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MORPHISMS FROM $S^2(C)$ TO $Gr(2, \mathbb{C}^4)$.

A. EL MAZOUNI, F. LAYTIMI, AND D.S. NAGARAJ

Abstract. In this note we study the natural vector bundle of rank 2 on $S^2(C)$, where $S^2(C)$ is the second symmetric power of non-singular irreducible non-degenerate space curve $C$.

Keywords: Vector bundles; Morphisms; Grassmannians; second symmetric power; curves.

1. Introduction

We denote by $\mathbb{P}^3$ the projective 3-space over the field $\mathbb{C}$ of complex numbers and $Gr(2, \mathbb{C}^4)$ be Grassmannian of two-dimentional quotient spaces of $\mathbb{C}^4$. Let $\mathcal{O}_{\mathbb{P}^3}(1)$ be the ample generator of the Pic($\mathbb{P}^3$). Let $C$ be an irreducible non-singular curve in $\mathbb{P}^3$. Assume $C$ is non-degenerate, i.e., it is not contained in hyper plane. Let $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Let $E_L$ denote the secant bundle of $L$, which is a rank 2 vector bundle on the second symmetric power $S^2(C)$ of the curve $C$. The bundle $E_L$ is generated by 4 linearly independent sections. Hence there is a morphism $S^2(C)$ to $Gr(2, \mathbb{C}^4)$. In this article we study this morphism. Also, we study the properties of the restriction of $E_L$ to some naturally imbedded curves in $S^2(C)$. Such a study was carried out by Schwarzenberger ([2]) when the curve $C$ is a rational normal curve in $\mathbb{P}^3$.

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2. Secant bundles

Let $C$ be a smooth irreducible curve of genus $g$. By $S^2(C)$ we denote the second symmetric power of $C$, i.e., $S^2(C)$ is the quotient of $C \times C$ by the involution $(x, y) \mapsto (y, x)$, and $S^2(C)$ is smooth irreducible surface. A point of $S^2(C)$ can be thought of an effective divisor of degree 2 on $C$ and $S^2(C)$ can be identified with the set of all effective divisor of degree two on $C$. Set

$$\Delta = \{(D, p) \in S^2(C) \times C | D = p + q\}.$$
Then $\Delta$ is a divisor on $S^2(C) \times C$ which is known as the universal divisor of degree two and the first projection induces a surjective morphism

$$p_1 : \Delta \to S^2(C).$$

This is a two sheeted ramified covering. Let

$$p_2 : S^2(C) \times C \to C$$

be the second projection. For any line bundle $M$ on $C$ denote by $E_M$ the locally free sheaf $(p_1)_*(p_2^*(M)|_\Delta)$ of rank two on $S^2(C)$. The exact sequence of $\mathcal{O}_{S^2(C) \times C}$ modules

(1) $0 \to p_2^*(M)(-\Delta) \to p_2^*(M) \to p_2^*(M)|_\Delta \to 0$

gives an exact sequence of $\mathcal{O}_{S^2(C)}$ modules

(2) $0 \to p_1_*(p_2^*(M)(-\Delta)) \to p_1_*(p_2^*(M)) \to E_M.$

By base change formula $p_1_*(p_2^*(M)) = H^0(C, M) \otimes \mathcal{O}_{S^2(C)}$. If there is a subspace $V$ of $H^0(C, M)$ such that the linear system given by $V$ is very ample on $C$ then we see that the induced map

$$V \otimes \mathcal{O}_{S^2(C)} \to E_M$$

of vector bundles is surjective. This defines a morphism

$$\phi : S^2(C) \to \text{Gr}(2, V).$$

If dim$(V) = 2$ implies $\text{Gr}(2, V)$ is a point and if dim$(V) = 3$ then $\text{Gr}(2, V) \simeq \mathbb{P}^2$ the morphism $\phi$ is finite and is a covering of degree $(d)$, where $d$ is the degree of $C$ in $\mathbb{P}^2$. Hence the first interesting case is the case dim$(V) = 4$, in that case as $\mathbb{P}(V) \simeq \mathbb{P}^3$. In the next section we treat this case.

**Remark 2.1.** Let $C$ be a non-singular irreducible curve of genus $g$ and $M$ be a line bundle of degree $d$. From the exact sequence (1) and Grothendick-Reimann-Rock one can compute the Chern classes of $E_M$:

$$C_1(E_M) = (d-g-1)x + \theta \text{ and } C_2(E_M) = \left(\frac{d-g}{2}\right)x^2 + (d-g)x.\theta + \frac{\theta}{2}$$

where $x$ is the cohomology class of image of $x \times C$ in $S^2(C)$ and $\theta$ is the cohomology class of the pull back of a theta divisor in $\text{Pic}^2(C)$ under the natural map of $S^2(C)$ into $\text{Pic}^2(C)$ [See, ACGH]. Note that the cohomology group $H^4(S^2(C), \mathbb{Z})$ is naturally isomorphic to $\mathbb{Z}$ and

$$C_2(E_M) = \frac{(d-g)(d-g-1)}{2} + (d-g)g + \frac{g(g-1)}{2} = \frac{d^2 - d(2g+1) + d.g}{2}.$$
3. Secant bundle associated to a space curve

Let $C$ be an irreducible non-singular curve in $\mathbb{P}^3$. Assume $C$ is non-degenerate, i.e., it is not contained in hyper plane. Let $L = \mathcal{O}_{\mathbb{P}^3}(1) |_C$. Let $E_L$ denote the secant bundle of $L$, which is a rank 2 vector bundle on the second symmetric power $S^2(C)$ of the curve $C$. Let $V$ be the sub linear system of $H^0(C, L)$ given by the imbedding of $C$ in $\mathbb{P}^3$. As seen in the previous section the surjection of vector bundles

$$V \otimes \mathcal{O}_{S^2(C)} \rightarrow E_L$$

defines a morphism

$$\phi : S^2(C) \rightarrow \text{Gr}(2, V).$$

First we prove the following well known result which we include here due the lack of a good reference.

**Theorem 3.1.** With the above notation the morphism $\phi$ is an imbedding then $C$ is one of the following

1) $C$ is a rational normal curve of degree three
2) $C$ is a smooth elliptic curve of degree four.

**Proof:** The curve $C$ is non-degenerate implies the degree of the curve is greater than or equal to three. First we show that if the degree of $C$ in $\mathbb{P}^3$ three or a smooth elliptic curve of degree four then it is an imbedding.

Case a): Assume degree of $C$ is three. Then by projecting it from a point on it to a plane not containing it we see that $C$ is isomorphic to $\mathbb{P}^1$ and the imbedding of $C$ in $\mathbb{P}^3$ is given by the complete linear system $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. In this case $S^2(C)$ is isomorphic to $\mathbb{P}^2$ and the line bundle $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(\Delta)$ of the universal divisor $\Delta$ is isomorphic to $p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(2))$. Hence the exact sequence (2) becomes

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathcal{O}_{\mathbb{P}^2} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(3)) \rightarrow E_{\mathcal{O}_{\mathbb{P}^1}(3)} \rightarrow 0.$$

From the exact sequence (3) we see that the line bundle $\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(2)$ and the induced map

$$\wedge^2 H^0(\mathcal{O}_{\mathbb{P}^1}(3)) \rightarrow H^0(\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)}))$$

is an isomorphism. Since $\mathcal{O}_{\mathbb{P}^2}(2)$ gives an imbedding of $\mathbb{P}^2$ in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(2)))$ and $\text{Gr}(2, H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$ is naturally imbedded in $\mathbb{P}(\wedge^2 H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$ we see that $\phi$ followed by the natural imbedding of $\text{Gr}(2, H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$ in $\mathbb{P}(\wedge^2 H^0(\mathcal{O}_{\mathbb{P}^1}(3)))$ is an imbedding. Hence $\phi$ is is an imbedding in this case.
case 2): Assume $C$ in $\mathbb{P}^3$ is a nonsingular elliptic curve of degree four. Since degree of the line bundle $L$ is four we see that by Riemann-Roch theorem $V = H^0(L)$ and $\dim(H^0(L^2))$ is eight. Hence there is a pencil of quadrics containing $C$. Since $C$ is non-degenerate we see that all the quadrics in this pencil are irreducible and $C$ is the scheme theoretic intersection of any two distinct members of this pencil. From this we can deduce that any line in $\mathbb{P}^3$ can meet $C$ in at most two points. Also, if $Z \subset C$ is a closed sub-sceme supported on a finite set such that the vector space dimension $\ell(O_Z)$ of $O_Z$ is three, then the natural map

$$H^0(L) \to O_Z$$

is surjective. Hence we see that the morphism

$$\phi : S^2(C) \to \text{Gr}(2, V)$$

is an imbedding.

If $C$ is a non degenerate non singular irreducible curve of degree four then $C$ is either a rational curve or an elliptic curve. This can be seen by projecting $C$ from a point on it to a plane not containing it. More over the curve $C$ is a rational curve if and only if the projected curve is singular curve. Hence the curve $C$ is rational if and only if there is line $\ell_0$ in $\mathbb{P}^3$ such that the length of the scheme $C \cap \ell_0$ is greater or equal to three. Since the scheme $C \cap \ell_0$ determines a sub scheme of length greater or equal to two on $S^2(C)$ we see that

$$\phi : S^2(C) \to \text{Gr}(2, V)$$

is not an imbedding.

Let $C$ be a irreducible non-singular curve in $\mathbb{P}^3$ of degree greater or equal to five. Then by using Castalnov bound on the genus of a curve in terms of its degree ([4]) we see that there are lines in $\mathbb{P}^3$ which intersect $C$ in a sub scheme of length greater or equal to three. Hence in this case

$$\phi : S^2(C) \to \text{Gr}(2, V)$$

is not an imbedding. This completes the proof of the theorem. □

Remark 3.2. Let $C$ be a smooth curve in $\mathbb{P}^3$ of degree $d$ greater than or equal to five. Then by Using Reimann-Roch theorem we see that projection $h$ of $C$ from a point $p$ on $C$ to a $\mathbb{P}^2$ not containing $p$ is not an imbedding because

$$h^*(O_{\mathbb{P}^2}(1)) = O_{\mathbb{P}^3}(1)|_{C(-p)}.$$ 

From this we get through every point of $C \in \mathbb{P}^3$ there are lines which intersect $C$ in more than two points (counting with multiplicity).
Remark 3.3. On the Grassmannian $\text{Gr}(2, \mathbb{C}^4)$ one has universal exact sequence:

\begin{equation}
0 \to S \to \mathbb{C}^4 \otimes \mathcal{O}_{\text{Gr}(2, \mathbb{C}^4)} \to Q \to 0,
\end{equation}

where $S$ and $Q$ are respectively the universal sub bundle and quotient bundle of rank two on $\text{Gr}(2, \mathbb{C}^4)$. The fiber of $Q$ (resp. of $S$) at a point $p \in \text{Gr}(2, \mathbb{C}^4)$ is the two dimensional quotient space (resp. subspace, which is the kernel of this quotient map) of $\mathbb{C}^4$ corresponding to the point $p$. It is known (see, page 197 and 411 of [4]) that the cohomology group $H^4(\text{Gr}(2, \mathbb{C}^4), \mathbb{Z})$ is equal to

\begin{equation}
\mathbb{Z}[c_2(Q)] \oplus \mathbb{Z}[c_2(S)],
\end{equation}

where $c_2(Q)$ (resp. $c_2(S)$) is the second Chern class of $Q$ (resp. of $S$). The morphism $\phi : S^2(C) \to \text{Gr}(2, \mathbb{C}^4)$ determines a cohomology class in $H^4(\text{Gr}(2, \mathbb{C}^4), \mathbb{Z})$ namely $\phi_*([S^2(C)])$. By the identification (5) the cohomology class $\phi_*([S^2(C)])$ is represented by a pair of integers $(a, b)$. Using the geometry of $\text{Gr}(2, \mathbb{C}^4)$ we see that $a$ is the number secants lines to $C$ which passes through a generic point of $\mathbb{P}^3$ and $b$ is the number of secant lines to $C$ which are contained in generic plane of $\mathbb{P}^3$ (see [4]). From this description we see that if $C$ is a rational normal curve of degree three than the cohomology class $\phi_*([S^2(C)])$ is $(1, 3)$ and if $C$ is an elliptic curve of degree four than the cohomology class $\phi_*([S^2(C)])$ is $(2, 6)$.

4. Properties of Secant bundle

Let $C, L$ and $V$ be as in previous section. When $C$ is a rational normal curve in $\mathbb{P}^3$ from the results obtained by Schwarzenberger in ([2]) it follows that restriction of $E_L$ is semi-stable for all lines in $S^2(C)$ which not tangent to the image of the diagonal of $C \times C$ and not semi-stable to those lines which are tangent to the image of the diagonal of $C \times C$. For a general curve $C$ we have the following:

**Theorem 4.1.** Let $C$ be a non-singular curve of degree $d \geq 4$ in $\mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Then $E_L|_{x \times C}$ is not semi-stable for every $x \in C$.

**Proof:** If $D = p + q$ be a point on the surface $S^2(C)$ then the fiber $E_L|_D$ of the vector bundle $E_L$ at $D$ can be identified with $L \otimes \mathcal{O}_D$, where $\mathcal{O}_D$ is the structure sheaf sub-scheme of $C$ of length two corresponding to $D$. Since every point of the curve $p \times C$ is of the form $p \times q$ for some $q \in C$ by looking at the linear system $V(-p) = \{ s \in V | s(p) = 0 \}$ we see that the there is a injective homomorphism

\begin{equation}
0 \to L(-p) \to E_L|_{p \times C}.
\end{equation}
But by (2.1) we see that $C_1(E_L|_{P \times C}) = d - 1$, where $d$ is the degree of the line bundle $L$ on $C$. Hence there there exists a line bundle $M$ of degree 0 on $C$ such that $E_L|_{P \times C}$ fits in an exact sequence

$$0 \to L(-p) \to E_L|_{P \times C} \to M \to 0.$$ 

This proves that $E_L|_{P \times C}$ is not semi-stable as degree of $M$ is less than half the degree of $E_L|_{P \times C}$.

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