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MORPHISMS FROM $S^2(C)$ TO $Gr(2, \mathbb{C}^4)$.

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Abstract. In this note we study the natural vector bundle of rank 2 on $S^2(C)$, where $S^2(C)$ is the second symmetric power of non-singular irreducible non-degenerate space curve C.

Keywords: Vector bundles; Morphisms; Grassmannians; second symmetric power; curves.

1. Introduction

We denote by $\mathbb{P}^3$ the projective 3-space over the field $\mathbb{C}$ of complex numbers and $\mathrm{Gr}(2, \mathbb{C}^4)$ be Grassmannian of two-dimentional quotient spaces of $\mathbb{C}^4$. Let $\mathcal{O}_{\mathbb{P}^3}(1)$ be the ample generator of the $\mathrm{Pic}(\mathbb{P}^3)$. Let C be an irreducible non-singular curve in $\mathbb{P}^3$. Assume C is non-degenerate, i.e., it is not contained in hyper plane. Let $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Let E_L denote the secant bundle of L, which is a rank 2 vector bundle on the second symmetric power $S^2(C)$ of the curve C. The bundle E_L is generated by 4 linearly independent sections. Hence there is a morphism $S^2(C)$ to $Gr(2, \mathbb{C}^4)$. In this article we study this morphism. Also, we study the properties of the restriction of E_L to some naturally imbedded curves in $S^2(C)$. Such a study was carried out by Schwarzenberger ([2]) when the curve C is a rational normal curve in $\mathbb{P}^3$.

Acknowledgments: The third author would like to thank Université Lille and Université d' Artois, France.

2. Secant bundles

Let C be a smooth irreducible curve of genus g. By $S^2(C)$ we denote the second symmetric power of C, i.e., $S^2(C)$ is the quotient of $C \times C$ by the involution $(x, y) \mapsto (y, x)$, and $S^2(C)$ is smooth irreducible surface. A point of $S^2(C)$ can be thought of an effective divisor of degree 2 on C and $S^2(C)$ can be identified with the set of all effective divisor of degree two on C. Set

$$\Delta = \{(D, p) \in S^2(C) \times C | D = p + q\}.$$

1991 *Mathematics Subject Classification.* 14F17.

Then Δ is a divisor on $S^2(C)\times C$ which is known as the universal divisor of degree two and the first projection induces a surjective morphism

$$p_1 : \Delta \to S^2(C).$$

This is a two sheeted ramified covering. Let

$$p_2 : S^2(C) \times C \to C$$

be the second projection. For any line bundle M on C denote by E_M the locally free sheaf $(p_1)_*(p_2^*(M)|_\Delta)$of rank two on $S^2(C)$. The exact sequence of $\mathcal{O}_{S^2(C)\times C}$ modules

$$0 \to p_2^*(M)(-\Delta) \to p_2^*(M) \to p_2^*(M)|_\Delta \to 0 \tag{1}$$

gives an exact sequence of $\mathcal{O}_{S^2(C)}$ modules

$$0 \to p_{1*}(p_2^*(M)(-\Delta)) \to p_{1*}(p_2^*(M)) \to E_M. \tag{2}$$

By base change formula $p_{1*}(p_2^*(M)) = H^0(C, M) \otimes \mathcal{O}_{S^2(C)}$. If there is a subspace V of $H^0(C, M)$ such that the linear system given by V is very ample on C then we see that the induced map

$$V \otimes \mathcal{O}_{S^2(C)} \to E_M$$

of vector bundles is surjective. This defines a morphism

$$\phi : S^2(C) \to \mathrm{Gr}(2, V).$$

If $\dim(V) = 2$ implies $\mathrm{Gr}(2, V)$ is a point and if $\dim(V) = 3$ then $\mathrm{Gr}(2, V) \simeq \mathbb{P}^2$ the morphism ϕ is finite and is a covering of degree $\binom{d}{2}$, where d is the degree of C in $\mathbb{P}^2$. Hence the first interesting case is the case $\dim(V) = 4$, in that case as $\mathbb{P}(V) \simeq \mathbb{P}^3$. In the next section we treat this case.

Remark 2.1. *Let C be a non-singular irreducible curve of genus g and M be a line bundle of degree d. From the exact sequence* (1) *and Grothendick-Reimann-Rock one can compute the Chern classes of E_M :*

$$C_1(E_M) = (d-g-1)x + \theta \ \text{ and } \ C_2(E_M) = \binom{d-g}{2}x^2 + (d-g)x.\theta + \frac{\theta}{2}$$

where x is the cohomology class of image of $x \times C$ in $S^2(C)$ and θ is the cohomology class of the pull back of a theta divisor in $\mathrm{Pic}^2(C)$ *under the natural map of $S^2(C)$ into* $\mathrm{Pic}^2(C)$ [See, ACGH]. *Note that the cohomology group* $\mathrm{H}^4(S^2(C), \mathbb{Z})$ *is naturally isomorphic to $\mathbb{Z}$ and*

$$C_2(E_M) = \frac{(d-g)(d-g-1)}{2} + (d-g)g + \frac{g(g-1)}{2} = \frac{d^2 - d(2g+1)}{2} + d.g.$$

3. Secant bundle associated to a space curve

Let C be a irreducible non-singular curve in $\mathbb{P}^3$. Assume C is non-degenerate, i.e., it is not contained in hyper plane. Let $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Let E_L denote the secant bundle of L, which is a rank 2 vector bundle on the second symmetric power $S^2(C)$ of the curve C. Let V be the sub linear system of $H^0(C, L)$ given by the imbedding of C in $\mathbb{P}^3$. As seen in the previous section the surjection of vector bundles

$$V \otimes \mathcal{O}_{S^2(C)} \to E_L$$

defines a morphism

$$\phi : S^2(C) \to \mathrm{Gr}(2, V).$$

First we prove the following well known result which we include here due the lack of a good reference.

Theorem 3.1. *With the above notation the morphism ϕ is an imbedding then C is one of the following*

1) *C is a rational normal curve of degree three*
2) *C is a smooth elliptic curve of degree four.*

Proof: The curve C is non-degenerate implies the degree of the curve is greater than or equal to three. First we show that if the degree of C in $\mathbb{P}^3$ three or a smooth elliptic curve of degree four then it is an imbedding.

Case a): Assume degree of C is three. Then by projecting it from a point on it to a plane not containing it we see that C is isomorphic to $\mathbb{P}^1$ and the imbedding of C in $\mathbb{P}^3$ is given by the complete linear system $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. In this case $S^2(C)$ is isomorphic to $\mathbb{P}^2$ and the line bundle $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(\Delta)$ of the universal divisor Δ is isomorphic to $p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(2))$. Hence the exact sequence (2) becomes

$$(3) \quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathcal{O}_{\mathbb{P}^2} \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3)) \to E_{\mathcal{O}_{\mathbb{P}^1}(3)} \to 0.$$

From the exact sequence (3) we see that the line bundle $\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(2)$ and the induced map

$$\overset{2}{\wedge}\, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3)) \to \mathrm{H}^0(\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)}))$$

is an isomorphism. Since $\mathcal{O}_{\mathbb{P}^2}(2)$ gives an imbedding of $\mathbb{P}^2$ in $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(2)))$ and $\mathrm{Gr}(2, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ is naturally imbedded in $\mathbb{P}(\overset{2}{\wedge}\, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ we see that ϕ followed by the natural imbedding of $\mathrm{Gr}(2, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ in $\mathbb{P}(\overset{2}{\wedge}\, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ is an imbedding. Hence ϕ is is an imbedding in this case.

case 2): Assume C in $\mathbb{P}^3$ is a nonsingular elliptic curve of degree four. Since degree of the line bundle L is four we see that by Riemann-Roch theorem $V = \mathrm{H}^0(L)$ and $\dim(\mathrm{H}^0(L^2))$ is eight. Hence there is a pencil of quadrics containing C. Since C is non-degenerate we see that all the quadrics in this pencil are irreducible and C is the scheme theoretic intersection of any two distinct members of this pencil. From this we can deduce that any line in $\mathbb{P}^3$ can meet C in at most two points. Also, if $Z \subset C$ is a closed sub-sceme supported on a finite set such that the vector space dimension $\ell(\mathcal{O}_Z)$ of $\mathcal{O}_Z$ is three, then the natural map

$$\mathrm{H}^0(L) \to \mathcal{O}_Z$$

is surjective. Hence we see that the morphism

$$\phi : S^2(C) \to \mathrm{Gr}(2, V)$$

is an imbedding.

If C is a non degenerate non singular irreducible curve of degree four then C is either a rational curve or an elliptic curve. This can be seen by projecting C from a point on it to a plane not containing it. More over the curve C is a rational curve if and only if the projected curve is singular curve. Hence the curve C is rational if and only if there is line ℓ_0 in $\mathbb{P}^3$ such that the length of the scheme $C \cap \ell_0$ is greater or equal to three. Since the scheme $C \cap \ell_0$ determines a sub scheme of length greater or equal to two on $S^2(C)$ we see that

$$\phi : S^2(C) \to \mathrm{Gr}(2, V)$$

is not an imbedding.

Let C be a irreducible non-singular curve in $\mathbb{P}^3$ of degree greater or equal to five. Then by using Castalnov bound on the genus of a curve in terms of its degree ([4]) we see that there are lines in $\mathbb{P}^3$ which intersect C in a sub scheme of length greater or equal to three. Hence in this case

$$\phi : S^2(C) \to \mathrm{Gr}(2, V)$$

is not an imbedding. This completes the proof of the theorem. □

Remark 3.2. *Let C be a smooth curve in $\mathbb{P}^3$ of degree d greater than or equal to five. Then by Using Reimann-Roch theorem we see that projection h of C from a point p on C to a $\mathbb{P}^2$ not containing p is not an imbedding because*

$$h^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^3}(1)|_C(-p).$$

¿From this we get through every point of $C \in \mathbb{P}^3$ there are lines which intersect C in more than two points (counting with multiplicity).

Remark 3.3. *On the Grassmaian $Gr(2, \mathbb{C}^4)$ one has universal exact sequence:*

$$0 \to S \to \mathbb{C}^4 \otimes \mathcal{O}_{Gr(2,\mathbb{C}^4)} \to Q \to 0, \tag{4}$$

where S and Q are respectively the universal sub bundle and quotient bundle of rank two on $Gr(2, \mathbb{C}^4)$. The fiber of Q (resp. of S) at a point $p \in Gr(2, \mathbb{C}^4)$ is the two dimensional quotient space (resp. subspace, which is the kernal of this quotient map) of $\mathbb{C}^4$ corresponding to the point p. It is known (see, page 197 and 411 of [4]) that the cohomology group $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$ is equal to

$$\mathbb{Z}[c_2(Q)] \oplus \mathbb{Z}[c_2(S)], \tag{5}$$

where $c_2(Q)$ (resp. $c_2(S)$) is the second Chern class of Q (resp. of S). The morphism $\phi : S^2(C) \to Gr(2, \mathbb{C}^4)$ determines a cohomology class in $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$ namely $\phi_*([S^2(C)])$. By the identification ((5)) the cohomology class $\phi_*([S^2(C)])$ is represented by a pair of integers (a, b). Using the geometry of $Gr(2, \mathbb{C}^4)$ we see that a is the number secants lines to C which passes through a generic point of $\mathbb{P}^3$ and b is the number of secant lines to C which are contained in generic plane of $\mathbb{P}^3$ (see [4]). From this description we see that if C is a rational normal curve of degree three than the cohomology class $\phi_*([S^2(C)])$ is $(1, 3)$ and if C is an elliptic curve of degree four than the cohomology class $\phi_*([S^2(C)])$ is $(2, 6)$.

4. Properties of Secant bundle

Let C, L and V be as in previous section. When C is a rational normal curve in $\mathbb{P}^3$ from the results obtained by Schwarzenberger in ([2]) it follows that restriction of E_L is semi-stable for all lines in $S^2(C)$ which not tangent to the image of the diagonal of $C \times C$ and not semi-stable to those lines which are tangent to the image of the diagonal of $C \times C$. For a general curve C we have the following:

Theorem 4.1. *Let C be a non-singular curve of degree $d \geq 4$ in $\mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Then $E_L|_{x \times C}$ is not semi-stable for every $x \in C$.*

Proof: If $D = p + q$ be a point on the surface $S^2(C)$ then the fiber $E_L|_D$ of the vector bundle E_L at D can be identified with $L \otimes \mathcal{O}_D$, where $\mathcal{O}_D$ is the structure sheaf sub-scheme of C of length two corresponding to D. Since every point of the curve $p \times C$ is of the form $p \times q$ for some $q \in C$ by looking at the linear system $V(-p) = \{s \in V | s(p) = 0$ we see that the there is a injective homomorphism

$$0 \to L(-p) \to E_L|_{p \times C}.$$

But by (2.1) we see that $C_1(E_L|_{p\times C}) = d-1$, where d is the degree of the line bundle L on C. Hence there there exists a line bundle M of degree 0 on C such that $E_L|_{p\times C}$ fits in an exact sequence

$$0 \to L(-p) \to E_L|_{p\times C} \to M \to 0.$$

This proves that $E_L|_{p\times C}$ is not semi-stable as degree of M is less than half the degree of $E_L|_{p\times C}$.

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