that every Lust-Piquard set is a Riesz set. On the other hand, Y. Katznelson (see [39], pages 37–38) proved that  $\mathbb{N}$  is not a Lust-Piquard set.

F. Lust-Piquard ([27], Theorem 2 and Theorem 4) showed that  $\Lambda = \mathbb{P} \cap (5\mathbb{Z} + 2)$ , where  $\mathbb{P}$  is the set of the prime numbers, is totally ergodic (a Lust-Piquard set, with our terminology) although  $\mathcal{C}_\Lambda$  contains  $c_0$ .

In the following theorem, we give another example of such a situation. Let us recall that  $H \subseteq \mathbb{Z}$  is a *Hilbert set* if there exist two sequences of integers  $(p_n)_{n \geq 1}$  and  $(q_n)_{n \geq 1}$ , with  $q_n \neq 0$ , such that

$$H = \bigcup_{n \geq 1} \left\{ p_n + \sum_{k=1}^n \varepsilon_k q_k ; \varepsilon_1, \dots, \varepsilon_n = 0 \text{ or } 1 \right\}.$$

It is shown in [22], Theorem 2, that  $\mathcal{C}_H$  contains  $c_0$  when  $H$  is a Hilbert set.

**Theorem 4.2** *There exists a Hilbert set  $H \subseteq \mathbb{N}$  which is a Lust-Piquard set.*

We begin with a lemma, which is implicit in [27], proof of Theorem 4.

**Lemma 4.3** *The family of Lust-Piquard sets in  $\Gamma$  is localizable for the Bohr topology.*

Let us recall that the *Bohr topology* of a discrete abelian group  $\Gamma$  is the topology of pointwise convergence, when  $\Gamma$  is seen as a subset of  $\mathcal{C}(G)$ ; it is also the natural topology on  $\Gamma$  as a subset of the dual group of  $G_d$ , the group  $G$  with the discrete topology. A class  $\mathcal{F}$  of subsets of  $\Gamma$  is *localizable for the Bohr topology* if  $\Lambda \in \mathcal{F}$  whenever for every  $\gamma \in \Gamma$  there is a neighbourhood  $V_\gamma$  of  $\gamma$  for the Bohr topology such that  $\Lambda \cap V_\gamma \in \mathcal{F}$ . This notion is due to Y. Meyer ([30]).

For the sake of completeness, we shall give a proof.

**Proof of Lemma 4.3.** We are going to prove that if  $V_\gamma$  is a neighbourhood of  $\gamma \in \Gamma$  such that  $\Lambda \cap V_\gamma$  is a Lust-Piquard set, then  $\overline{\gamma}f$  has a unique invariant mean for every  $f \in L_\Lambda^\infty$ , and that will prove the lemma.

By the regularity of the algebra  $L^1(G_d) = \ell_1(G) = \mathcal{M}_d(G)$ , there exists a discrete measure  $\nu \in \mathcal{M}_d(G)$  such that  $\widehat{\nu}(\gamma) = 1$  and  $\widehat{\nu} = 0$  outside  $V_\gamma$ . Since  $(\overline{\gamma}f) * (\overline{\gamma}\nu) \in L_{(\Lambda \cap V_\gamma) - \gamma}^\infty$ , and since  $(\Lambda \cap V_\gamma) - \gamma$  is a Lust-Piquard set, we have:

$$M((\overline{\gamma}f) * (\overline{\gamma}\nu)) = [(\overline{\gamma}f) * \widehat{(\overline{\gamma}\nu)}](0) = \widehat{f}(\gamma) \widehat{\nu}(\gamma) = \widehat{f}(\gamma).$$

But  $\overline{\gamma}\nu$  is a discrete measure, and we have, for every discrete measure  $\mu$ :

$$M(\mu * g) = M(g) \widehat{\mu}(0)$$

for every  $g \in L^\infty(G)$  and every invariant mean  $M$ . This is so since, if  $\mu = \sum_k a_k \delta_{x_k}$ , with  $\sum_k |a_k| < +\infty$ , we have

$$M(\mu * g) = \sum_k a_k M(g_{x_k}) = \sum_k a_k M(g).$$

Hence  $M(\overline{\gamma}f) = \widehat{f}(\gamma)$ , as required.  $\square$

**Proof of Theorem 4.2.** We are going to construct a Hilbert set  $H \subseteq \mathbb{N}$  which is discrete in  $\mathbb{Z}$  for the Bohr topology. For such a set, there is, for every  $k \in \mathbb{Z}$ , some Bohr-neighbourhood  $V_k$  of  $k$  such that  $H \cap V_k$  is finite. Therefore, we have  $L_{H \cap V_k}^\infty = \mathcal{C}_{H \cap V_k}$ , and so  $H \cap V_k$  is a Lust-Piquard set.

Let  $(d_n)_{n \geq 0}$  be a strictly increasing sequence of positive integers such that:

$$d_n \mid d_{n+1}, \quad n \geq 0, \quad \sum_{n=0}^{+\infty} \frac{2^{n+1}}{d_n} < 1.$$

For every  $k \in \mathbb{Z}$ , consider:

$$V(k) = k + d_{|k|}\mathbb{Z},$$

which is a Bohr-neighbourhood of  $k$ .

Now, we are going to show that we can choose, for every  $n \geq 0$ , an integer  $r_n \in \{0, 1, 2, \dots, d_n - 1\}$  such that, if

$$H_n = d_n + r_n + \left\{ \sum_{l=0}^{n-1} \varepsilon_l d_l; \varepsilon_l = 0 \text{ or } 1 \right\},$$

then

$$H_p \cap V(k) = \emptyset$$

for every  $k \in \mathbb{Z}$  and every  $p > |k|$ . The set  $H = \bigcup_{n \geq 0} H_n$  will be the required set.

We are going to do this by induction. First, we may choose an arbitrary  $r_0 \in \{0, 1, 2, \dots, d_0 - 1\}$ , and we set  $H_0 = \{d_0 + r_0\}$ . Suppose now that we have found  $r_1, r_2, \dots, r_{p-1}$  such that the previous conditions are fulfilled:

$$H_j \cap V(k) = \emptyset, \quad \text{for } 1 \leq j \leq p-1, \quad |k| < j.$$

To find  $r_p$ , note that  $m \in H_p \cap V(k)$  if and only if

$$m \in k + d_{|k|}\mathbb{Z} \tag{1}$$

and there exist  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{p-1} \in \{0, 1\}$  such that

$$m = d_p + r_p + \sum_{l=0}^{p-1} \varepsilon_l d_l. \tag{2}$$

Since, for  $0 \leq l < p$ , one has  $d_l \mid d_{l+1} \mid \cdots \mid d_p$ , conditions (1) and (2) are equivalent to  $r_p \equiv 0 \pmod{d_0}$ , for  $k = 0$ , and, for  $1 \leq l = |k| < p$ , to:

$$k \equiv r_p + \sum_{j=0}^{|k|-1} \varepsilon_j d_j \pmod{d_{|k|}}.$$

For each such  $k$  ( $0 \leq |k| < p$ ), there are

$$\frac{d_p}{d_{|k|}} \cdot 2^{|k|}$$

possible choices for  $r_p$ . As

$$\frac{d_p}{d_0} + 2 \sum_{l=1}^{p-1} 2^l \frac{d_p}{d_l} \leq \frac{d_p}{d_0} + 2 \sum_{l=1}^{+\infty} 2^l \frac{d_p}{d_l} < d_p,$$

by hypothesis, we can find an  $r_p \in \{0, 1, \dots, d_p - 1\}$  such that the set  $H_p$  constructed from it verifies  $H_p \cap V(k) = \emptyset$  for  $|k| < p$ . That ends the proof.  $\square$

**Remark 1.** Some particular Hilbert sets are the *IP*-sets, *i.e.* the sets  $F$  for which there exists a sequence  $(p_n)_{n \geq 1}$  of integers such that

$$F = \left\{ \sum_{k=1}^n \varepsilon_k p_k; \varepsilon_1, \dots, \varepsilon_n = 0 \text{ or } 1, n \geq 1 \right\}.$$

**Question.** Does there exist an *IP*-set  $F$  which is a Lust-Piquard set?

Every point of an *IP*-set  $F$  is non-isolated in  $F$  (see [10], Theorem 2.19; note that every point of an *IP*-set is inside the translation by this point of a sub-*IP*-set). Therefore we cannot use an argument similar to that of the previous theorem. Hilbert sets and *IP*-sets are different in several ways. For instance, every set  $\Lambda \subseteq \mathbb{Z}$  which has a positive uniform density contains a Hilbert set ([14], Theorem 11.11; [22], Theorem 4), but not necessarily an *IP*-set ([14], Theorem 11.6; [32], page 151). Another difference is that  $\mathcal{C}_\Lambda$  never has the Unconditional Metric Approximation Property if  $\Lambda \subseteq \mathbb{Z}$  is an *IP*-set ([23], Proposition 11), but can have this property when  $\Lambda$  is a Hilbert set ([23], Theorem 10).

**Remark 2.** Let  $\mathcal{F}$  be a class of subsets of  $\Gamma$ , which contains all the finite sets, and which is localizable for the Bohr topology. It follows from the proof of Theorem 4.2 that such a class must contain some Hilbert sets. In particular  $\mathcal{F}$  has to contain sets  $\Lambda$  such that  $\Lambda$  contains parallelepipeds of arbitrarily large dimensions. Note that this last assertion is actually implicit in [27]. Indeed, by Dirichlet's theorem,  $\sum_{n \in \mathbb{P} \cap (5\mathbb{Z}+2)} \frac{1}{n} = +\infty$ , and by [31], Corollary 2, we have  $\sum_{n \in \Lambda} \frac{1}{n} < +\infty$  when  $\Lambda$  does not contain parallelepipeds of arbitrarily large dimensions. It is known that the sets belonging to the following classes cannot contain parallelepipeds of arbitrarily large dimensions:

- a)  $\Lambda(p)$ -sets (see [31], Theorem 3, and [9], Theorem 4);
- b)  $UC$ -sets ([9], Theorem 4);
- c)  $p$ -Sidon sets ([15], Lemma 1);
- d) stationary sets ([18], Proposition 2.5);
- e)  $q$ -Rider sets (see [24] or [19] for the definition). Note that, for  $1 \leq q < 4/3$ ,  $q$ -Rider sets are  $p$ -Sidon sets, for every  $p > q/(2 - q)$  (see [20]), and so the result is in c). For  $4/3 \leq q < 2$ , there is no explicit published proof of that, and therefore we shall give one in Proposition 4.4, after this Remark.

Hence these classes are not localizable for the Bohr topology.

This remark shows that there is no hope to construct sets of the above classes by way of localization.

**Proposition 4.4** *If  $\Lambda$  is a  $q$ -Rider set,  $1 \leq q < 2$ , then  $\Lambda$  cannot contain parallelepipeds of arbitrarily large dimensions.*

**Proof.** A Sidon set (with constant less than 10, say) inside a parallelepiped  $P$  of size  $2^n$  cannot contain more than  $C n \log n$  elements ([16], Chapter 6, § 3, Theorem 5, page 71), whereas if  $P$  were contained in a  $q$ -Rider set, it should contain a quasi-independent (hence Sidon with constant less than 10) set of size at least  $C 2^{\varepsilon n}$ , with  $\varepsilon = (2 - q)/q$  ([36], or [37], Teorema 2.3).  $\square$

Note that another proof of Proposition 4.4 is implicit in [15]. Indeed the proof given in [15], Lemma 1, that  $p$ -Sidon sets share this property only uses the fact, proved in [4], Eq. (9), that if  $\Lambda$  is a  $p$ -Sidon set, then, with  $\alpha = 2p/(3p - 2)$ , there is a constant  $C > 0$  such that  $\|f\|_r \leq C \sqrt{r} \|\widehat{f}\|_\alpha$  for all  $r \geq 2$  (equivalently:  $\|f\|_{\Psi_2} \leq C' \|\widehat{f}\|_\alpha$ ) for every  $f \in \mathcal{C}_\Lambda$ . Now the fourth-named author proved that these inequalities characterize  $p$ -Rider sets ([36]; see also [37], Teorema 2.3).

## 5 Complemented subspaces

Since  $\Lambda$  is a Rosenthal set if  $L_\Lambda^\infty = \mathcal{C}_\Lambda$ , it is natural to ask whether  $\Lambda$  is a Rosenthal set if there exists a projection from  $L_\Lambda^\infty$  onto  $\mathcal{C}_\Lambda$ . We have not been able to answer this, even if this projection were to have norm 1 (see [12], where the condition that the space does not contain  $\ell_1$  is crucial), but we have a partial result. Recall that it is not known whether  $\mathcal{C}_\Lambda \not\supseteq c_0$  implies that  $\Lambda$  is a Rosenthal set.

**Theorem 5.1** *Let  $\Lambda \subseteq \Gamma$  be such that there exists a surjective projection  $P: L_\Lambda^\infty \rightarrow \mathcal{C}_\Lambda$ . Then  $\mathcal{C}_\Lambda$  does not contain  $c_0$ . Moreover, every Riemann-integrable function in  $L_\Lambda^\infty$  is actually in  $\mathcal{C}_\Lambda$ .*

Recall that a function  $h: G \rightarrow \mathbb{C}$  is *Riemann-integrable* if it is bounded and almost everywhere continuous. Actually, the last assertion of the proposition means that every element of  $L_\Lambda^\infty$  which contains a Riemann-integrable function contains also a continuous one.

**Proof.** 1) By [22], Proposition 14, if  $\mathcal{C}_\Lambda$  contains  $c_0$ , there is a sequence  $(f_n)_{n \geq 1}$  in  $\mathcal{C}_\Lambda$ , which is equivalent to the canonical basis of  $c_0$ , and whose  $w^*$  linear span  $F$  in  $L_\Lambda^\infty$  is isomorphic to  $\ell_\infty$ . The restriction  $P|_F$  is a projection from  $F$  onto a subspace of  $\mathcal{C}_\Lambda$  which contains  $E = \overline{\text{span}\{f_n; n \geq 1\}}$ .

Observe that  $E$  is a separable subspace of  $\mathcal{C}_\Lambda$ . So there exists a countable subset  $\Lambda_1 \subseteq \Lambda$  such that  $E \subseteq \mathcal{C}_{\Lambda_1}$ . Moreover, there exists a countable subgroup  $\Gamma_0 \subseteq \Gamma$  such that  $\Lambda_1$  is contained in  $\Gamma_0$ . Taking  $\Lambda_0 = \Lambda \cap \Gamma_0$ , we have  $E \subseteq \mathcal{C}_{\Lambda_0}$ , and  $\mathcal{C}_{\Lambda_0}$  is a separable space.

The set  $\Gamma_0$  being a subgroup, there exists a measure  $\mu$  on  $G$  whose Fourier transform is  $\widehat{\mu} = \mathbb{1}_{\Gamma_0}$ . The map  $f \mapsto f * \mu$  gives a projection from  $\mathcal{C}_\Lambda$  onto  $\mathcal{C}_{\Lambda_0}$ , and Sobczyk's theorem gives a projection from  $\mathcal{C}_{\Lambda_0}$  onto  $E$ . So there exists a projection from  $F \simeq \ell_\infty$  onto  $E \simeq c_0$ , which is a contradiction.

2) We first assume that the group  $G$  is metrizable, so that  $\mathcal{C}(G)$  is separable.

Let  $RI_\Lambda$  be the subspace of  $L_\Lambda^\infty$  consisting of Riemann-integrable functions (more precisely: the elements of  $L_\Lambda^\infty$  which have a Riemann-integrable representative).

Consider the restriction of  $P$  to  $RI_\Lambda$ . For  $f \in RI$ , the set

$$\{x \mapsto \xi(f_x); \xi \in L^\infty(G)^*, \|\xi\| \leq 1\}$$

is stable ([46], Theorem 15-6 c)). Let  $\mu \in (\mathcal{C}_\Lambda)^*$ , and set  $\varphi(x, y) = (P^*\mu_y)(f_x)$  for  $x, y \in G$ . The map  $x \in G \mapsto f_x \in L^\infty(G)$  is scalarly measurable ([45], Theorem 16) and  $y \mapsto P^*\mu_y$  is continuous for the  $w^*$ -topology. Moreover  $\{x \mapsto (P^*\mu_y)(f_x); y \in G\}$  is stable, so by [46], Theorem 10-2-1,  $\varphi$  is measurable. Measurability refers here to the completion of the product measure  $m \otimes m$  on  $G \times G$ , so in order to deduce that the map  $x \in G \mapsto \varphi(x, x) = (P^*\mu_x)(f_x)$  is measurable, we need the following lemma (note that our  $\varphi$  is bounded).

**Lemma 5.2** *Let  $G$  be a metrizable compact abelian group, and  $\varphi: G \times G \rightarrow \mathbb{C}$  a function such that:*

- 1)  $\varphi \in \mathcal{L}^\infty(G \times G)$ ;
- 2) the map  $y \mapsto \varphi(x, y)$  is continuous, for every  $x \in G$ .

*Then the map  $x \mapsto \varphi(x, x)$  is measurable.*

**Proof.**  $G$  being metrizable, there exists a bounded sequence  $(f_n)_n$  in  $L^1(G)$  such that

$$g(0) = \lim_{n \rightarrow \infty} \int_G f_n g \, dm, \quad \text{for every } g \in \mathcal{C}(G). \quad (3)$$

This sequence  $(f_n)_n$  represents an approximate identity.

For every  $n$ , the function  $(x, y) \mapsto f_n(x - y)\varphi(x, y)$  is integrable in  $G \times G$ . Define

$$F_n(x) = \int_G f_n(x - y) \varphi(x, y) \, dm(y) = \int_G f_n(t) \varphi(x, x - t) \, dm(t).$$

By Fubini's theorem  $F_n$  is defined almost everywhere, and is integrable. So  $F_n$  is measurable, for every  $n$ . The lemma follows since, by (3),

$$\varphi(x, x) = \lim_{n \rightarrow \infty} F_n(x), \quad \text{for every } x \in G. \quad \square$$

The fact that the map  $x \in G \mapsto (P^*\mu_x)(f_x) = \langle \mu, [P(f_x)]_{-x} \rangle$  is measurable means, since  $\mu$  is arbitrary, that  $x \mapsto [P(f_x)]_{-x} \in \mathcal{C}_\Lambda$  is scalarly measurable. Since we have assumed that  $\mathcal{C}(G)$  is separable, this map is strongly measurable, by Pettis's measurability theorem ([7], II §1, Theorem 2). Now we showed in the beginning of the proof that  $\mathcal{C}_\Lambda$  does not contain  $c_0$ ; so a result of J. Diestel [5] (see [7], II, §3, Theorem 7) says that this map is Pettis-integrable, which means that if we define  $Qf$  using

$$\langle Qf, \mu \rangle = \int_G \langle f_x, P^*(\mu_x) \rangle dx,$$

for every  $\mu \in (\mathcal{C}_\Lambda)^*$ , then  $Q$  maps  $RI_\Lambda$  into  $\mathcal{C}_\Lambda$ , and not only into its bidual (see the definition of Pettis-integrability in [7], II, §3, page 53, Definition 2, or in [46], Definition 4-2-1).

Thus  $Q$  is a projection from  $RI_\Lambda$  onto  $\mathcal{C}_\Lambda$  such that  $Q(f_x) = (Qf)_x$  for every  $f \in RI_\Lambda$  and every  $x \in G$ .

We want to prove that  $Qf = f$  for every  $f \in RI_\Lambda$ , and for that we have to see that  $\widehat{Qf}(\gamma) = \widehat{f}(\gamma)$  for every  $\gamma \in \Gamma$ . But it suffices to show that  $\widehat{Qf}(0) = \widehat{f}(0)$ , since, after replacing  $\Lambda$  by  $(\Lambda - \gamma)$  and  $Q$  by  $Q_\gamma: L_{\Lambda-\gamma}^\infty \rightarrow \mathcal{C}_{\Lambda-\gamma}$ , with  $Q_\gamma(g) = \overline{\gamma}Q(\gamma g)$ , we then get for  $f \in RI_\Lambda$  with  $g = \overline{\gamma}f$ :

$$\widehat{Qf}(\gamma) = [\overline{\gamma}(Qf)]^\wedge(0) = \widehat{Q_\gamma g}(0) = \widehat{g}(0) = (\overline{\gamma}f)^\wedge(0) = \widehat{f}(\gamma).$$

So, let  $f \in RI_\Lambda$ . Every Riemann-integrable function has a unique invariant mean ([39], Lemma 7; [44]); hence there are ([39], Proposition page 38; or [26], Proposition 1) convex combinations  $\sum_{k \in I_n} c_{n,k} f_{x_{n,k}}$ ,  $c_{n,k} > 0$ ,  $\sum_{k \in I_n} c_{n,k} = 1$ , of translates of  $f$  which converge in norm to the constant function  $\widehat{f}(0)\mathbf{1}$ . We have:

$$Q\left(\sum_{k \in I_n} c_{n,k} f_{x_{n,k}}\right) \xrightarrow{n \rightarrow +\infty} Q[\widehat{f}(0)\mathbf{1}] = \widehat{f}(0)\mathbf{1}.$$

But  $Q(\sum_{k \in I_n} c_{n,k} f_{x_{n,k}}) = \sum_{k \in I_n} c_{n,k} (Qf)_{x_{n,k}}$ , and its Fourier coefficient at 0 is:

$$\sum_{k \in I_n} c_{n,k} \widehat{Qf}(0) = \widehat{Qf}(0).$$

Therefore  $\widehat{Qf}(0) = \widehat{f}(0)$ . □

3) In order to finish the proof, we have to explain why we may assume that  $G$  is metrizable.

Let  $\Lambda$  be as in the theorem, and  $f \in RI_\Lambda$ . As explained in the proof of the first part of the theorem, there exists a countable subgroup  $\Gamma_0 \subseteq \Gamma$  such that  $f \in RI_{\Lambda_0}$ , for  $\Lambda_0 = \Lambda \cap \Gamma_0$ , and there exists a projection from  $L_{\Lambda_0}^\infty$  onto  $\mathcal{C}_{\Lambda_0}$ .

Let  $H$  be the annihilator of  $\Gamma_0$ ; that is,  $H$  is the following closed subgroup of  $G$ :

$$H = \Gamma_0^\perp = \{x \in G; \gamma(x) = 1, \forall \gamma \in \Gamma_0\}.$$

The quotient group  $G/H$  is metrizable since its dual group  $\Gamma_0$  is countable. Let  $\pi_H$  denote the quotient map from  $G$  onto  $G/H$ . It is known that the map  $g \mapsto g \circ \pi_H$  gives an isometry from  $L_{\Lambda_0}^\infty(G/H)$  onto  $L_{\Lambda_0}^\infty(G)$  sending  $\mathcal{C}_{\Lambda_0}(G/H)$  onto  $\mathcal{C}_{\Lambda_0}(G)$ .

In order to finish our reduction to the metrizable case we only have to see that this isometry sends  $RI_{\Lambda_0}(G/H)$  onto  $RI_{\Lambda_0}(G)$ . It is easy to see, via the map  $g \mapsto g \circ \pi_H$ , that having a Riemann-integrable function  $g: G/H \rightarrow \mathbb{C}$  is the same as having a Riemann-integrable function  $g: G \rightarrow \mathbb{C}$  with the property  $g(x+h) = g(x)$ , for every  $x \in G$  and every  $h \in H$ . Therefore the above isometry sends  $RI_{\Lambda_0}(G/H)$  into  $RI_{\Lambda_0}(G)$ . The surjectivity of this map is a consequence of the following proposition:

**Proposition 5.3** *Let  $f: G \rightarrow \mathbb{C}$  be a Riemann-integrable function such that  $\widehat{f}(\gamma) = 0$ , for every  $\gamma \in \Gamma \setminus \Gamma_0$ . Then there exists a Riemann-integrable function  $g: G \rightarrow \mathbb{C}$  such that:*

- a)  $f = g$  almost everywhere;
- b)  $g(x) = g(x+h)$ , for all  $x \in G$  and for all  $h \in H$ .

**Proof.** We can and we will assume that  $f$  is in fact real valued. Take an increasing sequence  $(K_n)_n$  of compact subsets of  $G$  such that, if  $B = \bigcup_n K_n$ , then:

- i)  $f$  is continuous at every point of  $B$ ;
- ii)  $m(G \setminus B) = 0$ .

Using the compactness of  $K_n$  and the continuity of  $f$  on  $B$ , one can find a neighbourhood  $W_n$  of 0 such that

$$|f(x) - f(x+y)| \leq \frac{1}{n}, \quad \text{for every } x \in K_n, \text{ and every } y \in W_n. \quad (4)$$

Let  $(V_n)_n$  be a decreasing sequence of open symmetric neighbourhoods of 0 such that  $V_n + V_n \subseteq W_n$ , for every  $n$ . For every  $n$ , define  $f_n$  as:

$$f_n(x) = \frac{1}{m(V_n)} \int_{V_n} f(x-y) dm(y), \quad x \in G.$$

$f_n$  is a continuous function since it is the convolution of  $f$  and

$$\psi_n = \frac{1}{m(V_n)} \mathbb{1}_{V_n}.$$

We also have

$$\widehat{f}_n(\gamma) = \widehat{f}(\gamma) \widehat{\psi}_n(\gamma) = 0, \quad \text{for all } \gamma \in \Gamma \setminus \Gamma_0.$$

Then the continuous function  $f_n$  only depends on the classes in  $G/H$ ; that is,

$$f_n(x) = f_n(x+h), \quad \text{for all } x \in G, \text{ all } h \in H \text{ and all } n.$$



Define

$$g(x) = \frac{1}{2} \left( \limsup_{n \rightarrow \infty} f_n(x) + \liminf_{n \rightarrow \infty} f_n(x) \right), \quad x \in G.$$

It is clear that  $g(x) = g(x + h)$ , for all  $x \in G$  and for all  $h \in H$ . Since  $V_n \subseteq W_n$ , we have from (4) that  $|f_n(x) - f(x)| \leq 1/n$ , for all  $x \in K_n$ . If  $x \in B = \bigcup_n K_n$ , then there exists  $N$  such that  $x \in K_n$ , for all  $n \geq N$ . Therefore  $|f_n(x) - f(x)| \leq 1/n$ , for all  $n \geq N$ , and  $g(x) = f(x)$ . So  $f = g$  almost everywhere.

In order to finish the proof we are going to see that every point of  $B$  is a point of continuity of  $g$ , and so  $g$  is Riemann-integrable. Let  $x$  be in  $B$ . Given  $\varepsilon > 0$ , there exists  $N$  such that  $1/N \leq \varepsilon$  and  $x \in K_n$ , for all  $n \geq N$ . We are going to prove

$$|g(x) - g(x + y)| \leq \varepsilon, \quad \text{for every } y \in V_N. \quad (5)$$

So  $g$  will be continuous at  $x$ .

Take  $n \geq N$ , and  $y \in V_N$ . For every  $z \in V_n$  we have  $x + y + z \in W_N$ , and  $|f(x) - f(x + y + z)| \leq 1/N$ . By the definition of  $f_n$  we get  $|f(x) - f_n(x + y)| \leq 1/N$ , for every  $n \geq N$ . Then we obtain (5) easily, since  $f(x) = g(x)$ .  $\square$

**Remarks.** 1) Actually the proof shows that if  $\Lambda$  is a Lust-Piquard set and if there exists a surjective projection  $Q: L_\Lambda^\infty \rightarrow \mathcal{C}_\Lambda$  which commutes with translations, then  $\Lambda$  is a Rosenthal set.

2) Talagrand's work [45] uses Martin's axiom, and, in [46], another axiom, called  $L$ . But these axioms do not intervene in the results we use (they are needed to obtain Riemann-integrability from the weak measurability of translations: see [46], Theorem 15-4).

3) F. Lust-Piquard and W. Schachermayer ([29], Corollary IV.4 and Proposition IV.15; see also [11], Theorem V.1, Corollary VI.18, and Example VIII.10) showed that if  $L^1(G)/L_{\Gamma \setminus (-\Lambda)}^1$  does not contain  $\ell_1$  (which is equivalent to  $L_\Lambda^\infty$  having the weak Radon-Nikodym property [46], Corollary (7-3-8)), then  $L_\Lambda^\infty = RI_\Lambda$ . Hence  $\Lambda$  must be a Rosenthal set if  $L_\Lambda^\infty$  has the weak Radon-Nikodym property and there exists a projection from  $L_\Lambda^\infty$  onto  $\mathcal{C}_\Lambda$ . However, a direct proof is available. For a more general result, see [11], Example following Proposition VII.6.

4) The first part of the proof is the same as the one used by A. Pełczyński ([34], Cor. 9.4 (a)) to show that  $A(\mathbb{D}) = \mathcal{C}_\mathbb{N}$  is not complemented in  $H^\infty = L_\mathbb{N}^\infty$ .

**Question.** When  $\Lambda$  is not a Rosenthal set, or, merely when  $\mathcal{C}_\Lambda$  contains  $c_0$ , how big can  $L_\Lambda^\infty/\mathcal{C}_\Lambda$  be?

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