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# MORPHISMS FROM $S^2(C)$ TO $Gr(2, \mathbb{C}^4)$ .

A. EL MAZOUNI, F. LAYTIMI, AND D.S. NAGARAJ

ABSTRACT. In this note we study the natural vector bundle of rank 2 on  $S^2(C)$ , where  $S^2(C)$  is the second symmetric power of non-singular irreducible non-degenerate space curve  $C$ .

**Keywords:** Vector bundles; Morphisms; Grassmannians; second symmetric power; curves.

## 1. INTRODUCTION

We denote by  $\mathbb{P}^3$  the projective 3-space over the field  $\mathbb{C}$  of complex numbers and  $Gr(2, \mathbb{C}^4)$  be Grassmannian of two-dimensional quotient spaces of  $\mathbb{C}^4$ . Let  $\mathcal{O}_{\mathbb{P}^3}(1)$  be the ample generator of the  $\text{Pic}(\mathbb{P}^3)$ . Let  $C$  be an irreducible non-singular curve in  $\mathbb{P}^3$ . Assume  $C$  is non-degenerate, i.e., it is not contained in hyper plane. Let  $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$ . Let  $E_L$  denote the secant bundle of  $L$ , which is a rank 2 vector bundle on the second symmetric power  $S^2(C)$  of the curve  $C$ . The bundle  $E_L$  is generated by 4 linearly independent sections. Hence there is a morphism  $S^2(C)$  to  $Gr(2, \mathbb{C}^4)$ . In this article we study this morphism. Also, we study the properties of the restriction of  $E_L$  to some naturally imbedded curves in  $S^2(C)$ . Such a study was carried out by Schwarzenberger ([2]) when the curve  $C$  is a rational normal curve in  $\mathbb{P}^3$ .

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## 2. SECANT BUNDLES

Let  $C$  be a smooth irreducible curve of genus  $g$ . By  $S^2(C)$  we denote the second symmetric power of  $C$ , i.e.,  $S^2(C)$  is the quotient of  $C \times C$  by the involution  $(x, y) \mapsto (y, x)$ , and  $S^2(C)$  is smooth irreducible surface. A point of  $S^2(C)$  can be thought of an effective divisor of degree 2 on  $C$  and  $S^2(C)$  can be identified with the set of all effective divisor of degree two on  $C$ . Set

$$\Delta = \{(D, p) \in S^2(C) \times C \mid D = p + q\}.$$

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Then  $\Delta$  is a divisor on  $S^2(C) \times C$  which is known as the universal divisor of degree two and the first projection induces a surjective morphism

$$p_1 : \Delta \rightarrow S^2(C).$$

This is a two sheeted ramified covering. Let

$$p_2 : S^2(C) \times C \rightarrow C$$

be the second projection. For any line bundle  $M$  on  $C$  denote by  $E_M$  the locally free sheaf  $(p_1)_*(p_2^*(M)|_\Delta)$  of rank two on  $S^2(C)$ . The exact sequence of  $\mathcal{O}_{S^2(C) \times C}$  modules

$$(1) \quad 0 \rightarrow p_2^*(M)(-\Delta) \rightarrow p_2^*(M) \rightarrow p_2^*(M)|_\Delta \rightarrow 0$$

gives an exact sequence of  $\mathcal{O}_{S^2(C)}$  modules

$$(2) \quad 0 \rightarrow p_{1*}(p_2^*(M)(-\Delta)) \rightarrow p_{1*}(p_2^*(M)) \rightarrow E_M.$$

By base change formula  $p_{1*}(p_2^*(M)) = H^0(C, M) \otimes \mathcal{O}_{S^2(C)}$ . If there is a subspace  $V$  of  $H^0(C, M)$  such that the linear system given by  $V$  is very ample on  $C$  then we see that the induced map

$$V \otimes \mathcal{O}_{S^2(C)} \rightarrow E_M$$

of vector bundles is surjective. This defines a morphism

$$\phi : S^2(C) \rightarrow \text{Gr}(2, V).$$

If  $\dim(V) = 2$  implies  $\text{Gr}(2, V)$  is a point and if  $\dim(V) = 3$  then  $\text{Gr}(2, V) \simeq \mathbb{P}^2$  the morphism  $\phi$  is finite and is a covering of degree  $\binom{d}{2}$ , where  $d$  is the degree of  $C$  in  $\mathbb{P}^2$ . Hence the first interesting case is the case  $\dim(V) = 4$ , in that case as  $\mathbb{P}(V) \simeq \mathbb{P}^3$ . In the next section we treat this case.

**Remark 2.1.** *Let  $C$  be a non-singular irreducible curve of genus  $g$  and  $M$  be a line bundle of degree  $d$ . From the exact sequence (1) and Grothendick-Reimann-Rock one can compute the Chern classes of  $E_M$  :*

$$C_1(E_M) = (d - g - 1)x + \theta \quad \text{and} \quad C_2(E_M) = \binom{d - g}{2}x^2 + (d - g)x.\theta + \frac{\theta}{2}$$

where  $x$  is the cohomology class of image of  $x \times C$  in  $S^2(C)$  and  $\theta$  is the cohomology class of the pull back of a theta divisor in  $\text{Pic}^2(C)$  under the natural map of  $S^2(C)$  into  $\text{Pic}^2(C)$  [ See, ACGH]. Note that the cohomology group  $H^4(S^2(C), \mathbb{Z})$  is naturally isomorphic to  $\mathbb{Z}$  and

$$C_2(E_M) = \frac{(d - g)(d - g - 1)}{2} + (d - g)g + \frac{g(g - 1)}{2} = \frac{d^2 - d(2g + 1)}{2} + d.g.$$

### 3. SECANT BUNDLE ASSOCIATED TO A SPACE CURVE

Let  $C$  be a irreducible non-singular curve in  $\mathbb{P}^3$ . Assume  $C$  is non-degenerate, i.e., it is not contained in hyper plane. Let  $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$ . Let  $E_L$  denote the secant bundle of  $L$ , which is a rank 2 vector bundle on the second symmetric power  $S^2(C)$  of the curve  $C$ . Let  $V$  be the sub linear system of  $H^0(C, L)$  given by the imbedding of  $C$  in  $\mathbb{P}^3$ . As seen in the previous section the surjection of vector bundles

$$V \otimes \mathcal{O}_{S^2(C)} \rightarrow E_L$$

defines a morphism

$$\phi : S^2(C) \rightarrow \text{Gr}(2, V).$$

First we prove the following well known result which we include here due the lack of a good reference.

**Theorem 3.1.** *With the above notation the morphism  $\phi$  is an imbedding then  $C$  is one of the following*

- 1)  $C$  is a rational normal curve of degree three
- 2)  $C$  is a smooth elliptic curve of degree four.

*Proof:* The curve  $C$  is non-degenerate implies the degree of the curve is greater than or equal to three. First we show that if the degree of  $C$  in  $\mathbb{P}^3$  three or a smooth elliptic curve of degree four then it is an imbedding.

Case a): Assume degree of  $C$  is three. Then by projecting it from a point on it to a plane not containing it we see that  $C$  is isomorphic to  $\mathbb{P}^1$  and the imbedding of  $C$  in  $\mathbb{P}^3$  is given by the complete linear system  $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ . In this case  $S^2(C)$  is isomorphic to  $\mathbb{P}^2$  and the line bundle  $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(\Delta)$  of the universal divisor  $\Delta$  is isomorphic to  $p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(2))$ . Hence the exact sequence (2) becomes

$$(3) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathcal{O}_{\mathbb{P}^2} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(3)) \rightarrow E_{\mathcal{O}_{\mathbb{P}^1}(3)} \rightarrow 0.$$

From the exact sequence (3) we see that the line bundle  $\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)})$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^2}(2)$  and the induced map

$$\overset{2}{\wedge} H^0(\mathcal{O}_{\mathbb{P}^1}(3)) \rightarrow H^0(\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)}))$$

is an isomorphism. Since  $\mathcal{O}_{\mathbb{P}^2}(2)$  gives an imbedding of  $\mathbb{P}^2$  in  $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(2)))$  and  $\text{Gr}(2, H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$  is naturally imbedded in  $\mathbb{P}(\overset{2}{\wedge} H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$  we see that  $\phi$  followed by the natural imbedding of  $\text{Gr}(2, H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$  in  $\mathbb{P}(\overset{2}{\wedge} H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$  is an imbedding. Hence  $\phi$  is is an imbedding in this case.

case 2): Assume  $C$  in  $\mathbb{P}^3$  is a nonsingular elliptic curve of degree four. Since degree of the line bundle  $L$  is four we see that by Riemann-Roch theorem  $V = H^0(L)$  and  $\dim(H^0(L^2))$  is eight. Hence there is a pencil of quadrics containing  $C$ . Since  $C$  is non-degenerate we see that all the quadrics in this pencil are irreducible and  $C$  is the scheme theoretic intersection of any two distinct members of this pencil. From this we can deduce that any line in  $\mathbb{P}^3$  can meet  $C$  in at most two points. Also, if  $Z \subset C$  is a closed sub-scheme supported on a finite set such that the vector space dimension  $\ell(\mathcal{O}_Z)$  of  $\mathcal{O}_Z$  is three, then the natural map

$$H^0(L) \rightarrow \mathcal{O}_Z$$

is surjective. Hence we see that the morphism

$$\phi : S^2(C) \rightarrow \text{Gr}(2, V)$$

is an imbedding.

If  $C$  is a non degenerate non singular irreducible curve of degree four then  $C$  is either a rational curve or an elliptic curve. This can be seen by projecting  $C$  from a point on it to a plane not containing it. Moreover the curve  $C$  is a rational curve if and only if the projected curve is singular curve. Hence the curve  $C$  is rational if and only if there is line  $\ell_0$  in  $\mathbb{P}^3$  such that the length of the scheme  $C \cap \ell_0$  is greater or equal to three. Since the scheme  $C \cap \ell_0$  determines a sub scheme of length greater or equal to two on  $S^2(C)$  we see that

$$\phi : S^2(C) \rightarrow \text{Gr}(2, V)$$

is not an imbedding.

Let  $C$  be a irreducible non-singular curve in  $\mathbb{P}^3$  of degree greater or equal to five. Then by using Castelnuov bound on the genus of a curve in terms of its degree ([4]) we see that there are lines in  $\mathbb{P}^3$  which intersect  $C$  in a sub scheme of length greater or equal to three. Hence in this case

$$\phi : S^2(C) \rightarrow \text{Gr}(2, V)$$

is not an imbedding. This completes the proof of the theorem.  $\square$

**Remark 3.2.** *Let  $C$  be a smooth curve in  $\mathbb{P}^3$  of degree  $d$  greater than or equal to five. Then by Using Reimann-Roch theorem we see that projection  $h$  of  $C$  from a point  $p$  on  $C$  to a  $\mathbb{P}^2$  not containing  $p$  is not an imbedding because*

$$h^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^3}(1)|_C(-p).$$

*From this we get through every point of  $C \in \mathbb{P}^3$  there are lines which intersect  $C$  in more than two points (counting with multiplicity).*

**Remark 3.3.** *On the Grassmaian  $Gr(2, \mathbb{C}^4)$  one has universal exact sequence:*

$$(4) \quad 0 \rightarrow S \rightarrow \mathbb{C}^4 \otimes \mathcal{O}_{Gr(2, \mathbb{C}^4)} \rightarrow Q \rightarrow 0,$$

where  $S$  and  $Q$  are respectively the universal sub bundle and quotient bundle of rank two on  $Gr(2, \mathbb{C}^4)$ . The fiber of  $Q$  (resp. of  $S$ ) at a point  $p \in Gr(2, \mathbb{C}^4)$  is the two dimensional quotient space (resp. subspace, which is the kernal of this quotient map) of  $\mathbb{C}^4$  corresponding to the point  $p$ . It is known (see, page 197 and 411 of [4]) that the cohomology group  $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$  is equal to

$$(5) \quad \mathbb{Z}[c_2(Q)] \oplus \mathbb{Z}[c_2(S)],$$

where  $c_2(Q)$  (resp.  $c_2(S)$ ) is the second Chern class of  $Q$  (resp. of  $S$ ). The morphism  $\phi : S^2(C) \rightarrow Gr(2, \mathbb{C}^4)$  determines a cohomology class in  $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$  namely  $\phi_*([S^2(C)])$ . By the identification ((5)) the cohomology class  $\phi_*([S^2(C)])$  is represented by a pair of integers  $(a, b)$ . Using the geometry of  $Gr(2, \mathbb{C}^4)$  we see that  $a$  is the number secants lines to  $C$  which passes through a generic point of  $\mathbb{P}^3$  and  $b$  is the number of secant lines to  $C$  which are contained in generic plane of  $\mathbb{P}^3$  (see [4]). From this description we see that if  $C$  is a rational normal curve of degree three than the cohomology class  $\phi_*([S^2(C)])$  is  $(1, 3)$  and if  $C$  is an elliptic curve of degree four than the cohomology class  $\phi_*([S^2(C)])$  is  $(2, 6)$ .

#### 4. PROPERTIES OF SECANT BUNDLE

Let  $C, L$  and  $V$  be as in previous section. When  $C$  is a rational normal curve in  $\mathbb{P}^3$  from the results obtained by Schwarzenberger in ([2]) it follows that restriction of  $E_L$  is semi-stable for all lines in  $S^2(C)$  which not tangent to the image of the diagonal of  $C \times C$  and not semi-stable to those lines which are tangent to the image of the diagonal of  $C \times C$ . For a general curve  $C$  we have the following:

**Theorem 4.1.** *Let  $C$  be a non-singular curve of degree  $d \geq 4$  in  $\mathbb{P}^3$  and  $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$ . Then  $E_L|_{x \times C}$  is not semi-stable for every  $x \in C$ .*

**Proof:** If  $D = p + q$  be a point on the surface  $S^2(C)$  then the fiber  $E_L|_D$  of the vector bundle  $E_L$  at  $D$  can be identified with  $L \otimes \mathcal{O}_D$ , where  $\mathcal{O}_D$  is the structure sheaf sub-scheme of  $C$  of length two corresponding to  $D$ . Since every point of the curve  $p \times C$  is of the form  $p \times q$  for some  $q \in C$  by looking at the linear system  $V(-p) = \{s \in V | s(p) = 0\}$  we see that there is an injective homomorphism

$$0 \rightarrow L(-p) \rightarrow E_L|_{p \times C}.$$

But by (2.1) we see that  $C_1(E_L|_{p \times C}) = d - 1$ , where  $d$  is the degree of the line bundle  $L$  on  $C$ . Hence there exists a line bundle  $M$  of degree 0 on  $C$  such that  $E_L|_{p \times C}$  fits in an exact sequence

$$0 \rightarrow L(-p) \rightarrow E_L|_{p \times C} \rightarrow M \rightarrow 0.$$

This proves that  $E_L|_{p \times C}$  is not semi-stable as degree of  $M$  is less than half the degree of  $E_L|_{p \times C}$ .

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