Compact composition operators on the Dirichlet space and capacity of sets of contact points

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza

Compact composition operators on the Dirichlet space and capacity of sets of contact points

Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza[*]

July 5, 2012

Abstract. *We prove that for every compact set $K \subseteq \partial\mathbb{D}$ of logarithmic capacity* $\operatorname{Cap} K = 0$*, there exists a Schur function φ both in the disk algebra $A(\mathbb{D})$ and in the Dirichlet space $\mathcal{D}_*$ such that the composition operator C_φ is in all Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$, and for which $K = \{\mathrm{e}^{it}\,;\ |\varphi(\mathrm{e}^{it})| = 1\} = \{\mathrm{e}^{it}\,;\ \varphi(\mathrm{e}^{it}) = 1\}$. We show that for every bounded composition operator C_φ on $\mathcal{D}_*$ and every $\xi \in \partial\mathbb{D}$, the logarithmic capacity of $\{\mathrm{e}^{it}\,;\ \varphi(\mathrm{e}^{it}) = \xi\}$ is 0. We show that every compact composition operator C_φ on $\mathcal{D}_*$ is compact on the Bergman-Orlicz space $\mathfrak{B}^{\Psi_2}$ and on the Hardy-Orlicz space H^{Ψ_2}; in particular, C_φ is in every Schatten class S_p, $p > 0$, both on the Hardy space H^2 and on the Bergman space $\mathfrak{B}^2$. On the other hand, there exists a Schur function φ such that C_φ is compact on H^{Ψ_2}, but which is not even bounded on $\mathcal{D}_*$. We prove that for every $p > 0$, there exists a symbol φ such that $C_\varphi \in S_p(\mathcal{D}_*)$, but $C_\varphi \notin S_q(\mathcal{D}_*)$ for any $q < p$, that there exists another symbol φ such that $C_\varphi \in S_q(\mathcal{D}_*)$ for every $q < p$, but $C_\varphi \notin S_p(\mathcal{D}_*)$. Also, there exists a Schur function φ such that C_φ is compact on $\mathcal{D}_*$, but in no Schatten class $S_p(\mathcal{D}_*)$.*

MSC 2010. Primary: 47B33 – Secondary: 28A12; 30C85; 31A15; 46E20; 46E22; 47B10.

Key-words. Bergman space – Bergman-Orlicz space – composition operator – Dirichlet space – Hardy space – Hardy-Orlicz space – logarithmic capacity – Schatten classes

1 Introduction, notation and background

1.1 Introduction

Recall that a Schur function is an analytic self-map of the open unit disk $\mathbb{D}$. Every Schur function φ generates a bounded composition operator C_φ on the

[*]Supported by a Spanish research project MTM 2009-08934.

Hardy space H^2, given by $C_\varphi(f) = f \circ \varphi$. Let us also introduce the set E_φ of contact points of the symbol with the unit circle (equipped with its normalized Haar measure m), namely:

$$E_\varphi = \{\mathrm{e}^{it}\,;\ |\varphi^*(\mathrm{e}^{it})| = 1\}. \tag{1.1}$$

In terms of E_φ, a well-known necessary condition for compactness of C_φ on H^2 is that $m(E_\varphi) = 0$. This set E_φ is otherwise more or less arbitrary. Indeed, it was proved in [7] that there exist compact composition operators C_φ on H^2 such that the Hausdorff dimension of E_φ is 1. This was generalized in [5]: for every Lebesgue-negligible compact set K of the unit circle $\mathbb{T}$, there is a Hilbert-Schmidt composition operator C_φ on H^2 such that $E_\varphi = K$, and in [18]:

Theorem 1.1 ([18]) *For every Lebesgue-negligible compact set K of the unit-circle $\mathbb{T}$ and every vanishing sequence (ε_n) of positive numbers, there is a composition operator C_φ on H^2 such that $E_\varphi = K$ and such that its approximation numbers satisfy $a_n(C_\varphi) \leq C\,\mathrm{e}^{-n\,\varepsilon_n}$.*

We are interested here in a different Hilbert space of analytic functions, on which not every Schur function defines a bounded composition operator, namely the Dirichlet space $\mathcal{D}$. Recall its definition: the Dirichlet space $\mathcal{D}$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, dA(z) < +\infty\,. \tag{1.2}$$

If $f(z) = \sum_{n=0}^\infty c_n z^n$, one has:

$$\|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^\infty n\,|c_n|^2\,. \tag{1.3}$$

Then $\|\ \|_{\mathcal{D}}$ is a norm on $\mathcal{D}$, making $\mathcal{D}$ a Hilbert space. Whereas every Schur function φ generates a bounded composition operator C_φ on the Hardy space H^2, it is no longer the case for the Dirichlet space (see [21], Proposition 3.12, for instance).

In [6], the study of compact composition operators on the Dirichlet space $\mathcal{D}$ associated with a Schur function φ in connection with the set E_φ was initiated. In particular, it is proved there that if the composition operator C_φ is Hilbert-Schmidt on $\mathcal{D}$, then the logarithmic capacity $\operatorname{Cap} E_\varphi$ of E_φ is 0, but, on the other hand, there are compact composition operators on $\mathcal{D}$ for which this capacity is positive. The optimality of this theorem was later proved in [5] under the following form:

Theorem 1.2 (O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi) *For every compact set K of the unit circle $\mathbb{T}$ with logarithmic capacity $\operatorname{Cap} K$ equal to 0, there exits a Hilbert-Schmidt composition operator C_φ on $\mathcal{D}$ such that $E_\varphi = K$.*

In this paper, we shall improve on this last result. We prove in Section 4 (Theorem 4.1) that for every compact set $K \subseteq \partial\mathbb{D}$ of logarithmic capacity $\operatorname{Cap} K = 0$, there exists a Schur function $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_*$ such that the composition operator C_φ is in all Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$, and for which $E_\varphi = K$ (and moreover $E_\varphi = \{\mathrm{e}^{it}\,;\ \varphi(\mathrm{e}^{it}) = 1\}$). On the other hand, in Section 2, we show (Theorem 2.1) that for every bounded composition operator C_φ on $\mathcal{D}_*$ and every $\xi \in \partial\mathbb{D}$, the logarithmic capacity of $E_\varphi(\xi) = \{\mathrm{e}^{it}\,;\ \varphi(\mathrm{e}^{it}) = \xi\}$ is 0.

In link with Hardy and Bergman spaces, we prove, in Section 2 yet, that every compact composition operator C_φ on $\mathcal{D}_*$ is compact on the Bergman-Orlicz space $\mathfrak{B}^{\Psi_2}$ and on the Hardy-Orlicz space H^{Ψ_2}. In particular, C_φ is in every Schatten class S_p, $p > 0$, both on the Hardy space H^2 and on the Bergman space $\mathfrak{B}^2$ (Theorem 2.5). However, there exists a Schur function φ such that C_φ is compact on H^{Ψ_2}, but which is not even bounded on $\mathcal{D}_*$ (Theorem 2.6).

In Section 3, we give a characterization of the membership of composition operators in the Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$ (actually in $S_p(\mathcal{D}_{\alpha,*})$, where $\mathcal{D}_{\alpha,*}$ is the weighted Dirichlet space). We deduce that for every $p > 0$, there exists a symbol φ such that $C_\varphi \in S_p(\mathcal{D}_*)$, but $C_\varphi \notin S_q(\mathcal{D}_*)$ for any $q < p$, and that there exists another symbol φ such that $C_\varphi \in S_q(\mathcal{D}_*)$ for every $q < p$, but $C_\varphi \notin S_p(\mathcal{D}_*)$ (Theorem 3.3). We also show that there exists a Schur function φ such that C_φ is compact on $\mathcal{D}_*$, but in no Schatten class $S_p(\mathcal{D}_*)$ (Theorem 3.4).

1.2 Notation and background.

We denote by $\mathbb{D}$ the unit open disk of the complex plane and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. A is the normalized area measure $dx\,dy/\pi$ of $\mathbb{D}$ and m the normalized Lebesgue measure $dt/2\pi$ on $\mathbb{T}$.

As said before, a Schur function is an analytic self-map of $\mathbb{D}$ and the associated composition operator is defined, formally, by $C_\varphi(f) = f \circ \varphi$. The function φ is called the symbol of C_φ.

The Dirichlet space $\mathcal{D}$ is defined above. We shall actually work, for convenience, with its subspace $\mathcal{D}_*$ of functions $f \in \mathcal{D}$ such that $f(0) = 0$. In this paper, we call $\mathcal{D}_*$ the *Dirichlet space*.

An orthonormal basis of $\mathcal{D}_*$ is formed by $e_n(z) = z^n/\sqrt{n}$, $n \geq 1$. The reproducing kernel on $\mathcal{D}_*$, defined by $f(a) = \langle f, K_a\rangle$ for every $f \in \mathcal{D}_*$, is given by $K_a(z) = \sum_{n=1}^\infty \overline{e_n(a)}\, e_n(z)$, so that:

$$K_a(z) = \log \frac{1}{1 - \overline{a}z}\, \cdot \tag{1.4}$$

Compactness of composition operators on $\mathcal{D}$ was characterized in terms of Carleson measure by D. Stegenga ([24]) and by B. McCluer and J. Shapiro in terms of angular derivative ([21]). Another characterization, more useful for us here, was given by N. Zorboska ([29], page 2020): for $\varphi \in \mathcal{D}$, C_φ is bounded on $\mathcal{D}$ if and only:

$$\sup_{h \in (0,2)} \sup_{|\xi|=1} \frac{1}{A[W(\xi,h)]} \int_{W(\xi,h)} n_\varphi(w)\, dA(w) < \infty\,, \tag{1.5}$$

where $W(\xi, h) = \{w \in \mathbb{D};\ 1 - |w| \leq h \text{ and } |\arg(w\bar{\xi})| \leq \pi h\}$ is the Carleson window of size $h \in (0, 2)$ center at $\xi \in \mathbb{T}$ and n_φ is the counting function of φ:

$$n_\varphi(w) = \sum_{\varphi(z)=w} 1 \, , \qquad w \in \varphi(\mathbb{D}) \, , \tag{1.6}$$

(we set $n_\varphi(w) = 0$ for $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$). In particular, every Schur function with bounded valence defines a bounded composition operator on $\mathcal{D}$.

Moreover, C_φ is compact if and only if:

$$\sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi,h)} n_\varphi(w) \, dA(w) \underset{h \to 0}{\longrightarrow} 0 \, . \tag{1.7}$$

For further informations on the Dirichlet space, one may consult the two surveys [1] and [23], for example.

1.2.1 Logarithmic capacity

The notion of logarithmic capacity is tied to the study of the Dirichlet space by the following seminal and sharp result of Beurling ([2]; see also [9]).

Theorem 1.3 (Beurling) *For every function $f(z) = \sum_{n=0}^\infty c_n z^n \in \mathcal{D}$, there exists a set $E \subseteq \partial\mathbb{D}$, with logarithmic capacity 0, such that, if $t \in \mathbb{T} \setminus E$, then the radial limit $f^*(\mathrm{e}^{it}) := \lim_{r \to 1^-} f(r\mathrm{e}^{it})$ exists (in $\mathbb{C}$). Moreover, the result is optimal: if a compact set $E \subseteq \mathbb{T}$ has zero logarithmic capacity, there exists $f(z) = \sum_{n=0}^\infty c_n z^n \in \mathcal{D}$ such that $f^*(\mathrm{e}^{it})$ does not exist on E.*

Let us recall some definitions (see [9], Chapitre III, [4], Chapter 21, § 7, or [23], Section 4, for example).

Let μ be a probability measure supported by a compact subset K of $\mathbb{T}$. The *potential* U_μ of μ is defined, for every $z \in \mathbb{C}$, by:

$$U_\mu(z) = \int_K \log \frac{\mathrm{e}}{|z - w|} \, d\mu(w) \, .$$

The *energy* I_μ of μ is defined by:

$$I_\mu = \int_K U_\mu(z) \, d\mu(z) = \iint_{K \times K} \log \frac{\mathrm{e}}{|z - w|} \, d\mu(w) \, d\mu(z) \, .$$

The *logarithmic capacity* of a Borel set $E \subseteq \mathbb{T}$ is:

$$\operatorname{Cap} E = \sup\nolimits_\mu \mathrm{e}^{-I_\mu} \, ,$$

where the supremum is over all Borel probability measures μ with compact support contained in E . Hence E is of logarithmic capacity 0 (which is the case we are interested in) if and only if $I_\mu = \infty$ for all probability measures compactly carried by E. The fact that $\operatorname{Cap} E = 0$ implies that E has null Lebesgue measure ([9], Chapitre III, Théorème I) (hence $\operatorname{Cap} E > 0$ if E is a non-void open subset of $\mathbb{T}$), but the converse is wrong, as shown by Cantor's middle-third set $\mathfrak{C}$. A compact set K such that $\operatorname{Cap} K = 0$ is totally disconnected ([4], Corollary 21.7.7).

If E is a compact set with $\operatorname{Cap} E > 0$, there is a unique probability measure compactly carried by E that minimizes the energy I_μ ([4], Theorem 21.10.2, or [9], Chapitre III, Proposition 4). Such a measure is called the *equilibrium measure* of E.

If μ is the equilibrium measure of the compact set K, we have Frostman's Theorem ([4], Theorem 21.7.12, or [9], Chapitre III, Proposition 5 and Proposition 6): $U_\mu(z) \leq I_\mu$ for every $z \in \mathbb{C}$ and

$$U_\mu(z) = I_\mu \quad \text{for almost all } z \in K\,. \tag{1.8}$$

Suppose that the compact set K has zero logarithmic capacity. For $\varepsilon > 0$, let $K_\varepsilon = \{z \in \mathbb{T}\,;\ \operatorname{dist}(z, K) \leq \varepsilon\}$, μ_ε its equilibrium measure, and I_{μ_ε} its energy. Then ([4], Proposition 21.7.15):

$$\lim_{\varepsilon \to 0} I_{\mu_\varepsilon} = \infty\,. \tag{1.9}$$

2 Bounded and compact composition operators

In [6], E. A. Gallardo-Gutiérrez and M. J. González showed that for every Hilbert-Schmidt composition operator C_φ on $\mathcal{D}_*$, the logarithmic capacity of the set $E_\varphi = \{\mathrm{e}^{i\theta} \in \partial\mathbb{D}\,;\ |\varphi(\mathrm{e}^{i\theta})| = 1\}$ is zero. On the other hand, they showed that there are compact composition operators on $\mathcal{D}_*$ for which E_φ has positive logarithmic capacity. We shall see that if we replace $|\varphi|$ by φ in the definition of E_φ, the result is very different.

Theorem 2.1 *For every bounded composition operator C_φ on $\mathcal{D}_*$ and every $\xi \in \partial\mathbb{D}$, the logarithmic capacity of $E_\varphi(\xi) = \{\mathrm{e}^{it}\,;\ \varphi(\mathrm{e}^{it}) = \xi\}$ is 0.*

We first state the following characterization of Hilbert-Schmidt composition operators on $\mathcal{D}_*$. This result is stated in [6], but not entirely proved.

Lemma 2.2 *Let $\varphi \in \mathcal{D}_*$ be an analytic self-map of $\mathbb{D}$. Then C_φ is Hilbert-Schmidt on $\mathcal{D}_*$ if and only if*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2}\, dA(z) < \infty\,. \tag{2.1}$$

Proof. Let $e_n(z) = z^n/\sqrt{n}$; then $(e_n)_{n\geq 1}$ is an orthonormal basis of $\mathcal{D}_*$ and

$$\sum_{n=1}^{\infty} \|C_\varphi(e_n)\|^2 = \sum_{n=1}^{\infty} \frac{\|\varphi^n\|^2}{n} = \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2}\, dA(z)\,.$$

Hence (2.1) is satisfied if C_φ is Hilbert-Schmidt. To get the converse, we need to show that (2.1) implies that C_φ is bounded on $\mathcal{D}_*$. Let $f \in \mathcal{D}_*$ and write $f(z) = \sum_{n=1}^\infty c_n z^n$. Then $C_\varphi f = \sum_{n=1}^\infty c_n \varphi^n$ and

$$\begin{aligned} \|C_\varphi f\| &\leq \sum_{n=1}^\infty |c_n| \, \|\varphi^n\| \leq \Big(\sum_{n=1}^\infty n \, |c_n|^2 \Big)^{1/2} \Big(\sum_{n=1}^\infty \frac{\|\varphi^n\|^2}{n} \Big)^{1/2} \\ &= \Big(\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} \, dA(z) \Big)^{1/2} \|f\| \, . \end{aligned}$$

Then (2.1) implies that C_φ is Hilbert-Schmidt. $\square$

Now Theorem 2.1 will follow from the next proposition.

Proposition 2.3 *There exists an analytic self-map σ of $\mathbb{D}$, belonging to $\mathcal{D}_*$ and to the disk algebra $A(\mathbb{D})$, such that $\sigma(1) = 1$ and $|\sigma(\xi)| < 1$ for $\xi \in \partial\mathbb{D} \setminus \{1\}$ and such that the associated composition operator C_σ is Hilbert-Schmidt on $\mathcal{D}_*$.*

Taking this proposition for granted for a while, we can prove the theorem.

Proof of Theorem 2.1. Making a rotation, we may, and do, assume that $\xi = 1$. Then, if σ is the map of Proposition 2.3, $C_\varphi C_\sigma = C_{\sigma \circ \varphi}$ is Hilbert-Schmidt. By [6], the set $E_{\sigma \circ \varphi}$ has zero logarithmic capacity. But σ has modulus 1 only at 1; hence $\mathrm{e}^{i\theta} \in E_{\sigma\circ\varphi}$ if and only if $\mathrm{e}^{i\theta} \in E_\varphi(1)$. $\square$

To prove Proposition 2.3, it will be convenient to use the following criteria, where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ ·

Lemma 2.4 *Let $f \in \mathcal{D}$ such that $\mathfrak{Re}\, f \geq 1$. Then if $\sigma = \varphi_a \circ \mathrm{e}^{-1/f}$, where $a = \mathrm{e}^{-1/f(0)}$, the composition operator C_σ is Hilbert-Schmidt on $\mathcal{D}_*$.*

Proof. Let $\sigma_0 = \mathrm{e}^{-1/f}$. If $u = \mathfrak{Re}\, f$ and $v = \mathfrak{Im}\, f$, one has:

$$|\sigma_0|^2 = \exp\Big(- \frac{2u}{u^2+v^2} \Big) \quad \text{and} \quad |\sigma_0'|^2 = \frac{{u'}^2 + {v'}^2}{(u^2+v^2)^2} \exp\Big(- \frac{2u}{u^2+v^2} \Big) \cdot$$

Then $|\sigma_0| < 1$ and so σ_0 is a self-map of $\mathbb{D}$. Since $u \geq 1 > 0$, one has $|\sigma_0'|^2 \leq ({u'}^2 + {v'}^2)/(u^2+v^2)^2 \leq {u'}^2 + {v'}^2 = |f'|^2$; hence $\sigma_0 \in \mathcal{D}$.

For $0 \leq x \leq 2$, one has $1 - \mathrm{e}^{-x} \geq x/4$. Therefore, since $u \geq 1$ implies $2u/(u^2+v^2) \leq 2/u \leq 2$, one has:

$$1 - |\sigma_0|^2 \geq \frac{u}{2(u^2+v^2)} \cdot$$

It follows that:

$$\frac{|\sigma_0'|^2}{(1-|\sigma_0|^2)^2} \leq \frac{{u'}^2 + {v'}^2}{(u^2+v^2)^2} \, \frac{4(u^2+v^2)^2}{u^2} \leq 4({u'}^2 + {v'}^2) = 4|f'|^2 \, .$$

Since $f \in \mathcal{D}$, $|f'|^2$ has a finite integral and therefore (2.1) is satisfied. It follows that C_{σ_0} is Hilbert-Schmidt on $\mathcal{D}$ and hence $C_\sigma = C_{\sigma_0} \circ C_{\varphi_a}$ is Hilbert-Schmidt on $\mathcal{D}_*$, since $\sigma(0) = 0$. $\square$

Proof of Proposition 2.3. Let Ω be the domain defined by:

$$\Omega = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 1 \text{ and } |\mathfrak{Im}\, z| < 1/(\mathfrak{Re}\, z)^2\}\,.$$

Let f be a conformal map from $\mathbb{D}$ onto Ω such that $f(1) = \infty$. Since $A(\Omega) < \infty$, we have $f \in \mathcal{D}$. By Lemma 2.4, the function $\sigma = \mathrm{e}^{-1/f}$ has the required properties. $\square$

For the next result, recall that an Orlicz function Ψ is a nondecreasing convex function such that $\Psi(0) = 0$ and $\Psi(x)/x \to \infty$ as x goes to infinity. We refer to [12] for the definition of Hardy-Orlicz and Bergman-Orlicz spaces. In the following result, one set $\Psi_2(x) = \exp(x^2) - 1$.

Theorem 2.5 *Every compact composition operator C_φ on $\mathcal{D}_*$ is compact on the Bergman-Orlicz space $\mathfrak{B}^{\Psi_2}$ and on the Hardy-Orlicz space H^{Ψ_2}. In particular, C_φ is in every Schatten class S_p, $p > 0$, both on the Hardy space H^2 and on the Bergman space $\mathfrak{B}^2$.*

Proof. Consider the normalized reproducing kernels $\tilde{K}_a = K_a/\|K_a\|$, $a \in \mathbb{D}$. When $|a|$ goes to 1, they tends to 0 uniformly on compact sets of $\mathbb{D}$; hence $\|C_\varphi^*(\tilde{K}_a)\|$ tends to 0, by compactness of the adjoint operator C_φ^*. But $C_\varphi^*(K_a) = K_{\varphi(a)}$ and $\|K_a\|^2 = \langle K_a, K_a\rangle = \log \frac{1}{1-|a|^2}$, so we get:

$$\lim_{|a|\to 1} \frac{\log \frac{1}{1-|\varphi(a)|^2}}{\log \frac{1}{1-|a|^2}} = 0\,. \tag{2.2}$$

This condition means that C_φ is compact on the Bergman-Orlicz space $\mathfrak{B}^{\Psi_2}$ ([12], page 69) and implies that C_φ is in all Schatten classes $S_p(\mathfrak{B}^2)$, $p > 0$ ([15]).

In the same way, it suffices to show that C_φ is compact on H^{Ψ_2}, because that implies that C_φ is in all Schatten classes $S_p(H^2)$ ([11], Theorem 5.2).

Compactness of C_φ on H^Ψ is equivalent to say ([12], Theorem 4.18) that:

$$\begin{aligned}\rho_\varphi(h) &:= \sup_{|\xi|=1} m\big(\{\mathrm{e}^{it}\,;\ \varphi(\mathrm{e}^{it}) \in W(\xi, h)\}\big)\\ &= o_{h\to 0}\left[\frac{1}{\Psi\big(A\Psi^{-1}(1/h)\big)}\right] \quad \text{for every } A > 0\,.\end{aligned}$$

When $\Psi = \Psi_2$, this means that $\rho_\varphi(h) = o\,(h^A)$ for every $A > 0$. Now, by [14], Theorem 4.2, this is also equivalent to say that:

$$\sup_{|\xi|=1} \frac{1}{A[W(\xi, h)]} \int_{W(\xi,h)} N_\varphi(w)\, dA(w) = o\,(h^A) \quad \text{for every } A > 0\,, \tag{2.3}$$

where N_φ is the Nevanlinna counting function of φ:

$$N_\varphi(w) = \sum_{\varphi(z)=w} (1 - |z|^2)\,, \qquad w \in \varphi(\mathbb{D})\,, \tag{2.4}$$

and $N_\varphi(w) = 0$ otherwise.

But (2.2) is equivalent to the fact that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that:

$$1 - |\varphi(z)| \geq \delta_\varepsilon (1 - |z|)^\varepsilon\,, \qquad \forall z \in \mathbb{D}\,. \tag{2.5}$$

Since $\varphi(0) = 0$, we have $|\varphi(z)| \leq |z|$, by Schwarz's lemma; hence one has $N_\varphi(w) \leq 2\delta_\varepsilon^{-1}(1 - |w|)^{1/\varepsilon} n_\varphi(w)$. It follows that (since $1 - |w| \leq h$ for $w \in W(\xi, h)$):

$$\frac{1}{A[W(\xi,h)]}\int_{W(\xi,h)} N_\varphi(w)\,dA(w) \leq 2\delta_\varepsilon^{-1} h^{1/\varepsilon} \frac{1}{A[W(\xi,h)]}\int_{W(\xi,h)} n_\varphi(w)\,dA(w)\,,$$

which is $o\,(h^{1/\varepsilon})$, uniformly for $|\xi| = 1$, by (1.7). $\square$

Remarks. 1) One may argue that compactness of C_φ on H^{Ψ_2} implies its compactness on $\mathfrak{B}^{\Psi_2}$ ([15], Proposition 4.1, or [17], Theorem 9). One may also use the forthcoming Corollary 3.2 saying that $C_\varphi \in S_p(H^2)$ implies that $C_\varphi \in S_p(\mathfrak{B}^2)$.

2) To show the compactness of C_φ on H^{Ψ_2}, we used its compactness on $\mathcal{D}_*$ twice. However, due to the fact that $\varepsilon > 0$ is arbitrary, we may replace $o\,(h^{1/\varepsilon})$ by $O\,(h^{1/\varepsilon})$; hence to end the proof, we only have to use (1.5), *i.e.* the boundedness of C_φ on $\mathcal{D}_*$, instead of (1.7).

Note that (2.2) does not suffice to have compactness on H^{Ψ_2} (in [12], Proposition 5.5, we construct a Blaschke product satisfying (2.2)).

In the opposite direction, we have the following result.

Theorem 2.6 *There exists a Schur function φ such that C_φ is compact on H^{Ψ_2}, but which is not even bounded on $\mathcal{D}_*$.*

To prove this theorem, we first begin with the following key lemma.

Lemma 2.7 *There exists a constant $\kappa_1 > 0$ such that for any $f \in \mathcal{H}(\mathbb{D})$ having radial limits f^* a.e. and which satisfies, for some $\alpha \in \mathbb{R}$:*

$$\begin{cases} \Im\mathrm{m}\, f(0) < \alpha & \textit{and} \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C}\,;\ 0 < \Re\mathrm{e}\, z < \pi\} \cup \{z \in \mathbb{C}\,;\ \Im\mathrm{m}\, z < \alpha\}, \end{cases} \tag{2.6}$$

we have, for all $y \geq \alpha$:

$$m\big(\{z \in \mathbb{T}\,;\ \Im\mathrm{m}\,[f^*(z)] \geq y\}\big) \leq \kappa_1 \mathrm{e}^{\alpha - y}.$$

Proof. Suppose that f satisfies (2.6), and define $f_1(z) = -if(z) + \frac{\pi}{2}i - \alpha$. Then either $\mathfrak{Re}\,[f_1(z)] < 0$, or $-\frac{\pi}{2} < \mathfrak{Im}\,[f_1(z)] < \frac{\pi}{2}$ for every $z \in \mathbb{D}$. Therefore, defining $h(z) = 1 + \exp[f_1(z)]$, we have $h\colon \mathbb{D} \to \mathbb{H}$, that is $\mathfrak{Re}\,[h(z)] > 0$ for every $z \in \mathbb{D}$.

Finally define $h_1(z) = h(z) - i\,\mathfrak{Im}\,[h(0)]$. Then $h_1\colon \mathbb{D} \to \mathbb{H}$ and $h_1(0) \in \mathbb{R}$ (and so $h_1(0) > 0$). Kolmogorov's inequality yields that, for some absolute constant C_1, one has, for every $\lambda > 0$:

$$m\big(\{z \in \mathbb{T}\,;\ |h_1^*(z)| \geq \lambda\}\big) \leq C_1\,\frac{h_1(0)}{\lambda}\,\cdot \tag{2.7}$$

Observe that, since $\mathfrak{Im}\,[f(0)] < \alpha$, we have $\mathfrak{Re}\,[f_1(0)] < 0$, and then:

$$|\mathfrak{Im}\,[h(0)]| < 1 \qquad \text{and} \qquad h_1(0) = \mathfrak{Re}\,[h(0)] < 2. \tag{2.8}$$

Suppose now that, for $y > \alpha$ and $z \in \mathbb{D}$, we have $\mathfrak{Im}\,[f(z)] > y$; then $\exp[f_1(z)] \in \mathbb{H}$, and $|h(z)| \geq |\exp[f_1(z)]| > \mathrm{e}^{y-\alpha}$. Taking radial limits we get, up to a set of null Lebesgue-measure:

$$\{z \in \mathbb{T}\,;\ \mathfrak{Im}\,[f^*(z)] \geq y\} \subseteq \{z \in \mathbb{T}\,;\ |h^*(z)| \geq \mathrm{e}^{y-\alpha}\}.$$

We consider two cases: $\mathrm{e}^{y-\alpha} \geq 2$ and $\mathrm{e}^{y-\alpha} < 2$. When $\mathrm{e}^{y-\alpha} \geq 2$, then $|h^*(z)| \geq \mathrm{e}^{y-\alpha}$ yields:

$$|h_1^*(z)| \geq \mathrm{e}^{y-\alpha} - |\mathfrak{Im}\,[h(0)]| > \mathrm{e}^{y-\alpha} - 1 \geq \frac{1}{2}\,\mathrm{e}^{y-\alpha},$$

by the first part of (2.8). Then, using (2.7) and the second part of (2.8), we have:

$$\begin{aligned} m\big(\{z \in \mathbb{T}\,;\ \mathfrak{Im}\,[f^*(z)] \geq y\}\big) &\leq m\big(\{z \in \mathbb{T}\,;\ |h_1^*(z)| > (1/2)\,\mathrm{e}^{y-\alpha}\}\big) \\ &\leq \frac{2\,C_1 h_1(0)}{\mathrm{e}^{y-\alpha}} \leq \frac{4C_1}{\mathrm{e}^{y-\alpha}}\,, \end{aligned}$$

and, in this case, the lemma is proved, if one takes $\kappa_1 \geq 4C_1$.

When $\mathrm{e}^{y-\alpha} < 2$, then $\mathrm{e}^{\alpha-y} > 1/2$, and, because:

$$m\big(\{z \in \mathbb{T}\,;\ \mathfrak{Im}\,[f^*(z)] \geq y\}\big) \leq 1 < \kappa_1 \mathrm{e}^{\alpha-y},$$

since $\kappa_1 > 2$, the lemma is proved. □

Now, we give a general construction of Schur functions with suitable properties.

Proposition 2.8 *Let $\mathfrak{g}\colon (0,\infty) \to (0,\infty)$ be a continuous non-increasing function such that:*

$$\lim_{t\to 0^+} \mathfrak{g}(t) = +\infty, \qquad \textit{and} \qquad \lim_{t\to+\infty} \mathfrak{g}(t) = 0.$$

Let $\mathfrak{h}\colon (0,\infty) \to (0,\infty]$ *be a lower semicontinuous function such that* $M := \sup\{\mathfrak{h}(t)\,;\ t \geq \pi\} < +\infty$ *and consider the simply connected domain:*

$$\Omega = \{x + iy\,;\ x \in (0,\infty) \quad \textit{and} \quad \mathfrak{g}(x) < y < \mathfrak{g}(x) + \mathfrak{h}(x)\}\,.$$

Let $\mathfrak{f}\colon \overline{\mathbb{D}} \to \overline{\Omega} \cup \{\infty\}$ *be a conformal mapping from* $\mathbb{D}$ *onto* Ω *such that* $\mathfrak{f}(0) = \pi + i(\mathfrak{g}(\pi) + \mathfrak{h}(\pi)/2)$.

Then the symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$ *defined by* $\varphi(z) = \exp[-\mathfrak{f}(z)]$, *for every* $z \in \mathbb{D}$, *satisfies, for some* $\varepsilon_0, k_0 > 0$*:*

1) For all $h \in (0,\varepsilon_0)$*:*

$$m(\{z \in \mathbb{T}\,;\ |\varphi^*(z)| > 1 - h\}) \leq k_0 \exp\big(-\mathfrak{g}(2h)\big)\,. \tag{2.9}$$

2) Assume that, for some $r \in (0,\infty]$ *and integers* $0 \leq n < N \leq \infty$, *one has* $\{\mathfrak{h}(t)\,;\ t \leq r\} \subseteq (2n\pi, 2N\pi]$. *Then, for all* $z \in \mathbb{D}$, *such that* $|z| > \mathrm{e}^{-r}$, *we have* $n \leq n_\varphi(z) \leq N$.

In particular, $\{z \in \mathbb{D}\,;\ |z| > \mathrm{e}^{-r}\} \subseteq \varphi(\mathbb{D}) \subseteq \mathbb{D} \setminus \{0\}$, *when* $n \geq 1$.

Remarks.

1. When $N = 1$, the map φ is univalent.
2. When $r = \infty$ and $n \geq 1$, we have $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$.
3. With $\mathfrak{g}(t) = 1/t$, the operator C_φ is compact on H^{Ψ_2}, therefore belongs to all Schatten classes $S_p(H^2)$, $p > 0$.
4. When $N < \infty$, the operator C_φ is bounded on the Dirichlet space.
5. When $n \geq 1$, the operator C_φ is not compact on the Dirichlet space (since the averages on the windows of the function n_φ cannot uniformly vanish).

Proof of Proposition 2.8. We shall apply Lemma 2.7 with $\alpha = M + \mathfrak{g}(\pi)$.

Suppose that, for $z \in \mathbb{T}$ and $0 < h < 1$, we have $|\varphi^*(z)| > 1 - h$. Then, if h is small enough,

$$\mathrm{e}^{-2h} < 1 - h < |\varphi^*(z)| = \exp\big(-\Re\mathrm{e}\,[f^*(z)]\big),$$

and therefore $2h > \Re\mathrm{e}\,[\mathfrak{f}^*(z)]$. But observe that $\mathfrak{f}^*(z) \in \overline{\Omega} \cup \{\infty\}$, and so, if $2h > \Re\mathrm{e}\,[f^*(z)]$, we necessarily have $\Im\mathrm{m}\,[\mathfrak{f}^*(z)] \geq \mathfrak{g}(2h)$. Again, if h is small enough, we have $y = \mathfrak{g}(2h) > \alpha$, and may apply the lemma to obtain:

$$m\big(\{z \in \mathbb{T}\,;\ |\varphi^*(z)| > 1 - h\}\big) \leq m\big(\{z \in \mathbb{T}\,;\ \Im\mathrm{m}\,[\mathfrak{f}^*(z)] \geq \mathfrak{g}(2h)\}\big) \leq \kappa_1 \mathrm{e}^{\alpha - \mathfrak{g}(2h)}.$$

We get (2.9).

On the other hand, let $Z \in \mathbb{D}$ such that $|Z| > \mathrm{e}^{-r}$, we can write $Z = \mathrm{e}^{-x}\mathrm{e}^{i\theta}$ with $x < r$. We can find $\theta_i's$ such that $\mathfrak{g}(x) < \theta_1 < \ldots < \theta_s < \mathfrak{g}(x) + \mathfrak{h}(x)$ and $\theta_j \equiv \theta[2\pi]$ with $n \leq s \leq N$. For each j, there exists a unique $z_j \in \mathbb{D}$, such that $\Re\mathrm{e}\,\mathfrak{f}(z_j) = x$ and $\Im\mathrm{m}\,\mathfrak{f}(z_j) = \theta_j$; hence $\varphi(z_j) = Z$. Moreover no other $z \in \mathbb{D}$ can satisfy $\varphi(z) = Z$. Hence $n_\varphi(Z) = s$. $\square$

Proof of Theorem 2.6. As said before, if one takes $\mathfrak{g}(t) = 1/t$ in Proposition 2.8, then C_φ is compact on H^{Ψ_2} and hence is in all Schatten classes $S_p(H^2)$, $p > 0$. On the other hand, if one choose also $\mathfrak{h}(t) = 1/t$, then, for every $r > 0$, $\{\mathfrak{h}(t)\,;\ t \le r\} = [1/r, \infty)$ and for $|z| > \mathrm{e}^{-r}$, we get that $n_\varphi(z) \ge [1/(2\pi r)]$ (the integer part of $1/(2\pi r)$). It follows that, for some constant $c > 0$, one has, with $\mathrm{e}^{-r} = 1 - h$:

$$\frac{1}{A[W(\xi,h)]} \int_{W(\xi,h)} n_\varphi(z)\, dA(z) \ge c\, \frac{1}{\log[1/(1-h)]} \underset{h\to 0}{\longrightarrow} \infty\,.$$

Therefore, C_φ is not bounded on $\mathcal{D}_*$, by (1.5). $\square$

Remarks. 1. Actually, as we may take $\mathfrak{g}$ growing as we wish, the proof shows, using [12], Theorem 4.18, that for every Orlicz function Ψ, one can find a Schur function φ such that C_φ is not bounded on $\mathcal{D}_*$, though compact on the Hardy-Orlicz space H^Ψ.

2. This construction also allows to produce a *univalent* map φ, with an arbitrary small Carleson function $\rho_\varphi(h) = \sup_{|\xi|=1} m\big(\{\mathrm{e}^{it}\,;\ \varphi^*(\mathrm{e}^{it}) \in W(\xi,h)\}\big)$, and such that C_φ is not compact on the Dirichlet space (note we cannot replace "compact" by "bounded" since any Schur function with a bounded valence is bounded on the Dirichlet space).

Indeed, take $\mathfrak{h}(t) = 2\pi$ and $\mathfrak{g}$ be $\mathcal{C}^1$: $\mathfrak{g}(t) = 1/t$ for instance. We have $N = 1$ and so φ is univalent. Now it suffices to notice that the range of the curve

$$\Gamma = \big\{\mathrm{e}^{-x-i\mathfrak{g}(x)}\,;\ x \in (0,\infty)\big\} = \big\{\big(t\cos(1/\ln(t)), t\sin(1/\ln(t))\big)\,;\ t \in (0,1)\big\} \subseteq \mathbb{D}$$

has a null area measure. The range of φ is $\mathbb{D} \setminus (\Gamma \cup \{0\})$ and for each $w \notin \Gamma$, we have $n_\varphi(w) = 1$ Then, for $h \in (0,1)$, we have:

$$\begin{aligned}\frac{1}{h^2} \int_{W(1,h)} n_\varphi(w)\, dA(w) &= \frac{1}{h^2} \int_{W(1,h)\setminus\Gamma} dA(w) = \frac{1}{h^2}\, A[W(1,h) \setminus \Gamma] \\ &= \frac{1}{h^2}\, A[W(1,h)] \approx 1\,,\end{aligned}$$

and so C_φ in not compact on $\mathcal{D}_*$, by (1.7). $\square$

3 Composition operators in Schatten classes

3.1 Characterization

In this section, we give a characterization of the membership in the Schatten classes of composition operators on $\mathcal{D}_*$. This characterization will be deduced from Luecking's one for composition operators on the Bergman space. Actually, we shall give it for weighted Dirichlet spaces $\mathcal{D}_{\alpha,*}$. Boundedness and compactness has been characterized by B. McCluer and J. Shapiro in [21] and, in other terms, by N. Zorboska in [29].

Recall that for $\alpha > -1$, the weighted Dirichlet space $\mathcal{D}_\alpha$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \, (1 - |z|^2)^\alpha \, dA(z) < \infty \, . \tag{3.1}$$

This is a Hilbert space for the norm given by:

$$\|f\|_\alpha^2 = |f(0)|^2 + (\alpha + 1) \int_{\mathbb{D}} |f'(z)|^2 \, (1 - |z|^2)^\alpha \, dA(z) < \infty \, . \tag{3.2}$$

The standard Dirichlet space $\mathcal{D}$ corresponds to $\alpha = 0$; the Hardy space H^2 to $\alpha = 1$ and the standard Bergman space to $\alpha = 2$. For more general weights, see [10].

We denote by $\mathcal{D}_{\alpha,*}$ the subspace of the $f \in \mathcal{D}_\alpha$ such that $f(0) = 0$.

If φ is a Schur function, one defines its *weighted Nevanlinna counting function* $N_{\varphi,\alpha}$ at $w \in \Omega := \varphi(\mathbb{D})$ as the number of pre-images of w with the weight $(1 - |z|)^\alpha$:

$$N_{\varphi,\alpha}(w) = \sum_{\varphi(z) = w} (1 - |z|^2)^\alpha \, . \tag{3.3}$$

For $w \in \mathbb{D} \setminus \varphi(\mathbb{D})$, we set $N_{\varphi,\alpha}(w) = 0$. One has $N_{\varphi,1} = N_\varphi$ and $N_{\varphi,0} = n_\varphi$.

With this notation, recall the change of variable formula:

$$\int_{\mathbb{D}} F[\varphi(z)] \, |\varphi'(z)|^2 \, (1 - |z|^2)^\alpha \, dA(z) = \int_\Omega F(w) \, N_{\varphi,\alpha}(w) \, dA(w) \, . \tag{3.4}$$

Denote by $R_{n,j}$, $n \geq 0$, $0 \leq j \leq 2^n - 1$, the Hastings-Luecking windows:

$$R_{n,j} = \Big\{ z \in \mathbb{D} \, ; \ 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \quad \text{and} \quad \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \Big\} \, .$$

We can now state.

Theorem 3.1 *Let $\alpha > -1$. Let φ be a Schur function and $p > 0$. Then $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ if and only if:*

$$\sum_{n=0}^\infty \sum_{j=0}^{2^n - 1} \bigg[2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w) \, dA(w) \bigg]^{p/2} < \infty \, . \tag{3.5}$$

If φ is univalent, (3.5) *can be replaced by the purely geometric condition:*

$$\sum_{n=0}^\infty \sum_{j=0}^{2^n - 1} \big[2^{n(\alpha+2)} A_\alpha(R_{n,j} \cap \Omega) \big]^{p/2} < \infty \, , \tag{3.6}$$

where A_α is the weighted measure $dA_\alpha(w) = (\alpha + 1) \, (1 - |w|^2)^\alpha dA(w)$.

Remark. Of course, every operator in a Schatten class is compact, but we may note that condition (3.5) implies the compactness of C_φ, by [29], Theorem 1 (and [13], Proposition 3.3).

Proof of Theorem 3.1. First, we compute $C_\varphi^* C_\varphi$. Let us fix f and g in the Dirichlet space $\mathcal{D}_{\alpha,*}$. We have:

$$\begin{aligned}(\alpha+1)\int_{\mathbb{D}} \big((C_\varphi^* C_\varphi)(f)\big)'(z)\,\overline{g'(z)}\,(1-|z|^2)^\alpha\,dA(z) &= \big\langle f\circ\varphi, g\circ\varphi\big\rangle_{\mathcal{D}_{\alpha,*}} \\ &= (\alpha+1)\int_{\mathbb{D}} (f'\circ\varphi)(z)\overline{(g'\circ\varphi)(z)}\,|\varphi'(z)|^2\,(1-|z|^2)^\alpha\,dA(z).\end{aligned}$$

By the change of variable formula, we get:

$$\int_{\mathbb{D}} \big((C_\varphi^* C_\varphi)(f)\big)'(z)\overline{g'(z)}\,(1-|z|^2)^\alpha dA = \int_{\mathbb{D}} f'(w)\,\overline{g'w)}\,N_{\varphi,\alpha}(w)\,dA(w)\,,$$

which is equivalent to:

$$\int_{\mathbb{D}} \big((C_\varphi^* C_\varphi)(f)\big)'(z)\,\overline{G(z)}\,(1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} f'(w)\,\overline{G(w)}\,N_{\varphi,\alpha}(w)\,dA(w)$$

for every function G belonging to the weighted Bergman space $\mathfrak{B}_\alpha^2$.

That means that $\big((C_\varphi^* C_\varphi)(f)\big)' - f'.N_{\varphi,\alpha}/(1-|w|^2)^\alpha$ is orthogonal to the weighted Bergman space $\mathfrak{B}_\alpha^2$. But $\big((C_\varphi^* C_\varphi)(f)\big)' \in \mathfrak{B}_\alpha^2$. Hence $\big((C_\varphi^* C_\varphi)(f)\big)'$ is the orthogonal projection onto $\mathfrak{B}_\alpha^2$ of the function $f'.N_{\varphi,\alpha}/(1-|w|^2)^\alpha$. Thus (see [27], § 6.4.1), we obtain that for every $z\in\mathbb{D}$:

$$\begin{aligned}\big((C_\varphi^* C_\varphi)(f)\big)'(z) &= (\alpha+1)\int_{\mathbb{D}} \frac{f'(w)}{(1-\bar{w}z)^{\alpha+2}}\,\frac{N_{\varphi,\alpha}(w)}{(1-|w|^2)^\alpha}\,(1-|w|^2)^\alpha\,dA(w) \\ &= (\alpha+1)\int_{\mathbb{D}} \frac{f'(w)}{(1-\bar{w}z)^{\alpha+2}}\,d\mu(w) \\ &= (\alpha+1)\,T_\mu(f')(z)\,,\end{aligned}$$

where μ is the positive measure A with weight $N_{\varphi,\alpha}$ and T_μ is the Toeplitz operator on $\mathfrak{B}_\alpha^2$ is introduced in [19] (let us point out that α in [19] corresponds to $-(\alpha+1)$ in our work).

In other words, introducing the map $\Delta(h)=h'$, which is an isometry from $\mathcal{D}_{\alpha,*}$ onto $\mathfrak{B}_\alpha^2$, we have $\Delta\circ(C_\varphi^* C_\varphi) = T_\mu\circ\Delta$. We have the following diagram:

$$\begin{tikzcd} \mathcal{D}_{\alpha,*} \arrow[r, "C_\varphi^* C_\varphi"] \arrow[d, "\Delta"'] & \mathcal{D}_{\alpha,*} \arrow[d, "\Delta"] \\ \mathfrak{B}_\alpha^2 \arrow[r, "T_\mu"] & \mathfrak{B}_\alpha^2 \end{tikzcd}$$

Hence the approximation numbers of T_μ (viewed as an operator on $\mathfrak{B}_\alpha^2$) and the ones of $C_\varphi^* C_\varphi$ (viewed as an operator on $\mathcal{D}_{\alpha,*}$) are the same. In particular, the membership in the Schatten classes are the same and the final result follows from the main theorem in [19]: $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ if and only if $C_\varphi^* C_\varphi \in S_{p/2}(\mathcal{D}_{\alpha,*})$ and that holds if and only if:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \big[2^{n(\alpha+2)} \mu(R_{n,j})\big]^{p/2} < \infty\,.$$

Hence $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ if and only if:

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \bigg[2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w)\, dA(w)\bigg]^{p/2} < \infty\,,$$

and that ends the proof of Theorem 3.1. $\square$

Remark. In the same way, we can obtain other characterizations for $\mathcal{D}_{\alpha,*}$ by using the ones for $\mathfrak{B}_\alpha^2$ given in [20] and [28]: $C_\varphi \in S_p(\mathfrak{B}_\alpha^2)$ if and only if $N_{\varphi,\alpha+2}(z)/\big(\log(1/|z|)\big)^{\alpha+2} \in L^{p/2}(\lambda)$, where $d\lambda(z) = (1-|z|^2)^{-2} dA(z)$ is the Möbius invariant measure on $\mathbb{D}$, and, when φ has bounded valence and $p \geq 2$, if and only if $(1-|z|^2)/\big(1-|\varphi(z)|^2\big) \in L^{p(\alpha+2)/2}(\lambda)$. Such a result can be found in [26].

3.2 Applications

We give several applications of the previous theorem.

Corollary 3.2 *Let $-1 < \alpha \leq \beta$, $p > 0$, and φ be a Schur function. Then $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$ implies that $C_\varphi \in S_p(\mathcal{D}_{\beta,*})$.*

In particular, $C_\varphi \in S_p(\mathcal{D}_*)$ implies that $C_\varphi \in S_p(H^2)$, which in turn implies that $C_\varphi \in S_p(\mathfrak{B}^2)$.

Proof. Assume that $C_\varphi \in S_p(\mathcal{D}_{\alpha,*})$. Then

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \bigg[2^{n(\alpha+2)} \int_{R_{n,j}} N_{\varphi,\alpha}(w)\, dA(w)\bigg]^{p/2} < \infty.$$

Since, thanks to Schwarz's lemma, $N_{\varphi,\beta}(w) \leq N_{\varphi,\alpha}(w)(1-|w|^2)^{\beta-\alpha}$, we have

$$N_{\varphi,\beta}(w) \leq (2.2^{-n})^{\beta-\alpha} N_{\varphi,\alpha}(w) \quad \text{for } w \in R_{n,j}.$$

It follows that

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \bigg[2^{n(\beta+2)} \int_{R_{n,j}} N_{\varphi,\beta}(w)\, dA(w)\bigg]^{p/2} < \infty\,,$$

and that proves Corollary 3.2. $\square$

It is known ([13]) that composition operators on H^2 separate Schatten classes, but the difficulty is that we must not only control the shape of $\varphi(\partial\mathbb{D})$, but also the parametrization $t \mapsto \varphi(\mathrm{e}^{it})$, even if φ is univalent. In the case of the Dirichlet space, this difficulty disappears, because only the areas come into play, and we can easily prove the following result.

Theorem 3.3 *The composition operators on $\mathcal{D}_*$ separate Schatten classes, in the following sense. Let $0 < p_1 < \infty$. Then, there exists a symbol φ such that:*

$$C_\varphi \in \Big(\bigcap_{p>p_1} S_p(\mathcal{D}_*)\Big) \setminus S_{p_1}(\mathcal{D}_*) \,.$$

Similarly, there exists a symbol φ such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \Big(\bigcup_{p<p_1} S_p(\mathcal{D}_*)\Big) \,.$$

In particular, for every $0 < p_1 < p_2 < \infty$, there exists φ such that $C_\varphi \in S_{p_2}(\mathcal{D}_*) \setminus S_{p_1}(\mathcal{D}_*)$.

Proof. Let $(h_n)_{n\geq 1}$, with $0 < h_n < 1$, be a sequence of real numbers with limit 0 to be adjusted, and J the Jordan curve formed by the segment $[0, 1]$ and the north and (truncated) north-east sides of the curvilinear rectangles

$$\{1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\} \times \{0 \leq \arg z < 2^{-n} h_n\}.$$

Let Ω_0 be the interior of J and $\Omega = \Omega_0 \cup D(0, 1/8)$. Let $\varphi\colon \mathbb{D} \to \Omega$ be a Riemann map such that $\varphi(0) = 0$. Since φ is univalent and bounded, it defines a symbol on $\mathcal{D}_*$, and the necessary and sufficient condition (3.6) for membership in $S_p(\mathcal{D}_*)$ reads:

$$\sum_{n=0}^{\infty} [4^n 4^{-n} h_n]^{p/2} = \sum_{n=0}^{\infty} {h_n}^{p/2} < \infty. \tag{3.7}$$

Indeed, it is clear that, for fixed n, the Hastings-Luecking windows $R_{n,j}$ satisfy:

$$R_{n,0} \cap \Omega \neq \emptyset; \quad R_{n,j} \cap \Omega = \emptyset \text{ for } 1 \leq j < 2^n.$$

Therefore, only the Hastings-Luecking windows $R_{n,0}$ matter. Since:

$$A(R_{n,0} \cap \Omega) = \iint_{1-2^{-n}\leq r<1-2^{-n-1},\ 0\leq\theta<2^{-n}h_n} r\, dr\, d\theta \approx 4^{-n} h_n \,,$$

we can test the criterion (3.7). Now, it is enough to take $h_n = (n + 1)^{-2/p_1}$ to get:

$$C_\varphi \in \Big(\bigcap_{p>p_1} S_p(\mathcal{D}_*)\Big) \setminus S_{p_1}(\mathcal{D}_*) \,.$$

Similarly, the choice $h_n = (n + 1)^{-2/p_1} [\log(n + 2)]^{-4/p_1}$, gives a symbol φ such that:

$$C_\varphi \in S_{p_1}(\mathcal{D}_*) \setminus \Big(\bigcup_{p<p_1} S_p(\mathcal{D}_*)\Big) \,.$$

This ends the proof. $\square$

T. Carroll and C. Cowen ([3] proved, but only for $\alpha > 0$, that there exist compact composition operators on $\mathcal{D}_\alpha$ which are in no Schatten class (see also [8]). In the next result, we shall see that this still true for $\alpha = 0$.

Theorem 3.4 *There exists a Schur function φ such that C_φ is compact on $\mathcal{D}_*$, but in no Schatten class $S_p(\mathcal{D}_*)$.*

Proof. It suffices to use the proof of Theorem 3.3 and to take, instead of the above h_n, $h_n = 1/\ln(n+2)$. $\square$

For the next application, which will be used in Section 4, we need to recall the definition of the cusp map χ, introduced in [15], and later used, with a slightly different definition in [18]. Actually, we have to modify it slightly again in order to have $\chi(0) = 0$. We first define:

$$\chi_0(z) = \frac{\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} - i}{-i\Big(\dfrac{z-i}{iz-1}\Big)^{1/2} + 1} \, ,$$

then:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi}\,\chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)} \, ,$$

and finally:

$$\chi(z) = 1 - \chi_3(z) \, ,$$

where $a = 1 - \frac{2}{\pi}\log(\sqrt{2} - 1) \in (1, 2)$ is chosen in order that $\chi(0) = 0$. The image Ω of the (univalent) cusp map is formed by the intersection of the inside of the disk $D\big(\frac{a}{2}, \frac{a}{2}\big)$ and the outside of the two disks $D\big(\frac{ia}{2}, \frac{a}{2}\big)$ and $D\big(-\frac{ia}{2}, \frac{a}{2}\big)$.

Corollary 3.5 *If χ is the cusp map, then C_χ belongs to all Schatten classes $S_p(\mathcal{D}_*)$, $p > 0$.*

Proof. Since χ is univalent, $\chi(0) = 0$, and $\Omega = \chi(\mathbb{D})$ has finite area, we have $\chi \in \mathcal{D}_*$. A little elementary geometry shows that, for some constant C, we have:

$$w \in \Omega,\ 0 < h < 1 \text{ and } |w| \geq 1 - h \quad \Longrightarrow \quad |\Im\mathrm{m}\, w| \leq C h^2. \tag{3.8}$$

It follows (changing C if necessary) that $R_{n,j} \cap \Omega$ is contained in a rectangle of sizes 2^{-n} and $C\,4^{-n}$ and with area $C\,8^{-n}$. Hence, for a given n, at most C of the Hastings-Luecking windows $R_{n,j}$ can intersect Ω. Therefore, the series in Theorem 3.1 reduces, up to constants, to the series:

$$\sum_{n=0}^{\infty} (4^n 8^{-n})^{p/2} = \sum_{n=0}^{\infty} 2^{-np} \, ,$$

which converges for every $p > 0$. $\square$

4 Logarithmic capacity and set of contact points

In view of the result of [6] mentioned in the introduction, if $\operatorname{Cap} K > 0$, there is no hope to find a symbol φ such that $E_\varphi = K$ and C_φ is Hilbert-Schmidt on $\mathcal{D}_*$. But as was later proved in [5], $\operatorname{Cap} K > 0$ is the only obstruction. We can improve on the results from [5] as follows: our composition operator is not only Hilbert-Schmidt, but in any Schatten class; moreover, we can replace $E_\varphi = K$ by $E_\varphi = E_\varphi(1) = K$.

Theorem 4.1 *For every compact set K of the unit circle $\mathbb{T}$ with logarithmic capacity $\operatorname{Cap} K = 0$, there exists a Schur function φ with the following properties:*

1) $\varphi \in A(\mathbb{D}) \cap \mathcal{D}_* := A$, the "Dirichlet algebra";

2) $E_\varphi = E_\varphi(1) = K$;

3) $C_\varphi \in \bigcap_{p>0} S_p(\mathcal{D}_*)$.

In fact, the approximation numbers of C_φ satisfy $a_n(C_\varphi) \leq a \exp(-b\sqrt{n})$.

This theorem actually results of the particular following case and the properties of the cusp map seen in Section 3.2.

Theorem 4.2 *For every compact set $K \subseteq \partial\mathbb{D}$ of logarithmic capacity $\operatorname{Cap} K = 0$, there exists a Schur function $q \in A(\mathbb{D}) \cap \mathcal{D}_*$ which peaks on K and such that the composition operator $C_q \colon \mathcal{D}_* \to \mathcal{D}_*$ is bounded (and even Hilbert-Schmidt).*

Recall that a function $q \in A(\mathbb{D})$, the disk algebra, is said to *peak* on a compact subset $K \subseteq \partial\mathbb{D}$ (and is called a *peaking function*) if:

$$q(z) = 1 \text{ if } z \in K\,; \quad |q(z)| < 1 \text{ if } z \in \overline{\mathbb{D}} \setminus K\,.$$

Proof of Theorem 4.1. We simply take for φ the composed map $\varphi = \chi \circ q$, where χ is the cusp map and q our peaking function. Recall that $\chi \in A(\mathbb{D})$ and that χ peaks on $\{1\}$. We take advantage of this fact by composing with q, for which $C_q \colon \mathcal{D}_* \to \mathcal{D}_*$ is bounded as well as C_χ (since χ is univalent). We clearly have $\varphi \in A(\mathbb{D})$, $\varphi(z) = \chi(1) = 1$ for $z \in K$, and $|\varphi(z)| < 1$ for $z \notin K$, since then $|q(z)| < 1$. Therefore $E_\varphi(1) = K$. Moreover, C_φ being bounded on $\mathcal{D}_*$, we have in particular $\varphi = C_\varphi(z) \in \mathcal{D}_*$. Since $C_\varphi = C_q \circ C_\chi$, we get 3), by Corollary 3.5.

In [16], we prove that $a_n(C_\chi) \leq a \exp(-b\sqrt{n})$. Since $a_n(C_\varphi) \leq \|C_q\|\, a_n(C_\chi)$, by the ideal property of approximation numbers, this ends the proof of Theorem 4.1. $\square$

In turn, the proof of Theorem 4.2 relies on the following crucial lemma.

Lemma 4.3 *Let $K \subseteq \partial\mathbb{D}$ be a compact set such that $\operatorname{Cap} K = 0$. Then, there exists a function $U\colon \overline{\mathbb{D}} \to \mathbb{R}^+ \cup \{\infty\}$, such that:*

1) $U(z) = \infty$ if and only if $z \in K$;
2) $U \geq 1$ on $\overline{\mathbb{D}}$;
3) U is continuous on $\overline{\mathbb{D}} \setminus K$, harmonic in $\mathbb{D}$ and $\int_{\mathbb{D}} |\nabla U|^2 \, dA < \infty$;
4) $\lim_{z \to K,\ z \in \overline{\mathbb{D}}} U(z) = \infty$;
5) the conjugate function $V = \tilde{U}$ is continuous on $\overline{\mathbb{D}} \setminus K$.

Proof of Theorem 4.2. Taking this lemma for granted, let us end the proof of the theorem. We set $f = U + iV$, $a = \mathrm{e}^{-1/f(0)}$ and $q = \varphi_a \circ \mathrm{e}^{-1/f}$, where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. In view of the third and fourth items of the lemma, we have $q \in A(\mathbb{D})$. Since $U \geq 1$, Lemma 2.4 shows that C_q is Hilbert-Schmidt on $\mathcal{D}_*$. Moreover, for $z \in K$, one has $f(z) = \infty$ and hence $q(z) = 1$ since $\varphi_a(1) = 1$ because $a \in \mathbb{R}$ (since $f(0) = U(0)$). On the other hand, when $z \notin K$, one has $|f(z)| < \infty$ and hence $|q(z)| < 1$. Therefore q peaks on K. $\square$

Proof of Lemma 4.3. This proof is strongly influenced by that of Theorem III, page 47, in [9]. Let:

$$L(z) = \log\Big(\frac{\mathrm{e}}{1-z}\Big) = P(z) + i\,Q(z), \tag{4.1}$$

with

$$P(z) = \log\frac{\mathrm{e}}{|1-z|} \text{ and } Q(z) = -\arg(1-z),\ |Q(z)| \leq \frac{\pi}{2}, \quad z \in \overline{\mathbb{D}} \setminus \{1\},$$

and write:

$$P(z) \sim \sum_{n\in\mathbb{Z}} \gamma_n \, z^n\,,$$

with

$$\gamma_n = 1/(2\,|n|) \quad \text{if } n \neq 0\,, \quad \text{and } \gamma_0 = 1\,.$$

For $0 < \varepsilon < 1/2$, let $K_\varepsilon = \{z \in \mathbb{T};\ \operatorname{dist}(z, K) \leq \varepsilon\}$, μ_ε its equilibrium measure, and U_ε the logarithmic potential of μ_ε, that is:

$$U_\varepsilon(z) = \int_{K_\varepsilon} \log\frac{\mathrm{e}}{|z-w|}\, d\mu_\varepsilon(w)\,,$$

that we could as well write (since $K_\varepsilon \subseteq \mathbb{T}$):

$$U_\varepsilon(z) = \int_{K_\varepsilon} P(z\,\bar{w})\, d\mu_\varepsilon(w)\,.$$

Let us set:

$$f_\varepsilon(z) = \int_{K_\varepsilon} L(z\,\bar{w})\, d\mu_\varepsilon(w) = U_\varepsilon(z) + iV_\varepsilon(z)\,, \tag{4.2}$$

with

$$V_\varepsilon(z) = \int_{K_\varepsilon} Q(z\,\bar{w})\,d\mu_\varepsilon(w)\,.$$

Then, if I_ε is the energy of μ_ε, one has (see [23], Section 4) $I_\varepsilon = 1+\sum_{n=1}^{\infty} \frac{|\widehat{\mu_\varepsilon}(n)|^2}{n}$, where $\widehat{\mu_\varepsilon}(n) = \int_{\mathbb{T}} \overline{w}^{\,n}\,d\mu_\varepsilon(w)$ is the n-th Fourier coefficient of μ_ε, and:

$$f_\varepsilon \in \mathcal{D} \quad \text{and} \quad \|f_\varepsilon\|_{\mathcal{D}}^2 = I_\varepsilon\,. \tag{4.3}$$

Note that $\|f_\varepsilon\|_{\mathcal{D}} \geq 1$.

We claim that there exist $\delta > 0$ and $0 < r < 1$ such that:

$$z \in \overline{\mathbb{D}} \text{ and } \operatorname{dist}(z, K) \leq \delta \quad \Longrightarrow \quad U_\varepsilon(rz) \geq I_\varepsilon/2 \tag{4.4}$$

Indeed, let $P_a(t) = \frac{1-|a|^2}{|\mathrm{e}^{it}-a|^2}$ be the Poisson kernel at $a \in \mathbb{D}$. Since U_ε is harmonic in $\mathbb{D}$ and integrable on $\mathbb{T}$ ([4], Proposition 19.5.2), one has, for every $z \in \mathbb{D}$:

$$U_\varepsilon(z) = \int_{-\pi}^{\pi} U_\varepsilon(\mathrm{e}^{it})\,P_z(t)\,\frac{dt}{2\pi}\,. \tag{4.5}$$

Let now $\delta \leq \varepsilon/4$, to be adjusted later, and take $1-\delta \leq r < 1$. Suppose that $\operatorname{dist}(z, K) \leq \delta$, with $z \in \overline{\mathbb{D}}$, and let $u \in K$ such that $|z-u| \leq \varepsilon/4$. Note that then $|rz-u| \leq (1-r) + |z-u| \leq \varepsilon/2$. It follows from (4.5) that:

$$I_\varepsilon - U_\varepsilon(r\,z) = \int_{-\pi}^{\pi} [I_\varepsilon - U_\varepsilon(\mathrm{e}^{it})]\,P_{rz}(t)\,\frac{dt}{2\pi}$$

(it is useful to recall that $U_\varepsilon(z) \leq I_\varepsilon$ for every $z \in \mathbb{C}$). Set:

$$J_1 = \int_{|\mathrm{e}^{it}-rz|\leq\varepsilon/2} [I_\varepsilon - U_\varepsilon(\mathrm{e}^{it})]\,P_{rz}(t)\,\frac{dt}{2\pi}$$

and

$$J_2 = \int_{|\mathrm{e}^{it}-rz|>\varepsilon/2} [I_\varepsilon - U_\varepsilon(\mathrm{e}^{it})]\,P_{rz}(t)\,\frac{dt}{2\pi}\,.$$

For the integral J_1, we have:

$$|\mathrm{e}^{it}-u| \leq |\mathrm{e}^{it}-rz| + |rz-u| \leq \varepsilon\,;$$

therefore $\mathrm{e}^{it} \in K_\varepsilon$. Since $U_\varepsilon = I_\varepsilon$ Lebesgue-almost everywhere on K_ε, by Frostman's Theorem, we get $J_1 = 0$.

For the integral J_2, we have:

$$P_{rz}(t) \leq \frac{2(1-r\,|z|)}{(\varepsilon/2)^2} \leq 2\,\frac{(1-r)+r(1-|z|)}{(\varepsilon/2)^2} \leq \frac{4\delta}{(\varepsilon/2)^2} = \frac{16\delta}{\varepsilon^2}\,;$$

hence (since $U_\varepsilon(\mathrm{e}^{it}) \geq 0$):

$$J_2 \leq \frac{16\delta}{\varepsilon^2}\,I_\varepsilon\,.$$

Therefore, if we choose $0 < \delta \le \varepsilon^2/32$, we get:

$$0 \le I_\varepsilon - U_\varepsilon(r\,z) \le I_\varepsilon/2\,,$$

which gives (4.4). $\square$

Now, as $\operatorname{Cap} K = 0$, we know from (1.9) that $\lim_{\varepsilon\to 0^+} I_\varepsilon = \infty$, and we can adjust a sequence $\varepsilon_j \to 0^+$ so that:

$$I_{\varepsilon_j} \ge 4\,j^6\,. \tag{4.6}$$

Using (4.4), we find two sequences $(\delta_j)_j$ and $(r_j)_j$, with $0 < \delta_j \to 0$ and $1 > r_j \to 1$, such that, for every $j \ge 1$,

$$z \in \overline{\mathbb{D}} \text{ and } \operatorname{dist}(z, K) \le \delta_j \quad \Longrightarrow \quad U_{\varepsilon_j}(r_j z) \ge I_{\varepsilon_j}/2. \tag{4.7}$$

Finally, let us set:

$$f_j(z) = f_{\varepsilon_j}(r_j z) \tag{4.8}$$

and

$$f = U + iV = 1 + \sum_{j=1}^{\infty} j^{-2} \frac{f_j}{\|f_j\|_{\mathcal{D}}}\,\cdot \tag{4.9}$$

The series defining f is absolutely convergent in $\mathcal{D}$. Note that $f(0)$ is real.

We now have:

1) f is continuous on $\overline{\mathbb{D}} \setminus K$.

Indeed, let $z \in \overline{\mathbb{D}}\setminus K$. Then, $\operatorname{dist}(z, K) > 0$ and there exists a neighbourhood ω of z in $\overline{\mathbb{D}}$, an integer $j_0 = j_0(z)$ and a positive number $\delta > 0$ such that:

$$w \in \omega \text{ and } j \ge j_0 \quad \Longrightarrow \quad \operatorname{dist}(r_j w, K_{\varepsilon_j}) \ge \delta.$$

We then have, for $w \in \omega$ and $j \ge j_0$:

$$\begin{aligned} |f_{\varepsilon_j}(w)| &= \left| \int_{K_{\varepsilon_j}} \log \frac{\mathrm{e}}{r_j w - u}\, d\mu_{\varepsilon_j}(u) \right| \\ &\le \int_{K_{\varepsilon_j}} \left(\log \frac{\mathrm{e}}{|r_j w - u|} + \frac{\pi}{2} \right) d\mu_{\varepsilon_j}(u) \le \log \frac{\mathrm{e}}{\delta} + \frac{\pi}{2} := C\,, \end{aligned}$$

since μ_{ε_j} is a probability measure supported by K_{ε_j}. Therefore, the series defining f is normally convergent on ω since its general term is dominated by $j^{-2}C$ on ω. Since the functions f_j are continuous on $\overline{\mathbb{D}}$, this shows that f is continuous at z.

2) $U(z) := \mathfrak{Re}\, f(z) \ge 1$.

This is obvious since, for every $z \in \overline{\mathbb{D}}$,

$$U_\varepsilon(z) := \mathfrak{Re}\, f_\varepsilon(z) = \int_{K_\varepsilon} \log \frac{\mathrm{e}}{|z - u|}\, d\mu_\varepsilon(u) \ge 0\,.$$

3) $\lim_{z\to K, z\in\overline{\mathbb{D}}} U(z) = \infty$.

Indeed, let $A > 0$. Take an integer $j \geq A$ and suppose that $\operatorname{dist}(z, K) \leq \delta_j$. Then, using the positivity of the U_{ε_k}'s as well as (4.3), (4.6) and (4.7), we have:

$$U(z) \geq j^{-2} \frac{U_{\varepsilon_j}(r_j z)}{\|f_{\varepsilon_j}\|_{\mathcal{D}}} \geq j^{-2} \frac{I_{\varepsilon_j}/2}{\sqrt{I_{\varepsilon_j}}} \geq j \geq A.$$

This ends the proof of our claims, and of Lemma 4.3. □

To end this paper, let us mention the following version of the classical Rudin-Carleson Theorem. Though it is not the main subject of this paper, it has the same flavor as Theorem 4.2. We do not give a proof, but only mention that it can be obtained by mixing the proofs of Theorems III.E.2 and III.E.6 in [25] (see pages 181–187).

Theorem 4.4 *Let K be a compact subset of $\mathbb{T}$ with $\operatorname{Cap} K = 0$. Given any continuous strictly positive function $s \in C(\mathbb{T})$ equal to 1 on K, we can find, for every $h \in C(K)$ and every $\varepsilon > 0$, a function $f \in A(\mathbb{D}) \cap \mathcal{D}$ such that $f_{|K} = h$ and:*

$$|f(\theta)| \leq (1+\varepsilon)\,\|h\|_\infty\, s(\theta)\,,\ \forall \theta \in \mathbb{T}\ ; \qquad \|f\|_{\mathcal{D}} \leq (1+\varepsilon)\,\|h\|_\infty\,.$$

References

[1] N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick, The Dirichlet space: a survey, *New York J. Math.* 17A (2011), 45–86.

[2] A. Beurling, Sur les ensembles exceptionnels, *Acta Math.* 72 (1940), 1–13.

[3] T. Carroll and C. C. Cowen, Compact composition operators not in the Schatten classes, *J. Operator Theory* 26, No. 1 (1991), 109–120.

[4] J. B. Conway, Functions of One Complex Variable II, Graduate Texts in Math. 159, Springer-Verlag (1995).

[5] O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi, Level sets and composition operators on the Dirichlet space, *J.Funct.Anal.* 260, No. 6 (2011), 1721–1733.

[6] E. A. Gallardo-Gutiérrez and M. J. González, Exceptional sets and Hilbert-Schmidt composition operators, *J. Funct. Anal.* 199, No. 2 (2003), 287–300.

[7] E. A. Gallardo-Gutiérrez and M. J. González, Hausdorff measures, capacities and compact composition operators, *Math. Z.* 253, No. 1 (2006), 63–74.

[8] M. M. Jones, Compact composition operators not in the Schatten classes, *Proc. Amer. Math. Soc.* 134, No. 7 (2006), 1947–1953.

[9] J. P. Kahane and R. Salem, Ensembles parfaits et séries trigonométriques, *nouvelle édition*, *Hermann* (1994).

[10] K. Kellay and P. Lefèvre, Compact composition operators on weighted Hilbert spaces of analytic functions, Journ. Math. Anal. Appl. 386 (2) (2012), 718–727.

[11] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on H^2 and Hardy-Orlicz spaces, *J. Math. Anal. Appl.* 354 (2009), 360–371.

[12] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, *Memoirs Amer. Math. Soc.* 207 (2010), No. 974.

[13] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some examples of compact composition operators on H^2, *J. Funct. Anal.* 255, No.11 (2008), 3098–3124.

[14] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Nevanlinna counting function and Carleson function of analytic maps, Math. Ann. 351 (2011), 305–326.

[15] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on Bergman-Orlicz spaces, *preprint 2009*

[16] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Approximation numbers of composition operators on the Dirichlet space, *in preparation.*

[17] D. Li, Compact composition operators on Hardy-Orlicz and Bergman-Orlicz spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM* 105, no. 2 (2011), 247–260.

[18] D. Li, H. Queffélec and L. Rodríguez-Piazza, Estimates for approximation numbers of some classes of composition operators, *preprint.*

[19] D. H. Luecking, Trace ideal criteria for Toeplitz operators, *J. Funct. Anal.* 73 (1987), 345–368.

[20] D. H. Luecking and K. H. Zhu, Composition operators belonging to the Schatten ideals, Amer. J. Math. 114, No. 5 (1992), 1127–1145.

[21] B. McCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* 38, no. 4 (1986), 878–906.

[22] B. P. Palka, An Introduction to Complex Function Theory, Undergraduate Texts in Mathematics, Springer-Verlag, New-York (1991).

[23] W. T. Ross, The classical Dirichlet space, Recent advances in operator-related function theory, 171–197, *Contemp. Math.* 393, Amer. Math. Soc., Providence, RI (2006).

[24] D. A. Stegenga, Multipliers on the Dirichlet space, Illinois J. Math. 24 (1) (1980), 113–139.

[25] P. Wojtaszczyk, Banach spaces for analysts, Cambridge Studies in Advanced Mathematics 25, Cambridge University Press, Cambridge (1991).

[26] Xu X. M., Schatten-class composition operators on weighted Dirichlet spaces, *Acta Anal. Funct. Appl.* 1, No. 1 (1999), 86–91.

[27] K. H. Zhu, Operator Theory in Function Spaces, Monographs and Textbooks in Pure and Applied Mathematics 139, Marcel Dekker, Inc., New York (1990).

[28] K. H. Zhu, Schatten class composition operators on weighted Bergman spaces of the disk, J. Operator Theory 46, No. 1 (2001), 173–181.

[29] N. Zorboska, Composition operators on weighted Dirichlet spaces, Proceed. Amer. Math. Soc. 126, No. 7 (1998), 2013–2023.

Pascal Lefèvre, *Univ Lille Nord de France, U-Artois,*
Laboratoire de Mathématiques de Lens EA 2462 & *Fédération CNRS Nord-Pas-de-Calais FR 2956,*
Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18,
F-62 300 LENS, FRANCE
pascal.lefevre@euler.univ-artois.fr

Daniel Li, *Univ Lille Nord de France, U-Artois,*
Laboratoire de Mathématiques de Lens EA 2462 & *Fédération CNRS Nord-Pas-de-Calais FR 2956,*
Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18,
F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec, *Univ Lille Nord de France,*
USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524, & *Fédération CNRS Nord-Pas-de-Calais FR 2956,*
F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza, *Universidad de Sevilla,*
Facultad de Matemáticas, Departamento de Análisis Matemático & *IMUS,*
Apartado de Correos 1160,
41 080 SEVILLA, SPAIN
piazza@us.es