



# Compact composition operators on the Dirichlet space and capacity of sets of contact points

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza

► **To cite this version:**

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza. Compact composition operators on the Dirichlet space and capacity of sets of contact points. 2012.

**HAL Id: hal-00714673**

**<https://hal-univ-artois.archives-ouvertes.fr/hal-00714673>**

Submitted on 5 Jul 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Compact composition operators on the Dirichlet space and capacity of sets of contact points

Pascal Lefèvre, Daniel Li, Hervé Queffélec,  
Luis Rodríguez-Piazza\*

July 5, 2012

**Abstract.** We prove that for every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap} K = 0$ , there exists a Schur function  $\varphi$  both in the disk algebra  $A(\mathbb{D})$  and in the Dirichlet space  $\mathcal{D}_*$  such that the composition operator  $C_\varphi$  is in all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ , and for which  $K = \{e^{it}; |\varphi(e^{it})| = 1\} = \{e^{it}; \varphi(e^{it}) = 1\}$ . We show that for every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  and every  $\xi \in \partial\mathbb{D}$ , the logarithmic capacity of  $\{e^{it}; \varphi(e^{it}) = \xi\}$  is 0. We show that every compact composition operator  $C_\varphi$  on  $\mathcal{D}_*$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  and on the Hardy-Orlicz space  $H^{\Psi_2}$ ; in particular,  $C_\varphi$  is in every Schatten class  $S_p$ ,  $p > 0$ , both on the Hardy space  $H^2$  and on the Bergman space  $\mathfrak{B}^2$ . On the other hand, there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , but which is not even bounded on  $\mathcal{D}_*$ . We prove that for every  $p > 0$ , there exists a symbol  $\varphi$  such that  $C_\varphi \in S_p(\mathcal{D}_*)$ , but  $C_\varphi \notin S_q(\mathcal{D}_*)$  for any  $q < p$ , that there exists another symbol  $\varphi$  such that  $C_\varphi \in S_q(\mathcal{D}_*)$  for every  $q < p$ , but  $C_\varphi \notin S_p(\mathcal{D}_*)$ . Also, there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but in no Schatten class  $S_p(\mathcal{D}_*)$ .

**MSC 2010.** Primary: 47B33 – Secondary: 28A12; 30C85; 31A15; 46E20; 46E22; 47B10.

**Key-words.** Bergman space – Bergman-Orlicz space – composition operator – Dirichlet space – Hardy space – Hardy-Orlicz space – logarithmic capacity – Schatten classes

## 1 Introduction, notation and background

### 1.1 Introduction

Recall that a Schur function is an analytic self-map of the open unit disk  $\mathbb{D}$ . Every Schur function  $\varphi$  generates a bounded composition operator  $C_\varphi$  on the

---

\*Supported by a Spanish research project MTM 2009-08934.

Hardy space  $H^2$ , given by  $C_\varphi(f) = f \circ \varphi$ . Let us also introduce the set  $E_\varphi$  of contact points of the symbol with the unit circle (equipped with its normalized Haar measure  $m$ ), namely:

$$(1.1) \quad E_\varphi = \{e^{it}; |\varphi^*(e^{it})| = 1\}.$$

In terms of  $E_\varphi$ , a well-known necessary condition for compactness of  $C_\varphi$  on  $H^2$  is that  $m(E_\varphi) = 0$ . This set  $E_\varphi$  is otherwise more or less arbitrary. Indeed, it was proved in [7] that there exist compact composition operators  $C_\varphi$  on  $H^2$  such that the Hausdorff dimension of  $E_\varphi$  is 1. This was generalized in [5]: for every Lebesgue-negligible compact set  $K$  of the unit circle  $\mathbb{T}$ , there is a Hilbert-Schmidt composition operator  $C_\varphi$  on  $H^2$  such that  $E_\varphi = K$ , and in [18]:

**Theorem 1.1 ([18])** *For every Lebesgue-negligible compact set  $K$  of the unit-circle  $\mathbb{T}$  and every vanishing sequence  $(\varepsilon_n)$  of positive numbers, there is a composition operator  $C_\varphi$  on  $H^2$  such that  $E_\varphi = K$  and such that its approximation numbers satisfy  $a_n(C_\varphi) \leq C e^{-n\varepsilon_n}$ .*

We are interested here in a different Hilbert space of analytic functions, on which not every Schur function defines a bounded composition operator, namely the Dirichlet space  $\mathcal{D}$ . Recall its definition: the Dirichlet space  $\mathcal{D}$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$(1.2) \quad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , one has:

$$(1.3) \quad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.$$

Then  $\|\cdot\|_{\mathcal{D}}$  is a norm on  $\mathcal{D}$ , making  $\mathcal{D}$  a Hilbert space. Whereas every Schur function  $\varphi$  generates a bounded composition operator  $C_\varphi$  on the Hardy space  $H^2$ , it is no longer the case for the Dirichlet space (see [21], Proposition 3.12, for instance).

In [6], the study of compact composition operators on the Dirichlet space  $\mathcal{D}$  associated with a Schur function  $\varphi$  in connection with the set  $E_\varphi$  was initiated. In particular, it is proved there that if the composition operator  $C_\varphi$  is Hilbert-Schmidt on  $\mathcal{D}$ , then the logarithmic capacity  $\text{Cap } E_\varphi$  of  $E_\varphi$  is 0, but, on the other hand, there are compact composition operators on  $\mathcal{D}$  for which this capacity is positive. The optimality of this theorem was later proved in [5] under the following form:

**Theorem 1.2 (O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi)**  
*For every compact set  $K$  of the unit circle  $\mathbb{T}$  with logarithmic capacity  $\text{Cap } K$  equal to 0, there exists a Hilbert-Schmidt composition operator  $C_\varphi$  on  $\mathcal{D}$  such that  $E_\varphi = K$ .*

In this paper, we shall improve on this last result. We prove in Section 4 (Theorem 4.1) that for every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_*$  such that the composition operator  $C_\varphi$  is in all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ , and for which  $E_\varphi = K$  (and moreover  $E_\varphi = \{e^{it}; \varphi(e^{it}) = 1\}$ ). On the other hand, in Section 2, we show (Theorem 2.1) that for every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  and every  $\xi \in \partial\mathbb{D}$ , the logarithmic capacity of  $E_\varphi(\xi) = \{e^{it}; \varphi(e^{it}) = \xi\}$  is 0.

In link with Hardy and Bergman spaces, we prove, in Section 2 yet, that every compact composition operator  $C_\varphi$  on  $\mathcal{D}_*$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  and on the Hardy-Orlicz space  $H^{\Psi_2}$ . In particular,  $C_\varphi$  is in every Schatten class  $S_p$ ,  $p > 0$ , both on the Hardy space  $H^2$  and on the Bergman space  $\mathfrak{B}^2$  (Theorem 2.5). However, there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , but which is not even bounded on  $\mathcal{D}_*$  (Theorem 2.6).

In Section 3, we give a characterization of the membership of composition operators in the Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$  (actually in  $S_p(\mathcal{D}_{\alpha,*})$ , where  $\mathcal{D}_{\alpha,*}$  is the weighted Dirichlet space). We deduce that for every  $p > 0$ , there exists a symbol  $\varphi$  such that  $C_\varphi \in S_p(\mathcal{D}_*)$ , but  $C_\varphi \notin S_q(\mathcal{D}_*)$  for any  $q < p$ , and that there exists another symbol  $\varphi$  such that  $C_\varphi \in S_q(\mathcal{D}_*)$  for every  $q < p$ , but  $C_\varphi \notin S_p(\mathcal{D}_*)$  (Theorem 3.3). We also show that there exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but in no Schatten class  $S_p(\mathcal{D}_*)$  (Theorem 3.4).

## 1.2 Notation and background.

We denote by  $\mathbb{D}$  the unit open disk of the complex plane and by  $\mathbb{T} = \partial\mathbb{D}$  the unit circle.  $A$  is the normalized area measure  $dx dy/\pi$  of  $\mathbb{D}$  and  $m$  the normalized Lebesgue measure  $dt/2\pi$  on  $\mathbb{T}$ .

As said before, a Schur function is an analytic self-map of  $\mathbb{D}$  and the associated composition operator is defined, formally, by  $C_\varphi(f) = f \circ \varphi$ . The function  $\varphi$  is called the symbol of  $C_\varphi$ .

The Dirichlet space  $\mathcal{D}$  is defined above. We shall actually work, for convenience, with its subspace  $\mathcal{D}_*$  of functions  $f \in \mathcal{D}$  such that  $f(0) = 0$ . In this paper, we call  $\mathcal{D}_*$  the *Dirichlet space*.

An orthonormal basis of  $\mathcal{D}_*$  is formed by  $e_n(z) = z^n/\sqrt{n}$ ,  $n \geq 1$ . The reproducing kernel on  $\mathcal{D}_*$ , defined by  $f(a) = \langle f, K_a \rangle$  for every  $f \in \mathcal{D}_*$ , is given by  $K_a(z) = \sum_{n=1}^{\infty} \overline{e_n(a)} e_n(z)$ , so that:

$$(1.4) \quad K_a(z) = \log \frac{1}{1 - \bar{a}z}.$$

Compactness of composition operators on  $\mathcal{D}$  was characterized in terms of Carleson measure by D. Stegenga ([24]) and by B. McCluer and J. Shapiro in terms of angular derivative ([21]). Another characterization, more useful for us here, was given by N. Zorboska ([29], page 2020): for  $\varphi \in \mathcal{D}$ ,  $C_\varphi$  is bounded on  $\mathcal{D}$  if and only:

$$(1.5) \quad \sup_{h \in (0,2)} \sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(w) dA(w) < \infty,$$

where  $W(\xi, h) = \{w \in \mathbb{D}; 1 - |w| \leq h \text{ and } |\arg(w\bar{\xi})| \leq \pi h\}$  is the Carleson window of size  $h \in (0, 2)$  center at  $\xi \in \mathbb{T}$  and  $n_\varphi$  is the counting function of  $\varphi$ :

$$(1.6) \quad n_\varphi(w) = \sum_{\varphi(z)=w} 1, \quad w \in \varphi(\mathbb{D}),$$

(we set  $n_\varphi(w) = 0$  for  $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$ ). In particular, every Schur function with bounded valence defines a bounded composition operator on  $\mathcal{D}$ .

Moreover,  $C_\varphi$  is compact if and only if:

$$(1.7) \quad \sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(w) dA(w) \xrightarrow{h \rightarrow 0} 0.$$

For further informations on the Dirichlet space, one may consult the two surveys [1] and [23], for example.

### 1.2.1 Logarithmic capacity

The notion of logarithmic capacity is tied to the study of the Dirichlet space by the following seminal and sharp result of Beurling ([2]; see also [9]).

**Theorem 1.3 (Beurling)** *For every function  $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{D}$ , there exists a set  $E \subseteq \partial\mathbb{D}$ , with logarithmic capacity 0, such that, if  $t \in \mathbb{T} \setminus E$ , then the radial limit  $f^*(e^{it}) := \lim_{r \rightarrow 1^-} f(re^{it})$  exists (in  $\mathbb{C}$ ). Moreover, the result is optimal: if a compact set  $E \subseteq \mathbb{T}$  has zero logarithmic capacity, there exists  $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{D}$  such that  $f^*(e^{it})$  does not exist on  $E$ .*

Let us recall some definitions (see [9], Chapitre III, [4], Chapter 21, § 7, or [23], Section 4, for example).

Let  $\mu$  be a probability measure supported by a compact subset  $K$  of  $\mathbb{T}$ . The potential  $U_\mu$  of  $\mu$  is defined, for every  $z \in \mathbb{C}$ , by:

$$U_\mu(z) = \int_K \log \frac{e}{|z - w|} d\mu(w).$$

The energy  $I_\mu$  of  $\mu$  is defined by:

$$I_\mu = \int_K U_\mu(z) d\mu(z) = \iint_{K \times K} \log \frac{e}{|z - w|} d\mu(w) d\mu(z).$$

The logarithmic capacity of a Borel set  $E \subseteq \mathbb{T}$  is:

$$\text{Cap } E = \sup_{\mu} e^{-I_\mu},$$

where the supremum is over all Borel probability measures  $\mu$  with compact support contained in  $E$ . Hence  $E$  is of logarithmic capacity 0 (which is the case we are interested in) if and only if  $I_\mu = \infty$  for all probability measures compactly carried by  $E$ . The fact that  $\text{Cap } E = 0$  implies that  $E$  has null Lebesgue

measure ([9], Chapitre III, Théorème I) (hence  $\text{Cap } E > 0$  if  $E$  is a non-void open subset of  $\mathbb{T}$ ), but the converse is wrong, as shown by Cantor's middle-third set  $\mathfrak{C}$ . A compact set  $K$  such that  $\text{Cap } K = 0$  is totally disconnected ([4], Corollary 21.7.7).

If  $E$  is a compact set with  $\text{Cap } E > 0$ , there is a unique probability measure compactly carried by  $E$  that minimizes the energy  $I_\mu$  ([4], Theorem 21.10.2, or [9], Chapitre III, Proposition 4). Such a measure is called the *equilibrium measure* of  $E$ .

If  $\mu$  is the equilibrium measure of the compact set  $K$ , we have Frostman's Theorem ([4], Theorem 21.7.12, or [9], Chapitre III, Proposition 5 and Proposition 6):  $U_\mu(z) \leq I_\mu$  for every  $z \in \mathbb{C}$  and

$$(1.8) \quad U_\mu(z) = I_\mu \quad \text{for almost all } z \in K.$$

Suppose that the compact set  $K$  has zero logarithmic capacity. For  $\varepsilon > 0$ , let  $K_\varepsilon = \{z \in \mathbb{T}; \text{dist}(z, K) \leq \varepsilon\}$ ,  $\mu_\varepsilon$  its equilibrium measure, and  $I_{\mu_\varepsilon}$  its energy. Then ([4], Proposition 21.7.15):

$$(1.9) \quad \lim_{\varepsilon \rightarrow 0} I_{\mu_\varepsilon} = \infty.$$

## 2 Bounded and compact composition operators

In [6], E. A. Gallardo-Gutiérrez and M. J. González showed that for every Hilbert-Schmidt composition operator  $C_\varphi$  on  $\mathcal{D}_*$ , the logarithmic capacity of the set  $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D}; |\varphi(e^{i\theta})| = 1\}$  is zero. On the other hand, they showed that there are compact composition operators on  $\mathcal{D}_*$  for which  $E_\varphi$  has positive logarithmic capacity. We shall see that if we replace  $|\varphi|$  by  $\varphi$  in the definition of  $E_\varphi$ , the result is very different.

**Theorem 2.1** *For every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  and every  $\xi \in \partial\mathbb{D}$ , the logarithmic capacity of  $E_\varphi(\xi) = \{e^{it}; \varphi(e^{it}) = \xi\}$  is 0.*

We first state the following characterization of Hilbert-Schmidt composition operators on  $\mathcal{D}_*$ . This result is stated in [6], but not entirely proved.

**Lemma 2.2** *Let  $\varphi \in \mathcal{D}_*$  be an analytic self-map of  $\mathbb{D}$ . Then  $C_\varphi$  is Hilbert-Schmidt on  $\mathcal{D}_*$  if and only if*

$$(2.1) \quad \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z) < \infty.$$

**Proof.** Let  $e_n(z) = z^n/\sqrt{n}$ ; then  $(e_n)_{n \geq 1}$  is an orthonormal basis of  $\mathcal{D}_*$  and

$$\sum_{n=1}^{\infty} \|C_\varphi(e_n)\|^2 = \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n} = \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} dA(z).$$

Hence (2.1) is satisfied if  $C_\varphi$  is Hilbert-Schmidt. To get the converse, we need to show that (2.1) implies that  $C_\varphi$  is bounded on  $\mathcal{D}_*$ . Let  $f \in \mathcal{D}_*$  and write  $f(z) = \sum_{n=1}^{\infty} c_n z^n$ . Then  $C_\varphi f = \sum_{n=1}^{\infty} c_n \varphi^n$  and

$$\begin{aligned} \|C_\varphi f\| &\leq \sum_{n=1}^{\infty} |c_n| \|\varphi^n\| \leq \left( \sum_{n=1}^{\infty} n |c_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n} \right)^{1/2} \\ &= \left( \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} dA(z) \right)^{1/2} \|f\|. \end{aligned}$$

Then (2.1) implies that  $C_\varphi$  is Hilbert-Schmidt.  $\square$

Now Theorem 2.1 will follow from the next proposition.

**Proposition 2.3** *There exists an analytic self-map  $\sigma$  of  $\mathbb{D}$ , belonging to  $\mathcal{D}_*$  and to the disk algebra  $A(\mathbb{D})$ , such that  $\sigma(1) = 1$  and  $|\sigma(\xi)| < 1$  for  $\xi \in \partial\mathbb{D} \setminus \{1\}$  and such that the associated composition operator  $C_\sigma$  is Hilbert-Schmidt on  $\mathcal{D}_*$ .*

Taking this proposition for granted for a while, we can prove the theorem.

**Proof of Theorem 2.1.** Making a rotation, we may, and do, assume that  $\xi = 1$ . Then, if  $\sigma$  is the map of Proposition 2.3,  $C_\varphi C_\sigma = C_{\sigma \circ \varphi}$  is Hilbert-Schmidt. By [6], the set  $E_{\sigma \circ \varphi}$  has zero logarithmic capacity. But  $\sigma$  has modulus 1 only at 1; hence  $e^{i\theta} \in E_{\sigma \circ \varphi}$  if and only if  $e^{i\theta} \in E_\varphi(1)$ .  $\square$

To prove Proposition 2.3, it will be convenient to use the following criteria, where  $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ .

**Lemma 2.4** *Let  $f \in \mathcal{D}$  such that  $\Re f \geq 1$ . Then if  $\sigma = \varphi_a \circ e^{-1/f}$ , where  $a = e^{-1/f(0)}$ , the composition operator  $C_\sigma$  is Hilbert-Schmidt on  $\mathcal{D}_*$ .*

**Proof.** Let  $\sigma_0 = e^{-1/f}$ . If  $u = \Re f$  and  $v = \Im f$ , one has:

$$|\sigma_0|^2 = \exp\left(-\frac{2u}{u^2+v^2}\right) \quad \text{and} \quad |\sigma_0'|^2 = \frac{u'^2+v'^2}{(u^2+v^2)^2} \exp\left(-\frac{2u}{u^2+v^2}\right).$$

Then  $|\sigma_0| < 1$  and so  $\sigma_0$  is a self-map of  $\mathbb{D}$ . Since  $u \geq 1 > 0$ , one has  $|\sigma_0'|^2 \leq (u'^2+v'^2)/(u^2+v^2)^2 \leq u'^2+v'^2 = |f'|^2$ ; hence  $\sigma_0 \in \mathcal{D}$ .

For  $0 \leq x \leq 2$ , one has  $1 - e^{-x} \geq x/4$ . Therefore, since  $u \geq 1$  implies  $2u/(u^2+v^2) \leq 2/u \leq 2$ , one has:

$$1 - |\sigma_0|^2 \geq \frac{u}{2(u^2+v^2)}.$$

It follows that:

$$\frac{|\sigma_0'|^2}{(1-|\sigma_0|^2)^2} \leq \frac{u'^2+v'^2}{(u^2+v^2)^2} \frac{4(u^2+v^2)^2}{u^2} \leq 4(u'^2+v'^2) = 4|f'|^2.$$

Since  $f \in \mathcal{D}$ ,  $|f'|^2$  has a finite integral and therefore (2.1) is satisfied. It follows that  $C_{\sigma_0}$  is Hilbert-Schmidt on  $\mathcal{D}$  and hence  $C_\sigma = C_{\sigma_0} \circ C_{\varphi_a}$  is Hilbert-Schmidt on  $\mathcal{D}_*$ , since  $\sigma(0) = 0$ .  $\square$

**Proof of Proposition 2.3.** Let  $\Omega$  be the domain defined by:

$$\Omega = \{z \in \mathbb{C}; \Re z > 1 \text{ and } |\Im z| < 1/(\Re z)^2\}.$$

Let  $f$  be a conformal map from  $\mathbb{D}$  onto  $\Omega$  such that  $f(1) = \infty$ . Since  $A(\Omega) < \infty$ , we have  $f \in \mathcal{D}$ . By Lemma 2.4, the function  $\sigma = e^{-1/f}$  has the required properties.  $\square$

For the next result, recall that an Orlicz function  $\Psi$  is a nondecreasing convex function such that  $\Psi(0) = 0$  and  $\Psi(x)/x \rightarrow \infty$  as  $x$  goes to infinity. We refer to [12] for the definition of Hardy-Orlicz and Bergman-Orlicz spaces. In the following result, one set  $\Psi_2(x) = \exp(x^2) - 1$ .

**Theorem 2.5** *Every compact composition operator  $C_\varphi$  on  $\mathcal{D}_*$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  and on the Hardy-Orlicz space  $H^{\Psi_2}$ . In particular,  $C_\varphi$  is in every Schatten class  $S_p$ ,  $p > 0$ , both on the Hardy space  $H^2$  and on the Bergman space  $\mathfrak{B}^2$ .*

**Proof.** Consider the normalized reproducing kernels  $\tilde{K}_a = K_a/\|K_a\|$ ,  $a \in \mathbb{D}$ . When  $|a|$  goes to 1, they tends to 0 uniformly on compact sets of  $\mathbb{D}$ ; hence  $\|C_\varphi^*(\tilde{K}_a)\|$  tends to 0, by compactness of the adjoint operator  $C_\varphi^*$ . But  $C_\varphi^*(K_a) = K_{\varphi(a)}$  and  $\|K_a\|^2 = \langle K_a, K_a \rangle = \log \frac{1}{1-|a|^2}$ , so we get:

$$(2.2) \quad \lim_{|a| \rightarrow 1} \frac{\log \frac{1}{1-|\varphi(a)|^2}}{\log \frac{1}{1-|a|^2}} = 0.$$

This condition means that  $C_\varphi$  is compact on the Bergman-Orlicz space  $\mathfrak{B}^{\Psi_2}$  ([12], page 69) and implies that  $C_\varphi$  is in all Schatten classes  $S_p(\mathfrak{B}^2)$ ,  $p > 0$  ([15]).

In the same way, it suffices to show that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , because that implies that  $C_\varphi$  is in all Schatten classes  $S_p(H^2)$  ([11], Theorem 5.2).

Compactness of  $C_\varphi$  on  $H^\Psi$  is equivalent to say ([12], Theorem 4.18) that:

$$\begin{aligned} \rho_\varphi(h) &:= \sup_{|\xi|=1} m(\{e^{it}; \varphi(e^{it}) \in W(\xi, h)\}) \\ &= o_{h \rightarrow 0} \left[ \frac{1}{\Psi(A\Psi^{-1}(1/h))} \right] \quad \text{for every } A > 0. \end{aligned}$$

When  $\Psi = \Psi_2$ , this means that  $\rho_\varphi(h) = o(h^A)$  for every  $A > 0$ . Now, by [14], Theorem 4.2, this is also equivalent to say that:

$$(2.3) \quad \sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} N_\varphi(w) dA(w) = o(h^A) \quad \text{for every } A > 0,$$



where  $N_\varphi$  is the Nevanlinna counting function of  $\varphi$ :

$$(2.4) \quad N_\varphi(w) = \sum_{\varphi(z)=w} (1 - |z|^2), \quad w \in \varphi(\mathbb{D}),$$

and  $N_\varphi(w) = 0$  otherwise.

But (2.2) is equivalent to the fact that for every  $\varepsilon > 0$  there exists  $\delta_\varepsilon > 0$  such that:

$$(2.5) \quad 1 - |\varphi(z)| \geq \delta_\varepsilon (1 - |z|)^\varepsilon, \quad \forall z \in \mathbb{D}.$$

Since  $\varphi(0) = 0$ , we have  $|\varphi(z)| \leq |z|$ , by Schwarz's lemma; hence one has  $N_\varphi(w) \leq 2\delta_\varepsilon^{-1}(1 - |w|)^{1/\varepsilon} n_\varphi(w)$ . It follows that (since  $1 - |w| \leq h$  for  $w \in W(\xi, h)$ ):

$$\frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} N_\varphi(w) dA(w) \leq 2\delta_\varepsilon^{-1} h^{1/\varepsilon} \frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(w) dA(w),$$

which is  $o(h^{1/\varepsilon})$ , uniformly for  $|\xi| = 1$ , by (1.7).  $\square$

**Remarks.** 1) One may argue that compactness of  $C_\varphi$  on  $H^{\Psi_2}$  implies its compactness on  $\mathfrak{B}^{\Psi_2}$  ([15], Proposition 4.1, or [17], Theorem 9). One may also use the forthcoming Corollary 3.2 saying that  $C_\varphi \in S_p(H^2)$  implies that  $C_\varphi \in S_p(\mathfrak{B}^2)$ .

2) To show the compactness of  $C_\varphi$  on  $H^{\Psi_2}$ , we used its compactness on  $\mathcal{D}_*$  twice. However, due to the fact that  $\varepsilon > 0$  is arbitrary, we may replace  $o(h^{1/\varepsilon})$  by  $O(h^{1/\varepsilon})$ ; hence to end the proof, we only have to use (1.5), *i.e.* the boundedness of  $C_\varphi$  on  $\mathcal{D}_*$ , instead of (1.7).

Note that (2.2) does not suffice to have compactness on  $H^{\Psi_2}$  (in [12], Proposition 5.5, we construct a Blaschke product satisfying (2.2)).

In the opposite direction, we have the following result.

**Theorem 2.6** *There exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $H^{\Psi_2}$ , but which is not even bounded on  $\mathcal{D}_*$ .*

To prove this theorem, we first begin with the following key lemma.

**Lemma 2.7** *There exists a constant  $\kappa_1 > 0$  such that for any  $f \in \mathcal{H}(\mathbb{D})$  having radial limits  $f^*$  a.e. and which satisfies, for some  $\alpha \in \mathbb{R}$ :*

$$(2.6) \quad \begin{cases} \Im f(0) < \alpha & \text{and} \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C}; 0 < \Re z < \pi\} \cup \{z \in \mathbb{C}; \Im z < \alpha\}, \end{cases}$$

*we have, for all  $y \geq \alpha$ :*

$$m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\}) \leq \kappa_1 e^{\alpha - y}.$$

**Proof.** Suppose that  $f$  satisfies (2.6), and define  $f_1(z) = -if(z) + \frac{\pi}{2}i - \alpha$ . Then either  $\Re[f_1(z)] < 0$ , or  $-\frac{\pi}{2} < \Im[f_1(z)] < \frac{\pi}{2}$  for every  $z \in \mathbb{D}$ . Therefore, defining  $h(z) = 1 + \exp[f_1(z)]$ , we have  $h: \mathbb{D} \rightarrow \mathbb{H}$ , that is  $\Re[h(z)] > 0$  for every  $z \in \mathbb{D}$ .

Finally define  $h_1(z) = h(z) - i\Im[h(0)]$ . Then  $h_1: \mathbb{D} \rightarrow \mathbb{H}$  and  $h_1(0) \in \mathbb{R}$  (and so  $h_1(0) > 0$ ). Kolmogorov's inequality yields that, for some absolute constant  $C_1$ , one has, for every  $\lambda > 0$ :

$$(2.7) \quad m(\{z \in \mathbb{T}; |h_1^*(z)| \geq \lambda\}) \leq C_1 \frac{h_1(0)}{\lambda}.$$

Observe that, since  $\Im[f(0)] < \alpha$ , we have  $\Re[f_1(0)] < 0$ , and then:

$$(2.8) \quad |\Im[h(0)]| < 1 \quad \text{and} \quad h_1(0) = \Re[h(0)] < 2.$$

Suppose now that, for  $y > \alpha$  and  $z \in \mathbb{D}$ , we have  $\Im[f(z)] > y$ ; then  $\exp[f_1(z)] \in \mathbb{H}$ , and  $|h(z)| \geq |\exp[f_1(z)]| > e^{y-\alpha}$ . Taking radial limits we get, up to a set of null Lebesgue-measure:

$$\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\} \subseteq \{z \in \mathbb{T}; |h^*(z)| \geq e^{y-\alpha}\}.$$

We consider two cases:  $e^{y-\alpha} \geq 2$  and  $e^{y-\alpha} < 2$ . When  $e^{y-\alpha} \geq 2$ , then  $|h^*(z)| \geq e^{y-\alpha}$  yields:

$$|h_1^*(z)| \geq e^{y-\alpha} - |\Im[h(0)]| > e^{y-\alpha} - 1 \geq \frac{1}{2}e^{y-\alpha},$$

by the first part of (2.8). Then, using (2.7) and the second part of (2.8), we have:

$$\begin{aligned} m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\}) &\leq m(\{z \in \mathbb{T}; |h_1^*(z)| > (1/2)e^{y-\alpha}\}) \\ &\leq \frac{2C_1 h_1(0)}{e^{y-\alpha}} \leq \frac{4C_1}{e^{y-\alpha}}, \end{aligned}$$

and, in this case, the lemma is proved, if one takes  $\kappa_1 \geq 4C_1$ .

When  $e^{y-\alpha} < 2$ , then  $e^{\alpha-y} > 1/2$ , and, because:

$$m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq y\}) \leq 1 < \kappa_1 e^{\alpha-y},$$

since  $\kappa_1 > 2$ , the lemma is proved.  $\square$

Now, we give a general construction of Schur functions with suitable properties.

**Proposition 2.8** *Let  $\mathbf{g}: (0, \infty) \rightarrow (0, \infty)$  be a continuous non-increasing function such that:*

$$\lim_{t \rightarrow 0^+} \mathbf{g}(t) = +\infty, \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathbf{g}(t) = 0.$$

Let  $\mathfrak{h}: (0, \infty) \rightarrow (0, \infty]$  be a lower semicontinuous function such that  $M := \sup\{\mathfrak{h}(t); t \geq \pi\} < +\infty$  and consider the simply connected domain:

$$\Omega = \{x + iy; x \in (0, \infty) \text{ and } \mathfrak{g}(x) < y < \mathfrak{g}(x) + \mathfrak{h}(x)\}.$$

Let  $\mathfrak{f}: \overline{\mathbb{D}} \rightarrow \overline{\Omega} \cup \{\infty\}$  be a conformal mapping from  $\mathbb{D}$  onto  $\Omega$  such that  $\mathfrak{f}(0) = \pi + i(\mathfrak{g}(\pi) + \mathfrak{h}(\pi)/2)$ .

Then the symbol  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  defined by  $\varphi(z) = \exp[-\mathfrak{f}(z)]$ , for every  $z \in \mathbb{D}$ , satisfies, for some  $\varepsilon_0, k_0 > 0$ :

1) For all  $h \in (0, \varepsilon_0)$ :

$$(2.9) \quad m(\{z \in \mathbb{T}; |\varphi^*(z)| > 1 - h\}) \leq k_0 \exp(-\mathfrak{g}(2h)).$$

2) Assume that, for some  $r \in (0, \infty]$  and integers  $0 \leq n < N \leq \infty$ , one has  $\{\mathfrak{h}(t); t \leq r\} \subseteq (2n\pi, 2N\pi]$ . Then, for all  $z \in \mathbb{D}$ , such that  $|z| > e^{-r}$ , we have  $n \leq n_\varphi(z) \leq N$ .

In particular,  $\{z \in \mathbb{D}; |z| > e^{-r}\} \subseteq \varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus \{0\}$ , when  $n \geq 1$ .

#### Remarks.

1. When  $N = 1$ , the map  $\varphi$  is univalent.
2. When  $r = \infty$  and  $n \geq 1$ , we have  $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$ .
3. With  $\mathfrak{g}(t) = 1/t$ , the operator  $C_\varphi$  is compact on  $H^{\Psi_2}$ , therefore belongs to all Schatten classes  $S_p(H^2)$ ,  $p > 0$ .
4. When  $N < \infty$ , the operator  $C_\varphi$  is bounded on the Dirichlet space.
5. When  $n \geq 1$ , the operator  $C_\varphi$  is not compact on the Dirichlet space (since the averages on the windows of the function  $n_\varphi$  cannot uniformly vanish).

**Proof of Proposition 2.8.** We shall apply Lemma 2.7 with  $\alpha = M + \mathfrak{g}(\pi)$ .

Suppose that, for  $z \in \mathbb{T}$  and  $0 < h < 1$ , we have  $|\varphi^*(z)| > 1 - h$ . Then, if  $h$  is small enough,

$$e^{-2h} < 1 - h < |\varphi^*(z)| = \exp(-\Re[f^*(z)]),$$

and therefore  $2h > \Re[f^*(z)]$ . But observe that  $\mathfrak{f}^*(z) \in \overline{\Omega} \cup \{\infty\}$ , and so, if  $2h > \Re[f^*(z)]$ , we necessarily have  $\Im[f^*(z)] \geq \mathfrak{g}(2h)$ . Again, if  $h$  is small enough, we have  $y = \mathfrak{g}(2h) > \alpha$ , and may apply the lemma to obtain:

$$m(\{z \in \mathbb{T}; |\varphi^*(z)| > 1 - h\}) \leq m(\{z \in \mathbb{T}; \Im[f^*(z)] \geq \mathfrak{g}(2h)\}) \leq \kappa_1 e^{\alpha - \mathfrak{g}(2h)}.$$

We get (2.9).

On the other hand, let  $Z \in \mathbb{D}$  such that  $|Z| > e^{-r}$ , we can write  $Z = e^{-x} e^{i\theta}$  with  $x < r$ . We can find  $\theta_j$ 's such that  $\mathfrak{g}(x) < \theta_1 < \dots < \theta_s < \mathfrak{g}(x) + \mathfrak{h}(x)$  and  $\theta_j \equiv \theta[2\pi]$  with  $n \leq s \leq N$ . For each  $j$ , there exists a unique  $z_j \in \mathbb{D}$ , such that  $\Re \mathfrak{f}(z_j) = x$  and  $\Im \mathfrak{f}(z_j) = \theta_j$ ; hence  $\varphi(z_j) = Z$ . Moreover no other  $z \in \mathbb{D}$  can satisfy  $\varphi(z) = Z$ . Hence  $n_\varphi(Z) = s$ .  $\square$

**Proof of Theorem 2.6.** As said before, if one takes  $\mathbf{g}(t) = 1/t$  in Proposition 2.8, then  $C_\varphi$  is compact on  $H^{\Psi_2}$  and hence is in all Schatten classes  $S_p(H^2)$ ,  $p > 0$ . On the other hand, if one choose also  $\mathbf{h}(t) = 1/t$ , then, for every  $r > 0$ ,  $\{\mathbf{h}(t); t \leq r\} = [1/r, \infty)$  and for  $|z| > e^{-r}$ , we get that  $n_\varphi(z) \geq [1/(2\pi r)]$  (the integer part of  $1/(2\pi r)$ ). It follows that, for some constant  $c > 0$ , one has, with  $e^{-r} = 1 - h$ :

$$\frac{1}{A[W(\xi, h)]} \int_{W(\xi, h)} n_\varphi(z) dA(z) \geq c \frac{1}{\log[1/(1-h)]} \xrightarrow{h \rightarrow 0} \infty.$$

Therefore,  $C_\varphi$  is not bounded on  $\mathcal{D}_*$ , by (1.5).  $\square$

**Remarks.** 1. Actually, as we may take  $\mathbf{g}$  growing as we wish, the proof shows, using [12], Theorem 4.18, that for every Orlicz function  $\Psi$ , one can find a Schur function  $\varphi$  such that  $C_\varphi$  is not bounded on  $\mathcal{D}_*$ , though compact on the Hardy-Orlicz space  $H^\Psi$ .

2. This construction also allows to produce a *univalent* map  $\varphi$ , with an arbitrary small Carleson function  $\rho_\varphi(h) = \sup_{|\xi|=1} m(\{e^{it}; \varphi^*(e^{it}) \in W(\xi, h)\})$ , and such that  $C_\varphi$  is not compact on the Dirichlet space (note we cannot replace “compact” by “bounded” since any Schur function with a bounded valence is bounded on the Dirichlet space).

Indeed, take  $\mathbf{h}(t) = 2\pi$  and  $\mathbf{g}$  be  $\mathcal{C}^1$ :  $\mathbf{g}(t) = 1/t$  for instance. We have  $N = 1$  and so  $\varphi$  is univalent. Now it suffices to notice that the range of the curve

$$\Gamma = \{e^{-x-ig(x)}; x \in (0, \infty)\} = \{(t \cos(1/\ln(t)), t \sin(1/\ln(t))); t \in (0, 1)\} \subseteq \mathbb{D}$$

has a null area measure. The range of  $\varphi$  is  $\mathbb{D} \setminus (\Gamma \cup \{0\})$  and for each  $w \notin \Gamma$ , we have  $n_\varphi(w) = 1$ . Then, for  $h \in (0, 1)$ , we have:

$$\begin{aligned} \frac{1}{h^2} \int_{W(1, h)} n_\varphi(w) dA(w) &= \frac{1}{h^2} \int_{W(1, h) \setminus \Gamma} dA(w) = \frac{1}{h^2} A[W(1, h) \setminus \Gamma] \\ &= \frac{1}{h^2} A[W(1, h)] \approx 1, \end{aligned}$$

and so  $C_\varphi$  is not compact on  $\mathcal{D}_*$ , by (1.7).  $\square$

## 3 Composition operators in Schatten classes

### 3.1 Characterization

In this section, we give a characterization of the membership in the Schatten classes of composition operators on  $\mathcal{D}_*$ . This characterization will be deduced from Luecking’s one for composition operators on the Bergman space. Actually, we shall give it for weighted Dirichlet spaces  $\mathcal{D}_{\alpha, *}$ . Boundedness and compactness has been characterized by B. McCluer and J. Shapiro in [21] and, in other terms, by N. Zorboska in [29].

Recall that for  $\alpha > -1$ , the weighted Dirichlet space  $\mathcal{D}_\alpha$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$(3.1) \quad \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

This is a Hilbert space for the norm given by:

$$(3.2) \quad \|f\|_\alpha^2 = |f(0)|^2 + (\alpha + 1) \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

The standard Dirichlet space  $\mathcal{D}$  corresponds to  $\alpha = 0$ ; the Hardy space  $H^2$  to  $\alpha = 1$  and the standard Bergman space to  $\alpha = 2$ . For more general weights, see [10].

We denote by  $\mathcal{D}_{\alpha,*}$  the subspace of the  $f \in \mathcal{D}_\alpha$  such that  $f(0) = 0$ .

If  $\varphi$  is a Schur function, one defines its *weighted Nevanlinna counting function*  $N_{\varphi,\alpha}$  at  $w \in \Omega := \varphi(\mathbb{D})$  as the number of pre-images of  $w$  with the weight  $(1 - |z|)^\alpha$ :

$$(3.3) \quad N_{\varphi,\alpha}(w) = \sum_{\varphi(z)=w} (1 - |z|^2)^\alpha.$$

For  $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$ , we set  $N_{\varphi,\alpha}(w) = 0$ . One has  $N_{\varphi,1} = N_\varphi$  and  $N_{\varphi,0} = n_\varphi$ .

With this notation, recall the change of variable formula:

$$(3.4) \quad \int_{\mathbb{D}} F[\varphi(z)] |\varphi'(z)|^2 (1 - |z|^2)^\alpha dA(z) = \int_{\Omega} F(w) N_{\varphi,\alpha}(w) dA(w).$$

Denote by  $R_{n,j}$ ,  $n \geq 0$ ,  $0 \leq j \leq 2^n - 1$ , the Hastings-Luecking windows:

$$R_{n,j} = \left\{ z \in \mathbb{D}; 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \quad \text{and} \quad \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \right\}.$$

We can now state.

**Theorem 3.1** *Let  $\alpha > -1$ . Let  $\varphi$  be a Schur function and  $p > 0$ . Then  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  if and only if:*

$$(3.5) \quad \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) dA(w) \right]^{p/2} < \infty.$$

If  $\varphi$  is univalent, (3.5) can be replaced by the purely geometric condition:

$$(3.6) \quad \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} [2^{n(\alpha+2)} A_\alpha(R_{n,j} \cap \Omega)]^{p/2} < \infty,$$

where  $A_\alpha$  is the weighted measure  $dA_\alpha(w) = (\alpha + 1) (1 - |w|^2)^\alpha dA(w)$ .

**Remark.** Of course, every operator in a Schatten class is compact, but we may note that condition (3.5) implies the compactness of  $C_\varphi$ , by [29], Theorem 1 (and [13], Proposition 3.3).

**Proof of Theorem 3.1.** First, we compute  $C_\varphi^* C_\varphi$ . Let us fix  $f$  and  $g$  in the Dirichlet space  $\mathcal{D}_{\alpha,*}$ . We have:

$$\begin{aligned} (\alpha + 1) \int_{\mathbb{D}} ((C_\varphi^* C_\varphi)(f))'(z) \overline{g'(z)} (1 - |z|^2)^\alpha dA(z) &= \langle f \circ \varphi, g \circ \varphi \rangle_{\mathcal{D}_{\alpha,*}} \\ &= (\alpha + 1) \int_{\mathbb{D}} (f' \circ \varphi)(z) \overline{(g' \circ \varphi)(z)} |\varphi'(z)|^2 (1 - |z|^2)^\alpha dA(z). \end{aligned}$$

By the change of variable formula, we get:

$$\int_{\mathbb{D}} ((C_\varphi^* C_\varphi)(f))'(z) \overline{g'(z)} (1 - |z|^2)^\alpha dA = \int_{\mathbb{D}} f'(w) \overline{g'(w)} N_{\varphi,\alpha}(w) dA(w),$$

which is equivalent to:

$$\int_{\mathbb{D}} ((C_\varphi^* C_\varphi)(f))'(z) \overline{G(z)} (1 - |z|^2)^\alpha dA(z) = \int_{\mathbb{D}} f'(w) \overline{G(w)} N_{\varphi,\alpha}(w) dA(w)$$

for every function  $G$  belonging to the weighted Bergman space  $\mathfrak{B}_\alpha^2$ .

That means that  $((C_\varphi^* C_\varphi)(f))' - f' \cdot N_{\varphi,\alpha} / (1 - |w|^2)^\alpha$  is orthogonal to the weighted Bergman space  $\mathfrak{B}_\alpha^2$ . But  $((C_\varphi^* C_\varphi)(f))' \in \mathfrak{B}_\alpha^2$ . Hence  $((C_\varphi^* C_\varphi)(f))'$  is the orthogonal projection onto  $\mathfrak{B}_\alpha^2$  of the function  $f' \cdot N_{\varphi,\alpha} / (1 - |w|^2)^\alpha$ . Thus (see [27], § 6.4.1), we obtain that for every  $z \in \mathbb{D}$ :

$$\begin{aligned} ((C_\varphi^* C_\varphi)(f))'(z) &= (\alpha + 1) \int_{\mathbb{D}} \frac{f'(w)}{(1 - \bar{w}z)^{\alpha+2}} \frac{N_{\varphi,\alpha}(w)}{(1 - |w|^2)^\alpha} (1 - |w|^2)^\alpha dA(w) \\ &= (\alpha + 1) \int_{\mathbb{D}} \frac{f'(w)}{(1 - \bar{w}z)^{\alpha+2}} d\mu(w) \\ &= (\alpha + 1) T_\mu(f')(z), \end{aligned}$$

where  $\mu$  is the positive measure  $A$  with weight  $N_{\varphi,\alpha}$  and  $T_\mu$  is the Toeplitz operator on  $\mathfrak{B}_\alpha^2$  is introduced in [19] (let us point out that  $\alpha$  in [19] corresponds to  $-(\alpha + 1)$  in our work).

In other words, introducing the map  $\Delta(h) = h'$ , which is an isometry from  $\mathcal{D}_{\alpha,*}$  onto  $\mathfrak{B}_\alpha^2$ , we have  $\Delta \circ (C_\varphi^* C_\varphi) = T_\mu \circ \Delta$ . We have the following diagram:

$$\begin{array}{ccc} \mathcal{D}_{\alpha,*} & \xrightarrow{C_\varphi^* C_\varphi} & \mathcal{D}_{\alpha,*} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathfrak{B}_\alpha^2 & \xrightarrow{T_\mu} & \mathfrak{B}_\alpha^2 \end{array}$$

Hence the approximation numbers of  $T_\mu$  (viewed as an operator on  $\mathfrak{B}_\alpha^2$ ) and the ones of  $C_\varphi^* C_\varphi$  (viewed as an operator on  $\mathcal{D}_{\alpha,*}$ ) are the same. In particular,

the membership in the Schatten classes are the same and the final result follows from the main theorem in [19]:  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  if and only if  $C_\varphi^* C_\varphi \in S_{p/2}(\mathcal{D}_{\alpha,*})$  and that holds if and only if:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} [2^{n(\alpha+2)} \mu(R_{n,j})]^{p/2} < \infty.$$

Hence  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  if and only if:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) dA(w) \right]^{p/2} < \infty,$$

and that ends the proof of Theorem 3.1.  $\square$

**Remark.** In the same way, we can obtain other characterizations for  $\mathcal{D}_{\alpha,*}$  by using the ones for  $\mathfrak{B}_\alpha^2$  given in [20] and [28]:  $C_\varphi \in S_p(\mathfrak{B}_\alpha^2)$  if and only if  $N_{\varphi,\alpha+2}(z)/(\log(1/|z|))^{\alpha+2} \in L^{p/2}(\lambda)$ , where  $d\lambda(z) = (1 - |z|^2)^{-2} dA(z)$  is the Möbius invariant measure on  $\mathbb{D}$ , and, when  $\varphi$  has bounded valence and  $p \geq 2$ , if and only if  $(1 - |z|^2)/(1 - |\varphi(z)|^2) \in L^{p(\alpha+2)/2}(\lambda)$ . Such a result can be found in [26].

## 3.2 Applications

We give several applications of the previous theorem.

**Corollary 3.2** *Let  $-1 < \alpha \leq \beta$ ,  $p > 0$ , and  $\varphi$  be a Schur function. Then  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$  implies that  $C_\varphi \in S_p(\mathcal{D}_{\beta,*})$ .*

*In particular,  $C_\varphi \in S_p(\mathcal{D}_*)$  implies that  $C_\varphi \in S_p(H^2)$ , which in turn implies that  $C_\varphi \in S_p(\mathfrak{B}^2)$ .*

**Proof.** Assume that  $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ . Then

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) dA(w) \right]^{p/2} < \infty.$$

Since, thanks to Schwarz's lemma,  $N_{\varphi,\beta}(w) \leq N_{\varphi,\alpha}(w)(1 - |w|^2)^{\beta-\alpha}$ , we have

$$N_{\varphi,\beta}(w) \leq (2 \cdot 2^{-n})^{\beta-\alpha} N_{\varphi,\alpha}(w) \quad \text{for } w \in R_{n,j}.$$

It follows that

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left[ 2^{n(\beta+2)} \int_{R_{n,j}} N_{\varphi,\beta}(w) dA(w) \right]^{p/2} < \infty,$$

and that proves Corollary 3.2.  $\square$

It is known ([13]) that composition operators on  $H^2$  separate Schatten classes, but the difficulty is that we must not only control the shape of  $\varphi(\partial\mathbb{D})$ , but also the parametrization  $t \mapsto \varphi(e^{it})$ , even if  $\varphi$  is univalent. In the case of the Dirichlet space, this difficulty disappears, because only the areas come into play, and we can easily prove the following result.

**Theorem 3.3** *The composition operators on  $\mathcal{D}_*$  separate Schatten classes, in the following sense. Let  $0 < p_1 < \infty$ . Then, there exists a symbol  $\varphi$  such that:*

$$C_\varphi \in \left( \bigcap_{p > p_1} S_p(\mathcal{D}_*) \right) \setminus S_{p_1}(\mathcal{D}_*).$$

Similarly, there exists a symbol  $\varphi$  such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \left( \bigcup_{p < p_1} S_p(\mathcal{D}_*) \right).$$

In particular, for every  $0 < p_1 < p_2 < \infty$ , there exists  $\varphi$  such that  $C_\varphi \in S_{p_2}(\mathcal{D}_*) \setminus S_{p_1}(\mathcal{D}_*)$ .

**Proof.** Let  $(h_n)_{n \geq 1}$ , with  $0 < h_n < 1$ , be a sequence of real numbers with limit 0 to be adjusted, and  $J$  the Jordan curve formed by the segment  $[0, 1]$  and the north and (truncated) north-east sides of the curvilinear rectangles

$$\{1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\} \times \{0 \leq \arg z < 2^{-n} h_n\}.$$

Let  $\Omega_0$  be the interior of  $J$  and  $\Omega = \Omega_0 \cup D(0, 1/8)$ . Let  $\varphi: \mathbb{D} \rightarrow \Omega$  be a Riemann map such that  $\varphi(0) = 0$ . Since  $\varphi$  is univalent and bounded, it defines a symbol on  $\mathcal{D}_*$ , and the necessary and sufficient condition (3.6) for membership in  $S_p(\mathcal{D}_*)$  reads:

$$(3.7) \quad \sum_{n=0}^{\infty} [4^n 4^{-n} h_n]^{p/2} = \sum_{n=0}^{\infty} h_n^{p/2} < \infty.$$

Indeed, it is clear that, for fixed  $n$ , the Hastings-Luecking windows  $R_{n,j}$  satisfy:

$$R_{n,0} \cap \Omega \neq \emptyset; \quad R_{n,j} \cap \Omega = \emptyset \text{ for } 1 \leq j < 2^n.$$

Therefore, only the Hastings-Luecking windows  $R_{n,0}$  matter. Since:

$$A(R_{n,0} \cap \Omega) = \iint_{1-2^{-n} \leq r < 1-2^{-n-1}, 0 \leq \theta < 2^{-n} h_n} r \, dr \, d\theta \approx 4^{-n} h_n,$$

we can test the criterion (3.7). Now, it is enough to take  $h_n = (n+1)^{-2/p_1}$  to get:

$$C_\varphi \in \left( \bigcap_{p > p_1} S_p(\mathcal{D}_*) \right) \setminus S_{p_1}(\mathcal{D}_*).$$

Similarly, the choice  $h_n = (n+1)^{-2/p_1} [\log(n+2)]^{-4/p_1}$ , gives a symbol  $\varphi$  such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \left( \bigcup_{p < p_1} S_p(\mathcal{D}_*) \right).$$

This ends the proof. □



T. Carroll and C. Cowen ([3] proved, but only for  $\alpha > 0$ , that there exist compact composition operators on  $\mathcal{D}_\alpha$  which are in no Schatten class (see also [8]). In the next result, we shall see that this still true for  $\alpha = 0$ .

**Theorem 3.4** *There exists a Schur function  $\varphi$  such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but in no Schatten class  $S_p(\mathcal{D}_*)$ .*

**Proof.** It suffices to use the proof of Theorem 3.3 and to take, instead of the above  $h_n$ ,  $h_n = 1/\ln(n+2)$ .  $\square$

For the next application, which will be used in Section 4, we need to recall the definition of the cusp map  $\chi$ , introduced in [15], and later used, with a slightly different definition in [18]. Actually, we have to modify it slightly again in order to have  $\chi(0) = 0$ . We first define:

$$\chi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1},$$

then:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where  $a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)$  is chosen in order that  $\chi(0) = 0$ . The image  $\Omega$  of the (univalent) cusp map is formed by the intersection of the inside of the disk  $D(\frac{a}{2}, \frac{a}{2})$  and the outside of the two disks  $D(\frac{ia}{2}, \frac{a}{2})$  and  $D(-\frac{ia}{2}, \frac{a}{2})$ .

**Corollary 3.5** *If  $\chi$  is the cusp map, then  $C_\chi$  belongs to all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ .*

**Proof.** Since  $\chi$  is univalent,  $\chi(0) = 0$ , and  $\Omega = \chi(\mathbb{D})$  has finite area, we have  $\chi \in \mathcal{D}_*$ . A little elementary geometry shows that, for some constant  $C$ , we have:

$$(3.8) \quad w \in \Omega, \quad 0 < h < 1 \text{ and } |w| \geq 1 - h \quad \implies \quad |\Im w| \leq Ch^2.$$

It follows (changing  $C$  if necessary) that  $R_{n,j} \cap \Omega$  is contained in a rectangle of sizes  $2^{-n}$  and  $C4^{-n}$  and with area  $C8^{-n}$ . Hence, for a given  $n$ , at most  $C$  of the Hastings-Luecking windows  $R_{n,j}$  can intersect  $\Omega$ . Therefore, the series in Theorem 3.1 reduces, up to constants, to the series:

$$\sum_{n=0}^{\infty} (4^n 8^{-n})^{p/2} = \sum_{n=0}^{\infty} 2^{-np},$$

which converges for every  $p > 0$ .  $\square$

## 4 Logarithmic capacity and set of contact points

In view of the result of [6] mentioned in the introduction, if  $\text{Cap } K > 0$ , there is no hope to find a symbol  $\varphi$  such that  $E_\varphi = K$  and  $C_\varphi$  is Hilbert-Schmidt on  $\mathcal{D}_*$ . But as was later proved in [5],  $\text{Cap } K > 0$  is the only obstruction. We can improve on the results from [5] as follows: our composition operator is not only Hilbert-Schmidt, but in any Schatten class; moreover, we can replace  $E_\varphi = K$  by  $E_\varphi = E_\varphi(1) = K$ .

**Theorem 4.1** *For every compact set  $K$  of the unit circle  $\mathbb{T}$  with logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $\varphi$  with the following properties:*

- 1)  $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_* := A$ , the “Dirichlet algebra”;
- 2)  $E_\varphi = E_\varphi(1) = K$ ;
- 3)  $C_\varphi \in \bigcap_{p>0} S_p(\mathcal{D}_*)$ .

*In fact, the approximation numbers of  $C_\varphi$  satisfy  $a_n(C_\varphi) \leq a \exp(-b\sqrt{n})$ .*

This theorem actually results of the particular following case and the properties of the cusp map seen in Section 3.2.

**Theorem 4.2** *For every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $q \in A(\mathbb{D}) \cap \mathcal{D}_*$  which peaks on  $K$  and such that the composition operator  $C_q: \mathcal{D}_* \rightarrow \mathcal{D}_*$  is bounded (and even Hilbert-Schmidt).*

Recall that a function  $q \in A(\mathbb{D})$ , the disk algebra, is said to *peak* on a compact subset  $K \subseteq \partial\mathbb{D}$  (and is called a *peaking function*) if:

$$q(z) = 1 \text{ if } z \in K; \quad |q(z)| < 1 \text{ if } z \in \overline{\mathbb{D}} \setminus K.$$

**Proof of Theorem 4.1.** We simply take for  $\varphi$  the composed map  $\varphi = \chi \circ q$ , where  $\chi$  is the cusp map and  $q$  our peaking function. Recall that  $\chi \in A(\mathbb{D})$  and that  $\chi$  peaks on  $\{1\}$ . We take advantage of this fact by composing with  $q$ , for which  $C_q: \mathcal{D}_* \rightarrow \mathcal{D}_*$  is bounded as well as  $C_\chi$  (since  $\chi$  is univalent). We clearly have  $\varphi \in A(\mathbb{D})$ ,  $\varphi(z) = \chi(1) = 1$  for  $z \in K$ , and  $|\varphi(z)| < 1$  for  $z \notin K$ , since then  $|q(z)| < 1$ . Therefore  $E_\varphi(1) = K$ . Moreover,  $C_\varphi$  being bounded on  $\mathcal{D}_*$ , we have in particular  $\varphi = C_\varphi(z) \in \mathcal{D}_*$ . Since  $C_\varphi = C_q \circ C_\chi$ , we get 3), by Corollary 3.5.

In [16], we prove that  $a_n(C_\chi) \leq a \exp(-b\sqrt{n})$ . Since  $a_n(C_\varphi) \leq \|C_q\| a_n(C_\chi)$ , by the ideal property of approximation numbers, this ends the proof of Theorem 4.1.  $\square$

In turn, the proof of Theorem 4.2 relies on the following crucial lemma.

**Lemma 4.3** *Let  $K \subseteq \partial\mathbb{D}$  be a compact set such that  $\text{Cap } K = 0$ . Then, there exists a function  $U: \overline{\mathbb{D}} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ , such that:*

- 1)  $U(z) = \infty$  if and only if  $z \in K$ ;
- 2)  $U \geq 1$  on  $\overline{\mathbb{D}}$ ;
- 3)  $U$  is continuous on  $\overline{\mathbb{D}} \setminus K$ , harmonic in  $\mathbb{D}$  and  $\int_{\mathbb{D}} |\nabla U|^2 dA < \infty$ ;
- 4)  $\lim_{z \rightarrow K, z \in \overline{\mathbb{D}}} U(z) = \infty$ ;
- 5) the conjugate function  $V = \tilde{U}$  is continuous on  $\overline{\mathbb{D}} \setminus K$ .

**Proof of Theorem 4.2.** Taking this lemma for granted, let us end the proof of the theorem. We set  $f = U + iV$ ,  $a = e^{-1/f(0)}$  and  $q = \varphi_a \circ e^{-1/f}$ , where  $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ . In view of the third and fourth items of the lemma, we have  $q \in A(\mathbb{D})$ . Since  $U \geq 1$ , Lemma 2.4 shows that  $C_q$  is Hilbert-Schmidt on  $\mathcal{D}_*$ . Moreover, for  $z \in K$ , one has  $f(z) = \infty$  and hence  $q(z) = 1$  since  $\varphi_a(1) = 1$  because  $a \in \mathbb{R}$  (since  $f(0) = U(0)$ ). On the other hand, when  $z \notin K$ , one has  $|f(z)| < \infty$  and hence  $|q(z)| < 1$ . Therefore  $q$  peaks on  $K$ .  $\square$

**Proof of Lemma 4.3.** This proof is strongly influenced by that of Theorem III, page 47, in [9]. Let:

$$(4.1) \quad L(z) = \log \left( \frac{e}{1-z} \right) = P(z) + iQ(z),$$

with

$$P(z) = \log \frac{e}{|1-z|} \text{ and } Q(z) = -\arg(1-z), \quad |Q(z)| \leq \frac{\pi}{2}, \quad z \in \overline{\mathbb{D}} \setminus \{1\},$$

and write:

$$P(z) \sim \sum_{n \in \mathbb{Z}} \gamma_n z^n,$$

with

$$\gamma_n = 1/(2|n|) \quad \text{if } n \neq 0, \quad \text{and } \gamma_0 = 1.$$

For  $0 < \varepsilon < 1/2$ , let  $K_\varepsilon = \{z \in \mathbb{T}; \text{dist}(z, K) \leq \varepsilon\}$ ,  $\mu_\varepsilon$  its equilibrium measure, and  $U_\varepsilon$  the logarithmic potential of  $\mu_\varepsilon$ , that is:

$$U_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{e}{|z-w|} d\mu_\varepsilon(w),$$

that we could as well write (since  $K_\varepsilon \subseteq \mathbb{T}$ ):

$$U_\varepsilon(z) = \int_{K_\varepsilon} P(z\bar{w}) d\mu_\varepsilon(w).$$

Let us set:

$$(4.2) \quad f_\varepsilon(z) = \int_{K_\varepsilon} L(z\bar{w}) d\mu_\varepsilon(w) = U_\varepsilon(z) + iV_\varepsilon(z),$$

with

$$V_\varepsilon(z) = \int_{K_\varepsilon} Q(z \bar{w}) d\mu_\varepsilon(w).$$

Then, if  $I_\varepsilon$  is the energy of  $\mu_\varepsilon$ , one has (see [23], Section 4)  $I_\varepsilon = 1 + \sum_{n=1}^{\infty} \frac{|\widehat{\mu_\varepsilon}(n)|^2}{n}$ , where  $\widehat{\mu_\varepsilon}(n) = \int_{\mathbb{T}} \bar{w}^n d\mu_\varepsilon(w)$  is the  $n$ -th Fourier coefficient of  $\mu_\varepsilon$ , and:

$$(4.3) \quad f_\varepsilon \in \mathcal{D} \quad \text{and} \quad \|f_\varepsilon\|_{\mathcal{D}}^2 = I_\varepsilon.$$

Note that  $\|f_\varepsilon\|_{\mathcal{D}} \geq 1$ .

We claim that there exist  $\delta > 0$  and  $0 < r < 1$  such that:

$$(4.4) \quad z \in \overline{\mathbb{D}} \text{ and } \text{dist}(z, K) \leq \delta \implies U_\varepsilon(rz) \geq I_\varepsilon/2$$

Indeed, let  $P_a(t) = \frac{1-|a|^2}{|e^{it}-a|^2}$  be the Poisson kernel at  $a \in \mathbb{D}$ . Since  $U_\varepsilon$  is harmonic in  $\mathbb{D}$  and integrable on  $\mathbb{T}$  ([4], Proposition 19.5.2), one has, for every  $z \in \mathbb{D}$ :

$$(4.5) \quad U_\varepsilon(z) = \int_{-\pi}^{\pi} U_\varepsilon(e^{it}) P_z(t) \frac{dt}{2\pi}.$$

Let now  $\delta \leq \varepsilon/4$ , to be adjusted later, and take  $1 - \delta \leq r < 1$ . Suppose that  $\text{dist}(z, K) \leq \delta$ , with  $z \in \overline{\mathbb{D}}$ , and let  $u \in K$  such that  $|z - u| \leq \varepsilon/4$ . Note that then  $|rz - u| \leq (1-r) + |z - u| \leq \varepsilon/2$ . It follows from (4.5) that:

$$I_\varepsilon - U_\varepsilon(rz) = \int_{-\pi}^{\pi} [I_\varepsilon - U_\varepsilon(e^{it})] P_{rz}(t) \frac{dt}{2\pi}$$

(it is useful to recall that  $U_\varepsilon(z) \leq I_\varepsilon$  for every  $z \in \mathbb{C}$ ). Set:

$$J_1 = \int_{|e^{it}-rz| \leq \varepsilon/2} [I_\varepsilon - U_\varepsilon(e^{it})] P_{rz}(t) \frac{dt}{2\pi}$$

and

$$J_2 = \int_{|e^{it}-rz| > \varepsilon/2} [I_\varepsilon - U_\varepsilon(e^{it})] P_{rz}(t) \frac{dt}{2\pi}.$$

For the integral  $J_1$ , we have:

$$|e^{it} - u| \leq |e^{it} - rz| + |rz - u| \leq \varepsilon;$$

therefore  $e^{it} \in K_\varepsilon$ . Since  $U_\varepsilon = I_\varepsilon$  Lebesgue-almost everywhere on  $K_\varepsilon$ , by Frostman's Theorem, we get  $J_1 = 0$ .

For the integral  $J_2$ , we have:

$$P_{rz}(t) \leq \frac{2(1-r|z|)}{(\varepsilon/2)^2} \leq 2 \frac{(1-r) + r(1-|z|)}{(\varepsilon/2)^2} \leq \frac{4\delta}{(\varepsilon/2)^2} = \frac{16\delta}{\varepsilon^2};$$

hence (since  $U_\varepsilon(e^{it}) \geq 0$ ):

$$J_2 \leq \frac{16\delta}{\varepsilon^2} I_\varepsilon.$$

Therefore, if we choose  $0 < \delta \leq \varepsilon^2/32$ , we get:

$$0 \leq I_\varepsilon - U_\varepsilon(rz) \leq I_\varepsilon/2,$$

which gives (4.4). □

Now, as  $\text{Cap } K = 0$ , we know from (1.9) that  $\lim_{\varepsilon \rightarrow 0^+} I_\varepsilon = \infty$ , and we can adjust a sequence  $\varepsilon_j \rightarrow 0^+$  so that:

$$(4.6) \quad I_{\varepsilon_j} \geq 4j^6.$$

Using (4.4), we find two sequences  $(\delta_j)_j$  and  $(r_j)_j$ , with  $0 < \delta_j \rightarrow 0$  and  $1 > r_j \rightarrow 1$ , such that, for every  $j \geq 1$ ,

$$(4.7) \quad z \in \overline{\mathbb{D}} \text{ and } \text{dist}(z, K) \leq \delta_j \implies U_{\varepsilon_j}(r_j z) \geq I_{\varepsilon_j}/2.$$

Finally, let us set:

$$(4.8) \quad f_j(z) = f_{\varepsilon_j}(r_j z)$$

and

$$(4.9) \quad f = U + iV = 1 + \sum_{j=1}^{\infty} j^{-2} \frac{f_j}{\|f_j\|_{\mathcal{D}}}.$$

The series defining  $f$  is absolutely convergent in  $\mathcal{D}$ . Note that  $f(0)$  is real.

We now have:

1)  $f$  is continuous on  $\overline{\mathbb{D}} \setminus K$ .

Indeed, let  $z \in \overline{\mathbb{D}} \setminus K$ . Then,  $\text{dist}(z, K) > 0$  and there exists a neighbourhood  $\omega$  of  $z$  in  $\overline{\mathbb{D}}$ , an integer  $j_0 = j_0(z)$  and a positive number  $\delta > 0$  such that:

$$w \in \omega \text{ and } j \geq j_0 \implies \text{dist}(r_j w, K_{\varepsilon_j}) \geq \delta.$$

We then have, for  $w \in \omega$  and  $j \geq j_0$ :

$$\begin{aligned} |f_{\varepsilon_j}(w)| &= \left| \int_{K_{\varepsilon_j}} \log \frac{e}{r_j w - u} d\mu_{\varepsilon_j}(u) \right| \\ &\leq \int_{K_{\varepsilon_j}} \left( \log \frac{e}{|r_j w - u|} + \frac{\pi}{2} \right) d\mu_{\varepsilon_j}(u) \leq \log \frac{e}{\delta} + \frac{\pi}{2} := C, \end{aligned}$$

since  $\mu_{\varepsilon_j}$  is a probability measure supported by  $K_{\varepsilon_j}$ . Therefore, the series defining  $f$  is normally convergent on  $\omega$  since its general term is dominated by  $j^{-2}C$  on  $\omega$ . Since the functions  $f_j$  are continuous on  $\overline{\mathbb{D}}$ , this shows that  $f$  is continuous at  $z$ .

2)  $U(z) := \Re f(z) \geq 1$ .

This is obvious since, for every  $z \in \overline{\mathbb{D}}$ ,

$$U_\varepsilon(z) := \Re f_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{e}{|z - u|} d\mu_\varepsilon(u) \geq 0.$$

3)  $\lim_{z \rightarrow K, z \in \mathbb{D}} U(z) = \infty$ .

Indeed, let  $A > 0$ . Take an integer  $j \geq A$  and suppose that  $\text{dist}(z, K) \leq \delta_j$ . Then, using the positivity of the  $U_{\varepsilon_k}$ 's as well as (4.3), (4.6) and (4.7), we have:

$$U(z) \geq j^{-2} \frac{U_{\varepsilon_j}(r_j z)}{\|f_{\varepsilon_j}\|_{\mathcal{D}}} \geq j^{-2} \frac{I_{\varepsilon_j}/2}{\sqrt{I_{\varepsilon_j}}} \geq j \geq A.$$

This ends the proof of our claims, and of Lemma 4.3.  $\square$

To end this paper, let us mention the following version of the classical Rudin-Carleson Theorem. Though it is not the main subject of this paper, it has the same flavor as Theorem 4.2. We do not give a proof, but only mention that it can be obtained by mixing the proofs of Theorems III.E.2 and III.E.6 in [25] (see pages 181–187).

**Theorem 4.4** *Let  $K$  be a compact subset of  $\mathbb{T}$  with  $\text{Cap } K = 0$ . Given any continuous strictly positive function  $s \in C(\mathbb{T})$  equal to 1 on  $K$ , we can find, for every  $h \in C(K)$  and every  $\varepsilon > 0$ , a function  $f \in A(\mathbb{D}) \cap \mathcal{D}$  such that  $f|_K = h$  and:*

$$|f(\theta)| \leq (1 + \varepsilon) \|h\|_{\infty} s(\theta), \quad \forall \theta \in \mathbb{T}; \quad \|f\|_{\mathcal{D}} \leq (1 + \varepsilon) \|h\|_{\infty}.$$

## References

- [1] N. ARCOZZI, R. ROCHBERG, E. T. SAWYER AND B. D. WICK, The Dirichlet space: a survey, *New York J. Math.* 17A (2011), 45–86.
- [2] A. BEURLING, Sur les ensembles exceptionnels, *Acta Math.* 72 (1940), 1–13.
- [3] T. CARROLL AND C. C. COWEN, Compact composition operators not in the Schatten classes, *J. Operator Theory* 26, No. 1 (1991), 109–120.
- [4] J. B. CONWAY, *Functions of One Complex Variable II*, Graduate Texts in Math. 159, Springer-Verlag (1995).
- [5] O. EL-FALLAH, K. KELLAY, M. SHABANKHAH AND H. YOUSSEFI, Level sets and composition operators on the Dirichlet space, *J. Funct. Anal.* 260, No. 6 (2011), 1721–1733.
- [6] E. A. GALLARDO-GUTIÉRREZ AND M. J. GONZÁLEZ, Exceptional sets and Hilbert-Schmidt composition operators, *J. Funct. Anal.* 199, No. 2 (2003), 287–300.
- [7] E. A. GALLARDO-GUTIÉRREZ AND M. J. GONZÁLEZ, Hausdorff measures, capacities and compact composition operators, *Math. Z.* 253, No. 1 (2006), 63–74.
- [8] M. M. JONES, Compact composition operators not in the Schatten classes, *Proc. Amer. Math. Soc.* 134, No. 7 (2006), 1947–1953.

- [9] J. P. KAHANE AND R. SALEM, Ensembles parfaits et séries trigonométriques, *nouvelle édition*, Hermann (1994).
- [10] K. KELLAY AND P. LEFÈVRE, Compact composition operators on weighted Hilbert spaces of analytic functions, *Journ. Math. Anal. Appl.* 386 (2) (2012), 718–727.
- [11] P. LEFÈVRE, D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Compact composition operators on  $H^2$  and Hardy-Orlicz spaces, *J. Math. Anal. Appl.* 354 (2009), 360–371.
- [12] P. LEFÈVRE, D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Composition operators on Hardy-Orlicz spaces, *Memoirs Amer. Math. Soc.* 207 (2010), No. 974.
- [13] P. LEFÈVRE, D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Some examples of compact composition operators on  $H^2$ , *J. Funct. Anal.* 255, No.11 (2008), 3098–3124.
- [14] P. LEFÈVRE, D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Nevanlinna counting function and Carleson function of analytic maps, *Math. Ann.* 351 (2011), 305–326.
- [15] P. LEFÈVRE, D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Compact composition operators on Bergman-Orlicz spaces, *preprint 2009*
- [16] P. LEFÈVRE, D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Approximation numbers of composition operators on the Dirichlet space, *in preparation*.
- [17] D. LI, Compact composition operators on Hardy-Orlicz and Bergman-Orlicz spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* 105, no. 2 (2011), 247–260.
- [18] D. LI, H. QUEFFÉLEC AND L. RODRÍGUEZ-PIAZZA, Estimates for approximation numbers of some classes of composition operators, *preprint*.
- [19] D. H. LUECKING, Trace ideal criteria for Toeplitz operators, *J. Funct. Anal.* 73 (1987), 345–368.
- [20] D. H. LUECKING AND K. H. ZHU, Composition operators belonging to the Schatten ideals, *Amer. J. Math.* 114, No. 5 (1992), 1127–1145.
- [21] B. MCCLUER AND J. SHAPIRO, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* 38, no. 4 (1986), 878–906.
- [22] B. P. PALKA, An Introduction to Complex Function Theory, Undergraduate Texts in Mathematics, Springer-Verlag, New-York (1991).

- [23] W. T. ROSS, The classical Dirichlet space, Recent advances in operator-related function theory, 171–197, *Contemp. Math.* 393, Amer. Math. Soc., Providence, RI (2006).
- [24] D. A. STEGENGA, Multipliers on the Dirichlet space, *Illinois J. Math.* 24 (1) (1980), 113–139.
- [25] P. WOJTASZCZYK, Banach spaces for analysts, Cambridge Studies in Advanced Mathematics 25, Cambridge University Press, Cambridge (1991).
- [26] XU X. M., Schatten-class composition operators on weighted Dirichlet spaces, *Acta Anal. Funct. Appl.* 1, No. 1 (1999), 86–91.
- [27] K. H. ZHU, Operator Theory in Function Spaces, Monographs and Textbooks in Pure and Applied Mathematics 139, Marcel Dekker, Inc., New York (1990).
- [28] K. H. ZHU, Schatten class composition operators on weighted Bergman spaces of the disk, *J. Operator Theory* 46, No. 1 (2001), 173–181.
- [29] N. ZORBOSKA, Composition operators on weighted Dirichlet spaces, *Proceed. Amer. Math. Soc.* 126, No. 7 (1998), 2013–2023.

Pascal Lefèvre, *Univ Lille Nord de France, U-Artois,*  
*Laboratoire de Mathématiques de Lens EA 2462 & Fédération CNRS Nord-Pas-de-Calais FR 2956,*  
*Faculté des Sciences Jean Perrin, Rue Jean Souwraz, S.P. 18,*  
*F-62 300 LENS, FRANCE*  
*pascal.lefevre@euler.univ-artois.fr*

Daniel Li, *Univ Lille Nord de France, U-Artois,*  
*Laboratoire de Mathématiques de Lens EA 2462 & Fédération CNRS Nord-Pas-de-Calais FR 2956,*  
*Faculté des Sciences Jean Perrin, Rue Jean Souwraz, S.P. 18,*  
*F-62 300 LENS, FRANCE*  
*daniel.li@euler.univ-artois.fr*

Hervé Queffelec, *Univ Lille Nord de France,*  
*USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524, & Fédération CNRS Nord-Pas-de-Calais FR 2956,*  
*F-59 655 VILLENEUVE D’ASCQ Cedex, FRANCE*  
*Herve.Queffelec@univ-lille1.fr*

Luis Rodríguez-Piazza, *Universidad de Sevilla,*  
*Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS,*  
*Apartado de Correos 1160,*  
*41 080 SEVILLA, SPAIN*  
*piazza@us.es*