# MORPHISMS FROM S2(C) TO Gr(2,C4)

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# MORPHISMS FROM $S^2(C)$ TO $Gr(2, \mathbb{C}^4)$.

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Abstract. In this note we study the natural vector bundle of rank 2 on $S^2(C)$, where $S^2(C)$ is the second symmetric power of non-singular irreducible non-degenerate space curve $C$.

**Keywords:** Vector bundles; Morphisms; Grassmannians; second symmetric power; curves.

## 1. Introduction

We denote by $\mathbb{P}^3$ the projective 3-space over the field $\mathbb{C}$ of complex numbers and $\mathrm{Gr}(2, \mathbb{C}^4)$ be Grassmannian of two-dimentional quotient spaces of $\mathbb{C}^4$. Let $\mathcal{O}_{\mathbb{P}^3}(1)$ be the ample generator of the $\mathrm{Pic}(\mathbb{P}^3)$. Let $C$ be an irreducible non-singular curve in $\mathbb{P}^3$. Assume $C$ is non-degenerate, i.e., it is not contained in hyper plane. Let $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Let $E_L$ denote the secant bundle of $L$, which is a rank 2 vector bundle on the second symmetric power $S^2(C)$ of the curve $C$. The bundle $E_L$ is generated by 4 linearly independent sections. Hence there is a morphism $S^2(C)$ to $Gr(2, \mathbb{C}^4)$. In this article we study this morphism. Also, we study the properties of the restriction of $E_L$ to some naturally imbedded curves in $S^2(C)$. Such a study was carried out by Schwarzenberger ([2]) when the curve $C$ is a rational normal curve in $\mathbb{P}^3$.

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## 2. Secant bundles

Let $C$ be a smooth irreducible curve of genus $g$. By $S^2(C)$ we denote the second symmetric power of $C$, i.e., $S^2(C)$ is the quotient of $C \times C$ by the involution $(x, y) \mapsto (y, x)$, and $S^2(C)$ is smooth irreducible surface. A point of $S^2(C)$ can be thought of an effective divisor of degree 2 on $C$ and $S^2(C)$ can be identified with the set of all effective divisor of degree two on $C$. Set

$$\Delta = \{(D, p) \in S^2(C) \times C | D = p + q\}.$$

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Then $\Delta$ is a divisor on $S^2(C)\times C$ which is known as the universal divisor of degree two and the first projection induces a surjective morphism

$$p_1 : \Delta \to S^2(C).$$

This is a two sheeted ramified covering. Let

$$p_2 : S^2(C) \times C \to C$$

be the second projection. For any line bundle $M$ on $C$ denote by $E_M$ the locally free sheaf $(p_1)_*(p_2^*(M)|_\Delta)$of rank two on $S^2(C)$. The exact sequence of $\mathcal{O}_{S^2(C)\times C}$ modules

$$(1) \qquad 0 \to p_2^*(M)(-\Delta) \to p_2^*(M) \to p_2^*(M)|_\Delta \to 0$$

gives an exact sequence of $\mathcal{O}_{S^2(C)}$ modules

$$(2) \qquad 0 \to p_{1*}(p_2^*(M)(-\Delta)) \to p_{1*}(p_2^*(M)) \to E_M.$$

By base change formula $p_{1*}(p_2^*(M)) = H^0(C, M) \otimes \mathcal{O}_{S^2(C)}$. If there is a subspace $V$ of $H^0(C, M)$ such that the linear system given by $V$ is very ample on $C$ then we see that the induced map

$$V \otimes \mathcal{O}_{S^2(C)} \to E_M$$

of vector bundles is surjective. This defines a morphism

$$\phi : S^2(C) \to \mathrm{Gr}(2, V).$$

If $\dim(V) = 2$ implies $\mathrm{Gr}(2, V)$ is a point and if $\dim(V) = 3$ then $\mathrm{Gr}(2, V) \simeq \mathbb{P}^2$ the morphism $\phi$ is finite and is a covering of degree $\binom{d}{2}$, where $d$ is the degree of $C$ in $\mathbb{P}^2$. Hence the first interesting case is the case $\dim(V) = 4$, in that case as $\mathbb{P}(V) \simeq \mathbb{P}^3$. In the next section we treat this case.

**Remark 2.1.** *Let $C$ be a non-singular irreducible curve of genus $g$ and $M$ be a line bundle of degree $d$. From the exact sequence* (1) *and Grothendick-Reimann-Rock one can compute the Chern classes of $E_M$ :*

$$C_1(E_M) = (d-g-1)x + \theta \ \text{ and } \ C_2(E_M) = \binom{d-g}{2}x^2 + (d-g)x.\theta + \frac{\theta}{2}$$

*where $x$ is the cohomology class of image of $x \times C$ in $S^2(C)$ and $\theta$ is the cohomology class of the pull back of a theta divisor in* $\mathrm{Pic}^2(C)$ *under the natural map of $S^2(C)$ into* $\mathrm{Pic}^2(C)$ [ See, ACGH]. *Note that the cohomology group* $\mathrm{H}^4(S^2(C), \mathbb{Z})$ *is naturally isomorphic to $\mathbb{Z}$ and*

$$C_2(E_M) = \frac{(d-g)(d-g-1)}{2} + (d-g)g + \frac{g(g-1)}{2} = \frac{d^2 - d(2g+1)}{2} + d.g.$$

## 3. Secant bundle associated to a space curve

Let $C$ be a irreducible non-singular curve in $\mathbb{P}^3$. Assume $C$ is non-degenerate, i.e., it is not contained in hyper plane. Let $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Let $E_L$ denote the secant bundle of $L$, which is a rank 2 vector bundle on the second symmetric power $S^2(C)$ of the curve $C$. Let $V$ be the sub linear system of $H^0(C, L)$ given by the imbedding of $C$ in $\mathbb{P}^3$. As seen in the previous section the surjection of vector bundles

$$V \otimes \mathcal{O}_{S^2(C)} \to E_L$$

defines a morphism

$$\phi : S^2(C) \to \mathrm{Gr}(2, V).$$

First we prove the following well known result which we include here due the lack of a good reference.

**Theorem 3.1.** *With the above notation the morphism $\phi$ is an imbedding then $C$ is one of the following*

1) *$C$ is a rational normal curve of degree three*
2) *$C$ is a smooth elliptic curve of degree four.*

*Proof:* The curve $C$ is non-degenerate implies the degree of the curve is greater than or equal to three. First we show that if the degree of $C$ in $\mathbb{P}^3$ three or a smooth elliptic curve of degree four then it is an imbedding.

Case a): Assume degree of $C$ is three. Then by projecting it from a point on it to a plane not containing it we see that $C$ is isomorphic to $\mathbb{P}^1$ and the imbedding of $C$ in $\mathbb{P}^3$ is given by the complete linear system $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$. In this case $S^2(C)$ is isomorphic to $\mathbb{P}^2$ and the line bundle $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(\Delta)$ of the universal divisor $\Delta$ is isomorphic to $p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(2))$. Hence the exact sequence (2) becomes

$$(3)\quad 0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathcal{O}_{\mathbb{P}^2} \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3)) \to E_{\mathcal{O}_{\mathbb{P}^1}(3)} \to 0.$$

From the exact sequence (3) we see that the line bundle $\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(2)$ and the induced map

$$\overset{2}{\wedge}\, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3)) \to \mathrm{H}^0(\det(E_{\mathcal{O}_{\mathbb{P}^1}(3)}))$$

is an isomorphism. Since $\mathcal{O}_{\mathbb{P}^2}(2)$ gives an imbedding of $\mathbb{P}^2$ in $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(2)))$ and $\mathrm{Gr}(2, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ is naturally imbedded in $\mathbb{P}(\overset{2}{\wedge}\ \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ we see that $\phi$ followed by the natural imbedding of $\mathrm{Gr}(2, \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ in $\mathbb{P}(\overset{2}{\wedge}\ \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(3))$ is an imbedding. Hence $\phi$ is is an imbedding in this case.

case 2): Assume $C$ in $\mathbb{P}^3$ is a nonsingular elliptic curve of degree four. Since degree of the line bundle $L$ is four we see that by Riemann-Roch theorem $V = \mathrm{H}^0(L)$ and $\dim(\mathrm{H}^0(L^2))$ is eight. Hence there is a pencil of quadrics containing $C$. Since $C$ is non-degenerate we see that all the quadrics in this pencil are irreducible and $C$ is the scheme theoretic intersection of any two distinct members of this pencil. From this we can deduce that any line in $\mathbb{P}^3$ can meet $C$ in at most two points. Also, if $Z \subset C$ is a closed sub-sceme supported on a finite set such that the vector space dimension $\ell(\mathcal{O}_Z)$ of $\mathcal{O}_Z$ is three, then the natural map

$$\mathrm{H}^0(L) \to \mathcal{O}_Z$$

is surjective. Hence we see that the morphism

$$\phi : S^2(C) \to \mathrm{Gr}(2, V)$$

is an imbedding.

If $C$ is a non degenerate non singular irreducible curve of degree four then $C$ is either a rational curve or an elliptic curve. This can be seen by projecting $C$ from a point on it to a plane not containing it. More over the curve $C$ is a rational curve if and only if the projected curve is singular curve. Hence the curve $C$ is rational if and only if there is line $\ell_0$ in $\mathbb{P}^3$ such that the length of the scheme $C \cap \ell_0$ is greater or equal to three. Since the scheme $C \cap \ell_0$ determines a sub scheme of length greater or equal to two on $S^2(C)$ we see that

$$\phi : S^2(C) \to \mathrm{Gr}(2, V)$$

is not an imbedding.

Let $C$ be a irreducible non-singular curve in $\mathbb{P}^3$ of degree greater or equal to five. Then by using Castalnov bound on the genus of a curve in terms of its degree ([4]) we see that there are lines in $\mathbb{P}^3$ which intersect $C$ in a sub scheme of length greater or equal to three. Hence in this case

$$\phi : S^2(C) \to \mathrm{Gr}(2, V)$$

is not an imbedding. This completes the proof of the theorem. □

**Remark 3.2.** *Let $C$ be a smooth curve in $\mathbb{P}^3$ of degree $d$ greater than or equal to five. Then by Using Reimann-Roch theorem we see that projection $h$ of $C$ from a point $p$ on $C$ to a $\mathbb{P}^2$ not containing $p$ is not an imbedding because*

$$h^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^3}(1)|_C(-p).$$

*¿From this we get through every point of $C \in \mathbb{P}^3$ there are lines which intersect $C$ in more than two points (counting with multiplicity).*

**Remark 3.3.** *On the Grassmaian $Gr(2,\mathbb{C}^4)$ one has universal exact sequence:*

$$0 \to S \to \mathbb{C}^4 \otimes \mathcal{O}_{Gr(2,\mathbb{C}^4)} \to Q \to 0, \tag{4}$$

*where $S$ and $Q$ are respectively the universal sub bundle and quotient bundle of rank two on $Gr(2,\mathbb{C}^4)$. The fiber of $Q$ (resp. of $S$) at a point $p \in Gr(2,\mathbb{C}^4)$ is the two dimensional quotient space (resp. subspace, which is the kernal of this quotient map) of $\mathbb{C}^4$ corresponding to the point $p$. It is known (see, page 197 and 411 of* [4]*) that the cohomology group $H^4(Gr(2,\mathbb{C}^4),\mathbb{Z})$ is equal to*

$$\mathbb{Z}[c_2(Q)] \oplus \mathbb{Z}[c_2(S)], \tag{5}$$

*where $c_2(Q)$ (resp. $c_2(S)$) is the second Chern class of $Q$ (resp. of $S$). The morphism $\phi : S^2(C) \to Gr(2,\mathbb{C}^4)$ determines a cohomology class in $H^4(Gr(2,\mathbb{C}^4),\mathbb{Z})$ namely $\phi_*([S^2(C)])$. By the identification ( (5)) the cohomology class $\phi_*([S^2(C)])$ is represented by a pair of integers $(a,b)$. Using the geometry of $Gr(2,\mathbb{C}^4)$ we see that $a$ is the number secants lines to $C$ which passes through a generic point of $\mathbb{P}^3$ and $b$ is the number of secant lines to $C$ which are contained in generic plane of $\mathbb{P}^3$ (see* [4]*). From this description we see that if $C$ is a rational normal curve of degree three than the cohomology class $\phi_*([S^2(C)])$ is $(1,3)$ and if $C$ is an elliptic curve of degree four than the cohomology class $\phi_*([S^2(C)])$ is $(2,6)$.*

## 4. Properties of Secant bundle

Let $C, L$ and $V$ be as in previous section. When $C$ is a rational normal curve in $\mathbb{P}^3$ from the results obtained by Schwarzenberger in ([2]) it follows that restriction of $E_L$ is semi-stable for all lines in $S^2(C)$ which not tangent to the image of the diagonal of $C \times C$ and not semi-stable to those lines which are tangent to the image of the diagonal of $C \times C$. For a general curve $C$ we have the following:

**Theorem 4.1.** *Let $C$ be a non-singular curve of degree $d \geq 4$ in $\mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(1)|_C$. Then $E_L|_{x \times C}$ is not semi-stable for every $x \in C$.*

**Proof:** If $D = p + q$ be a point on the surface $S^2(C)$ then the fiber $E_L|_D$ of the vector bundle $E_L$ at $D$ can be identified with $L \otimes \mathcal{O}_D$, where $\mathcal{O}_D$ is the structure sheaf sub-scheme of $C$ of length two corresponding to $D$. Since every point of the curve $p \times C$ is of the form $p \times q$ for some $q \in C$ by looking at the linear system $V(-p) = \{s \in V | s(p) = 0$ we see that the there is a injective homomorphism

$$0 \to L(-p) \to E_L|_{p \times C}.$$

But by (2.1) we see that $C_1(E_L|_{p\times C}) = d-1$, where $d$ is the degree of the line bundle $L$ on $C$. Hence there there exists a line bundle $M$ of degree 0 on $C$ such that $E_L|_{p\times C}$ fits in an exact sequence

$$0 \to L(-p) \to E_L|_{p\times C} \to M \to 0.$$

This proves that $E_L|_{p\times C}$ is not semi-stable as degree of $M$ is less than half the degree of $E_L|_{p\times C}$.

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